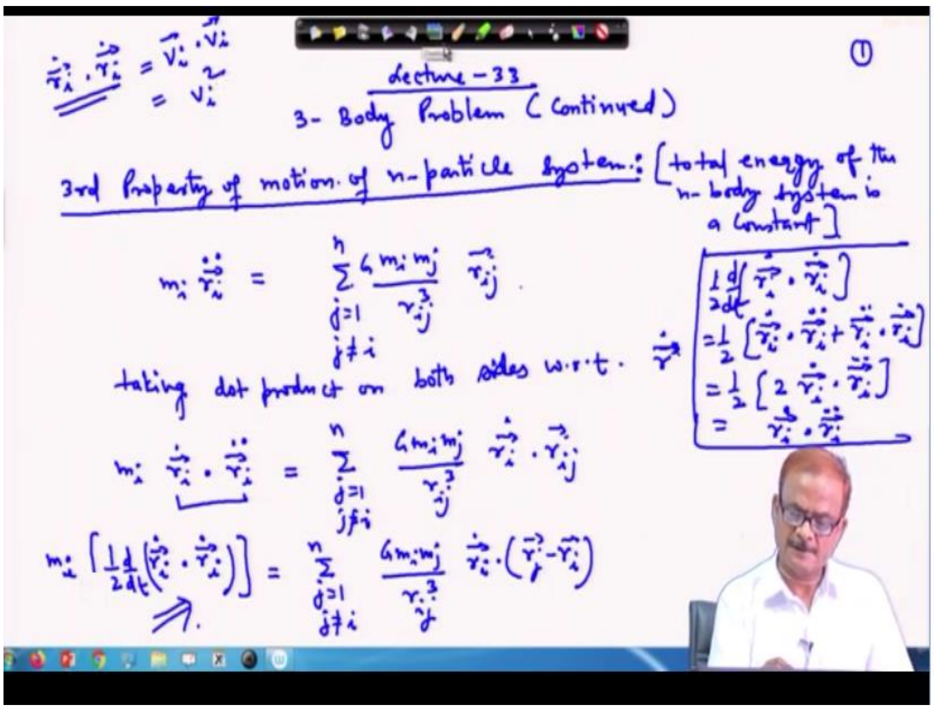
Space Flight Mechanics
Prof. Manoranjan Sinha
Department of Aerospace Engineering
Indian Institute of Technology – Kharagpur

Lecture 33
3-Body Problem (Contd.)

Welcome to lecture number 33. We have been discussing about the 3-particle system, the n-body problem. Out of that, the 3-body is one of the case. 2-body problem already we have discussed. So in that case, we were looking into the general property of the motion, so today in this lecture, we are looking at the third and the last property of general property of motion.

(Refer Slide Time: 00:41)



So here the third property of the motion, this is total energy of the n-body system, which is a constant. So again we start with the equation of motion; we have written here earlier r_{ij}^3. Now we are taking dot product on both sides with respect to $\dot{r}$. Now here in this equation, as per our 2-body problem which we were discussing, so at that time we have taken care of this kind of problem.

So this can be written as

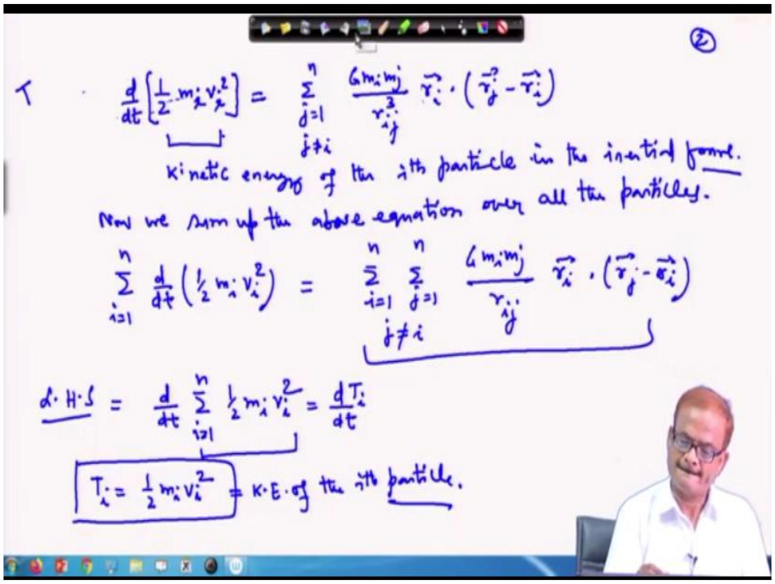
$$m_i\left(\vec{\dot{r}}_i \cdot \vec{\ddot{r}}_i\right) = \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{G m_i m_j}{r_{ij}^3}\, \vec{\dot{r}}_i \cdot \vec{r}_{ij}$$

1/2 d/dt ($\dot{r}_i . \dot{r}_i$) and dot product with ($r_j - r_i$) We expand it. This part we can verify

$$\frac{1}{2}\frac{d}{dt}\ (\dot{r}_i . \dot{r}_i) = \dot{r}_i . \ddot{r}_i$$

Therefore, whatever we have written here it is true and also you can observe that $\dot{r}_i$, this dot product we are taking. So this quantity is nothing but $\dot{v}_i\ v_i$.. So that means it is v_i^2 term.

(Refer Slide Time: 04:59)



From the left hand side, then we have 1/2 times m_i d/dt (v_i^2) This term we have written there, because m_i is a constant. So either putting it outside or inside does not matter and 1/2 also we will take it inside. So in one step, we will wind up all those things. We can write here ½ d/dt (1/2 m_i v_i^2) and summation sign later on we will put for all the particles. Right now, we have not done this.

This is the situation here right now from this equation

$$\sum_{i=1}^{n} \frac{d}{dt}(\frac{1}{2} m_i v_i^2)\ = \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{G m_i m_j}{r_{ij}^3}\ \vec{r}_i .\ (\vec{r}_j - \vec{r}_i)$$

Now on the left hand side, what we can identify that this is the kinetic energy of the i_{th} particle in the initial frame. But right hand side still it is not clear and we need to work out.

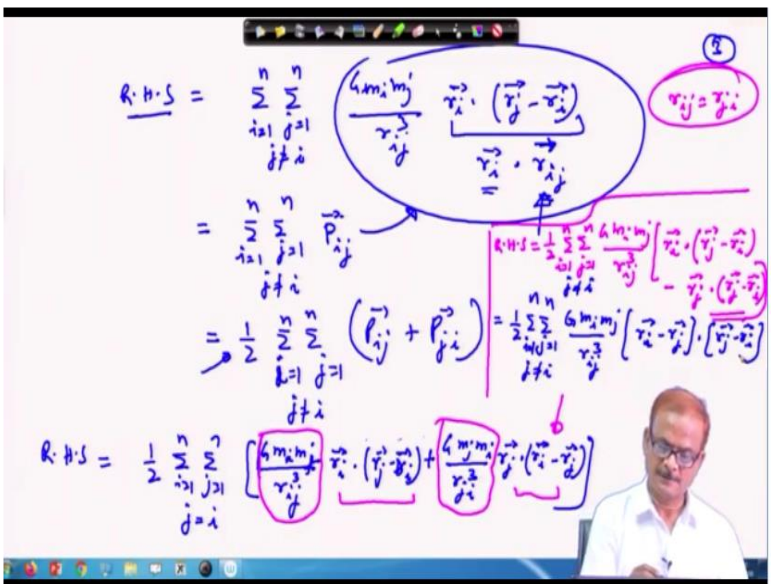
What we do, that we sum up overall the particles. Now we sum the above equation over all the particles. If we do that, this summation d/dt (1/2 m_i v_i^2)and here i = 1 to n, j not equal to y. Now this is the situation. Right hand side, we have to treat it separately. On the left hand side, first we will process. So LHS, we can rearrange it and write it this way i = 1 to n, 1/2 (m_i v_i^2), so this is d/dt and what this is, the individual kinetic energy.

And this term, kinetic energy usually we represent it by t. So I will write here Ti. So left hand side is dTi by dt where

$$T_i \ = \frac{1}{2} m_i \, v_i^2$$

kinetic energy of the i_{th} particle. We have to treat the right hand side.

(Refer Slide Time: 09:38)



On the RHS, we have G times m_i m_j r_{ij}^3 and then $\dot{r}_i - r_j - r_i$ and this quantity is nothing but your $\dot{r}_i$. r_{ij}. Now here we have to work a little cautiously. Suppose the way this whole term is appearing here, we write it as another function j not equal to i and here let us say that we write this as something. Let us represent it by some function p_{ij}. Here p_{ij} is this quantity and this is because of the symbols we are using here, the subscripts we are using here p_{ij}, m_i m_j, r_{ij}, r_i, r_{ij}.

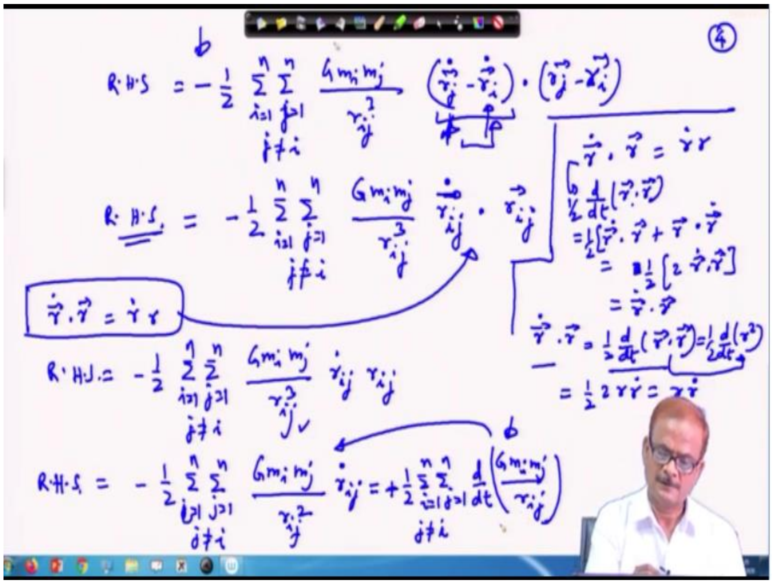
This way it is just depending on i and j and following the earlier notation again, this can be written as j not equal to y, p_{ij} + p_{ji} and 1/2 introducing here. So wherever you have i that we need to replace it by j and wherever j is there, that we need to replace by i. So if we do that the RHS gets reduced to 1/2 i = 1 to n, j = 1 to n, j not equal to y and here the change will take place. So therefore the above equation becomes m_i m_j divided by r_{ji} whole cube.

We are changing $\dot{r}_j$ r_i– r_j. This is one of the term. This term is just p_{ji}. So this term is here or let us first write the p_{ij} term and thereafter we will write the p_{ji} term. Let me rub it out and write first. So first we write the p_{ij} term from the above g m_i m_j divided by r_{ij}^3, then $\dot{r}_i$. Now I will write it in expanded format or it is better to write here itself in the expanded format, one step we can save. So this is $r_j - r_i$+ gm_j m_i divided by r_{ji}^3 $\dot{r}_j$ r_i– r_j.

So if we observe from this place, this term is common to both of them. This particular term is present on both the sides. Only thing, this term we have to take care of. So the RHS then gets reduced to 1/2 summation over j = 1 to n. Remember these are the techniques used space flight mechanics or the celestial mechanics and this will help you solve many problems. This is not just unit to this one, you will require this technique in many places.

So here we will take out gm_i m_j divided by r_{ij}^3 because r_{ij} = r_{ji}, the magnitude of the distance, it remains i to j or j to i, it remains the same. So this can be taken outside and what we are left with inside is $\dot{r}_i$ $r_j - r_i$– $\dot{r}_j$ $r_j r_j - r_i$. Here we have changed the sign. This is r_i– r_j, so this we have made here as $r_j - r_i$. Then we can observe that this gets reduced to j not equal to y, gm_i m_j r_{ij}^3 and r_j – r_i, it can be taken outside the bracket. So this is r_i– $\dot{r}_j$ $r_j - r_i$. We go on the next page.

(Refer Slide Time: 16:36)



So therefore, the RHS then becomes 1/2 j = 1 to n, i = 1 to n j not equal to y m_i m_j g r_{ij}^3 and (r_j – r_i) and here we have one is (r_i– r_j), another one is (r_j – r_i). So this sign we change here. This make it (r_j – r_i), unnecessarily one step we have to take more, so (r_j – r_i) and this $m_i$nus sign we will bring it here. We are exchanging it, r_i was here in this place, so from here we have brought here in this place.

Therefore, the sign comes here and minus sign comes here, j not equal to i, G m_i m_j divided by r_{ij}^3 and you can see that this quantity is one part we have missed out here. See while working, we often do the mistake and this mistake is longstanding. Here we have this $\dot{r}_i$. We have here $\dot{r}_i$. So we have missed the sign altogether from beginning to the end. So this is your $\dot{r}_i$. Here also, this is your $\dot{r}_i$. Left hand side, it is okay.

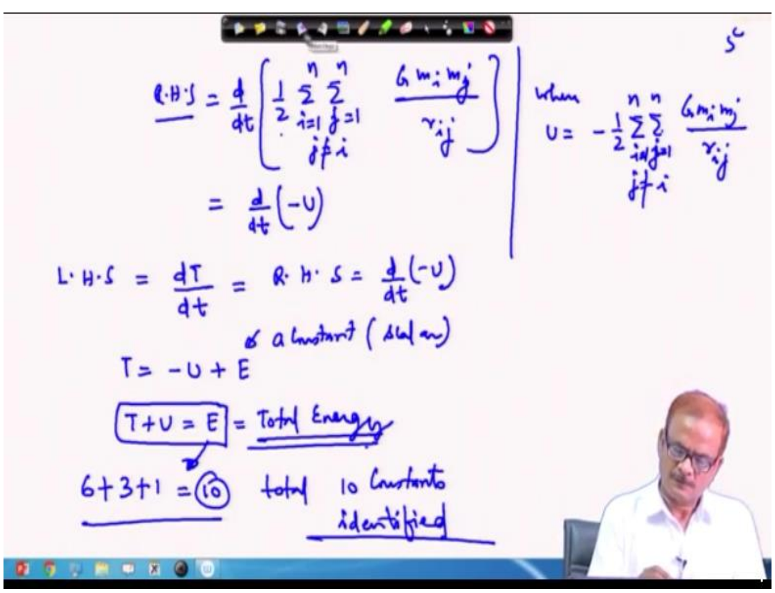
On this place, this is $\dot{r}_i$, here also we write it as $\dot{r}_i$. Now this quantity, we have written as p_{ij}. So this is okay, this part is okay, but here we need to put $\dot{r}_i$. So in this place, this will be $\dot{r}_j$. So here this will be $\dot{r}_i$ and this will be r_j. You can see that we will have dot placed over this. So from this part, finally we are coming to this place and once we have written this, this is $\dot{r}_{ij}$, this is $\dot{r}_{ij}$, because this is a vector from i to j, the symbol we are using $\dot{r}_{ij}$.

This is your right hand side and if you remember, for 2-body problem, we have used the technique that $\dot{r}$ r, the same technique we can utilize here and solve this problem. Why this is so? Because we have started with this and you can see that this can be written as. This gets reduced to this form, so this is fine. On the right hand side, the stuff is now, from here what we can observe that this can be written as 1/2 d/dt ($\dot{r}$) = 1/2 ; This we have derived here.

So 1/2 d/dt, this is r^2. This quantity becomes r^2 here. So 1/2 and once we differentiate it, so this gives us 2 times r$\dot{r}$. So this is r$\dot{r}$. Therefore, the quantity we have here this is nothing but $\dot{r}$ r and we utilize this information here in this place and therefore our right hand side this gets reduced to – 1/2 j not equal to I, $Gm_i m_j r_{ij}^3$ and this gets reduced to $\dot{r_{ij}}$ and r_{ij} and r_{ij} here is also in the denominator.

We can cancel it out and this can be again re-arranged as j = 1 to n, i = 1 to n, j not equal to i, G Fm_i m_j r_{ij}^2 $\dot{r_{ij}}$ and here little bit of work and we will be able to solve this problem 1/2 1 to n summation j = 1 to n, j not equal to i and the quantity here, this we can write as G$m_i m_j$ divided by r_{ij} d/dt and what we see that the minus sign, which is why we were carrying here, so this gets converted into plus, because once you differentiate this, you get this quantity here. So automatically, the minus sign will appear. So this is our right hand side. So we go on to the next page.

(Refer Slide Time: 23:55)



The RHS then gets reduced to 1/2 summation j = 1 to n and i = 1 to n, j not equal to i, m_i m_j and r_{ij} d/dt, we can pull outside. This d/dt, because it is differential operator, these are linear operator basically, so we will be able to take it outside. This you know from here basic mathematics, so I need not get into all these things. So we take it out of the summation sign and once we take it out of the summation sign on 1/2, we will include inside.

So 1/2 we have taken here in this place. So 1/2 comes here and Gm_i m_j and this is r_{ij}. This is what we are getting. Therefore, we write this quantity on the right hand side as –u, where u = -1/2 summation j = 1 to n, I = 1 to n, j not equal to i divided by r_{ij}. Now the left hand side, we have got. LHS we have looked into and what we have got here. LHS was dTi / dt and then we have to sum it over, all the particles. Here we have to put summation also.

So here summation is missing here in this place, i = 1 to n. Therefore, we can write this as d by dt summation T_i, i = 1 to n and this we can write as

$$\frac{dTi}{dt} = \frac{d}{dt}(-U)$$

where T_i is the kinetic energy of the i_{th} particle. So this is the part here. Therefore, our left hand side is dT/dt and this equal to right hand side equal to d/dt (– U) and therefore, once we integrate it, so this gets reduced to –u and +E, where E is a constant and this is a scalar constant.

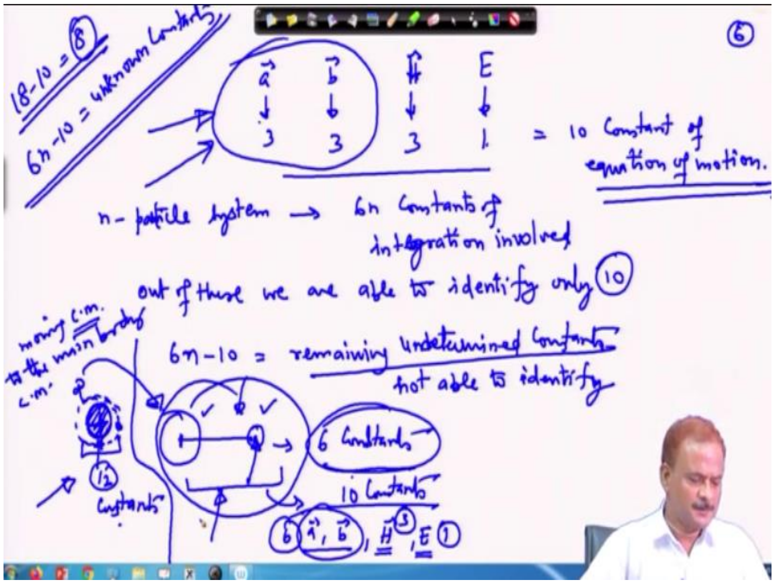
Because kinetic energy is a scalar quantity and therefore, this

$$T + U = E$$

So this is that your total energy and therefore, the last constant we have identified no more all the efforts to solve this problem more than that. People have not been able to do it. So this is the last constant you can identify. So totally we have got 6 + 3 and 1 from this place, total of 10 constants we are able to identify. So total 10 constants identified.

So total energy of the system of particles, it remains constant. Total energy of this, it is moving under mutual gravitational acceleration and obviously this is moving from under mutual gravitational acceleration and it is free from all the external forces and moreover, we are assuming that there is no dissipation of energy anywhere.

(Refer Slide Time: 28:49)



Once we have identified this constant, we got here a 3 scalar constants involved in this and v again 3 were involved, in h 3 constants were involved and in E one constant is involved. So this makes it total of 10 constants 10 constants of motion. No more we can identify and this is the general property of the motion. Now for in particle system, we have total 6n constants of integration involved and out of this we are able to identify only 10.

That means 6n – 10, these are the remaining unknown or undetermined constants. So we are not able to identify. So the 2-particle system, we have solved by assuming that one particle is heavy and another particle is quite small, so that the center of mass can be moved to the center of mass of the main body or the bigger body. So in that case, we were able to identify those, or either for the relative motion, one particle is here, another particle is here.

And then with respect to this, how this particle is moving, so for that case we have solved and we got 6 constants and for this where one is very small and one is large, so in that case we have got total 12 constants and how we have done that? This we have done by moving the center of mass or barycenter to the main body center of mass and then applying the technique of finding the 6 other constants, because the center of mass motion, that involves already.

We know that this involves 6 constants. So 6 constants are identified and thereafter only you are left with finding out the parameter related to the orbit of this particle, because there you are assuming that it is coinciding with the center of mass of these 2 systems, one is the heavier one and another one very small, whose mass is negligible as compared to the primary body. So 6 constants are identified this way and another 6 constants are identified in this way, where we describe the relative motion of one particle with respect to other.

So total of 12 constants were identified in that case, but if we have a situation where both the particles are heavy, in that case only we will be able to identify a total of 10 constants and those 10 constants are what? They are a, v, as described here in this place and plus the angular momenta, which involves 3 constants and plus energy, so that makes it 10. This involves 1, this involves 3, and this involves 6, so total of 10 constants.

So more than 10 constants we cannot identify even for 2-body problem, if both the particles are heavy. If we are taking the 3-body problem, so in the 3-body problem, it is explicitly clear that only we will be able to identify these 10 constants and rest 18 – 10 equal to 8 constants remain unknown. If we have n-particle system, so 6n – 10 these are the unknown constants, we are not able to identify. So whatever we have solved 3 particles onward system, it is not solvable.

You cannot get a closed form solution, at least for 2-particle system for the relative motion considering one particle with respect to other, we have got the constants involved. Those 6 constants we were able to decipher and we were able to get a closed form solution, but for the 3, you cannot get it. so under restricted condition, we will be able to solve this 3-particle system and there is no question of solving the 4 and other higher system where the higher number of particles are more than 3 are involved.

So for the 3, some restricted solutions are available and in the past researchers have done a lot of work on that and this problem has been solved. So we stop here and then we will continue in the next lecture. Thank you.