Space Flight Mechanics

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Lecture 6
Gravitational Central Force Motion (Contd.)

Welcome to the lecture number 6. We have been discussing about the central force motion, especially with respect to the gravitational force or either the particle moving under the gravitational force. So we are going to get the equation of motion.

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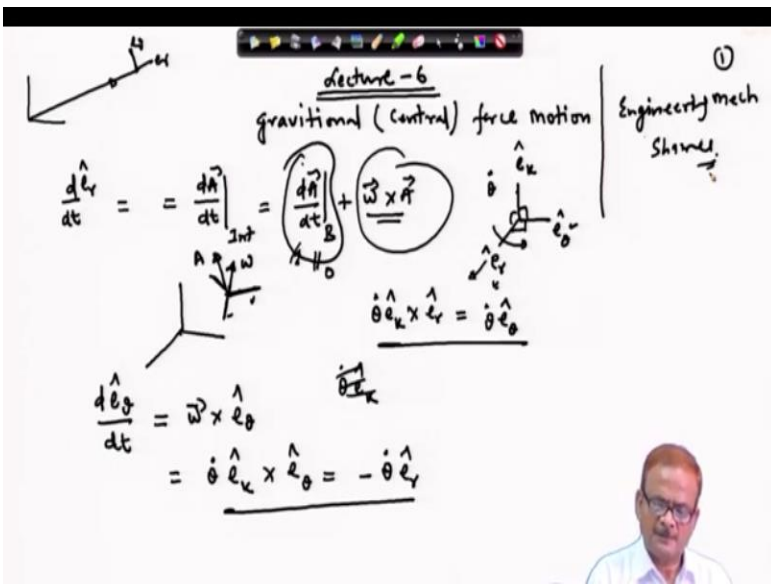

From the Newton's second law, we are trying to get the equation of motion in the geometric form or either way, earlier I have told that we are trying to derive the Kepler's equation. So one thing you remember, in mechanics, there is a transport theorem, which says that dA/ dt in the inertial frame it can be written as

$$\frac{d\vec{A}}{dt}\bigg|_{int} = \frac{d\vec{A}}{dt}\bigg|_{B} + (\vec{\omega} \times \vec{A})$$

So that means, if I have a frame like this and with respect to this another frame is rotating with angular velocity omega and the vector A in this frame, this is fixed in the body frame.

This is called the body frame; this is called the inertial frame. If it is fixed, this quantity will be the set to 0 and then we get

$$\frac{d\vec{A}}{dt} = \left(\vec{\omega} \times \vec{A}\right)$$

Using this also, you can see that here in this case, suppose we have $\widehat{e_r}, \widehat{e_\theta}$ and let us say that this is $\widehat{e_k}$. These are perpendicular to each other forming the right hand shear. So $\vec{\omega}$ is here in this direction, this is your ***r*** vector. So the ***r*** vector is rotating. So this is rotating here in this direction.

So $\dot{\theta}$ is along this direction. So $\vec{\omega}$ becomes $\dot{\theta}\widehat{e_k}$and A here in this case either it can be e_θ or it can be e_r. That means, what I am trying to do that says on the left hand side, if I write $\frac{d\widehat{e_r}}{dt}$, so rate of change of the e_r vector. So how it will look like? So here, A we need to replace it by e_r. So e_k times $e_r$is nothing but $\widehat{e_\theta}$, from here we can see.

Similarly, the other part we have been doing from the basic geometry that can also be worked out. So in that case we are looking for

$$\frac{d\widehat{e_\theta}}{dt} = (\vec{\omega} \times \widehat{e_\theta})$$
$$= \dot{\theta}\,\widehat{e_r}$$

Now here, you can see that this will be opposite to the e_r direction. What we have derived here, all these things, it can be done very easily using this method, because this is a unit vector and it is fixed in the body.

So this does not change. That means your r is here and with respect to the r vector, this is always fixed. At this point, this is the e_r and this is e_θ. So in inertial frame this is rotating. This is another way of looking, so I will tell you that you explore it more, go to the book by engineering mechanics by Beer Johnston or either you can look into the book Shames even Irving Shames. This may be a little simpler to work with. So this already we have done. We are going to advance further.

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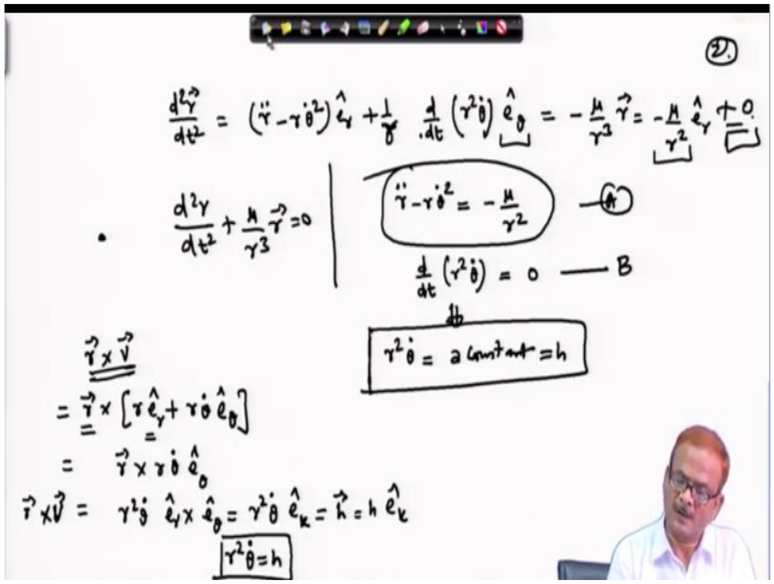

So once we have got our equation that

$$\frac{d^2\,\vec{r}}{dt^2} = \left(\ddot{r} - r\dot{\theta}^2\right)\widehat{e_r} + \frac{1}{r}\frac{d\left(r^2\dot{\theta}\right)}{dt}\widehat{e_\theta}$$

Now going back to our gravitational force motion, we go back and look into the equation we have written, this particular equation,

$$\frac{d^2\,\vec{r}}{dt^2} + \frac{\mu}{r^3}\vec{r} \;\; = 0$$

So therefore, this quantity can be written as $-\frac{\mu}{r^3}$ using this and either other way, this $\frac{\mu}{r^2}\widehat{e_r}$. So what we observe from this place, that $\ddot{r} - r\dot{\theta}^2$, this quantity is nothing but $-\frac{\mu}{r^2}$ and

$$\frac{d\left(r^2\dot{\theta}\right)}{dt} \; = 0$$

because there is no term related to e theta on the right hand side. In the vector, we equate each term corresponding to the orthogonal vector.

These are the e_θ and e_r are the orthogonal vectors, so corresponding parts we have to equate on both sides. So corresponding to this, there is no term here. This is plus 0. So therefore, this part gets eliminated here. We write it like this. So this is your equation B and this implies

$$r^2\dot{\theta} = constant = h$$

Now what this constant is, just have a look of this. We have $\vec{r} \times \vec{v}$,

$$\vec{r} \times \left(\mathrm{r}\,\hat{e_r} + \mathrm{r}\,\dot{\theta}\,\widehat{e_\theta}\right)$$

So this gives, this part, this part that becomes 0

$$\vec{r} \times \vec{v} = r^2\dot{\theta}\,\hat{e_r} \times \widehat{e_\theta}$$

. So this is $\vec{r} \times \mathrm{r}\,\dot{\theta}\,\widehat{e_\theta}$ and this is $r^2\dot{\theta}\,\hat{e_r} \times \widehat{e_\theta}$ and this is nothing but $\widehat{e_k}$. So this is your $\vec{r} \times \vec{v}$, which you have written as h. So we can write this h as

$$\vec{h} = \mathrm{h}\,\widehat{e_k}$$

because ***h*** is perpendicular to both the ***r*** and ***v*** vector. So this implies that

$$r^2\dot{\theta} = h$$

This part you should remember. This will be used frequently throughout your solving of the problems, $r^2\dot{\theta}$, this is a constant. This we are left with this equation to solve.

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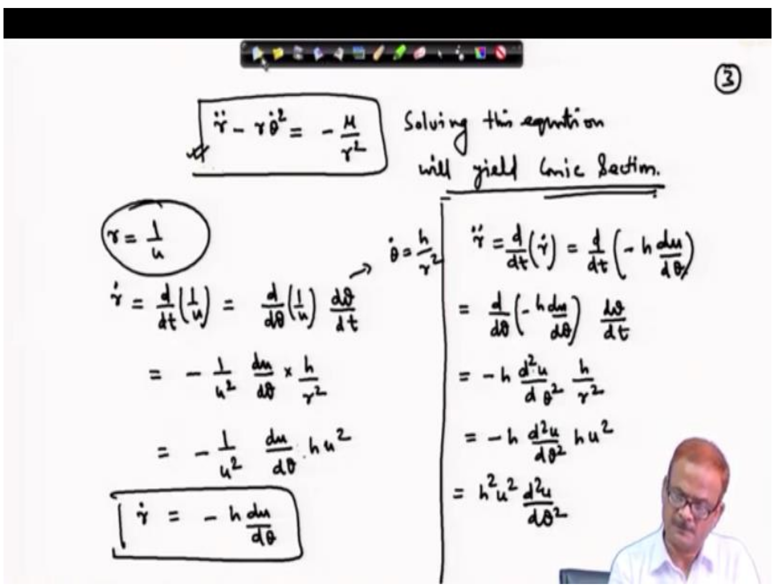

So we have here

$$\ddot{r} - r\dot{\theta}^2 = -\frac{\mu}{r^2}$$

If we insert into this $\dot{\theta}$ part from this place, which is h by r^2, so everything will be in terms of then *r*. So our objective is to solve this equation, if we solve this equation, conic section. So already we have worked out rest other things, now we need to concentrate here on this particular part. So let us write. Now in this format, it is a little difficult to solve.

Whatever I am going to work out, this is little long, but this is bit simple. There are other methods also available, using which you can solve this equation and you can get the conic section equation. So anytime if you forget everything, you just go by this method and you will be able to work it out. Let us write

$$r = \frac{1}{u}$$

and therefore $\dot{r}$, this will be equal to

$$\dot{r} = \frac{d}{dt}\left(\frac{1}{u}\right)$$

and we will write this as

$$= \frac{d}{d\theta}\left(\frac{1}{u}\right)\frac{d\theta}{dt}$$

we use this from this place $\dot{\theta}$is here. This quantity is $\dot{\theta}$equal to h/r^2. So this is h/r^2 times h, these are not cross product. This is just multiplication and we can remove it, h times 1/r is u. So this is u^2.

$$\dot{r} = -h\frac{du}{d\theta}$$

Next we have to get the other quantity

$$\ddot{r} = \frac{d}{dt}(\dot{r})$$

$$= \frac{d}{dt}\left(-h\frac{du}{d\theta}\right)$$

Same way, we can write it

$$= \frac{d}{d\theta}\left(-h\frac{du}{d\theta}\right)\frac{d\theta}{dt}$$

is nothing but h/r^2. This is

$$= -h\frac{d^2u}{d\theta^2}\frac{h}{r^2}$$

is nothing but from this place

$$= \frac{h^2}{r^2}\frac{d^2u}{d\theta^2}$$

Now we can try to solve this equation with this information.

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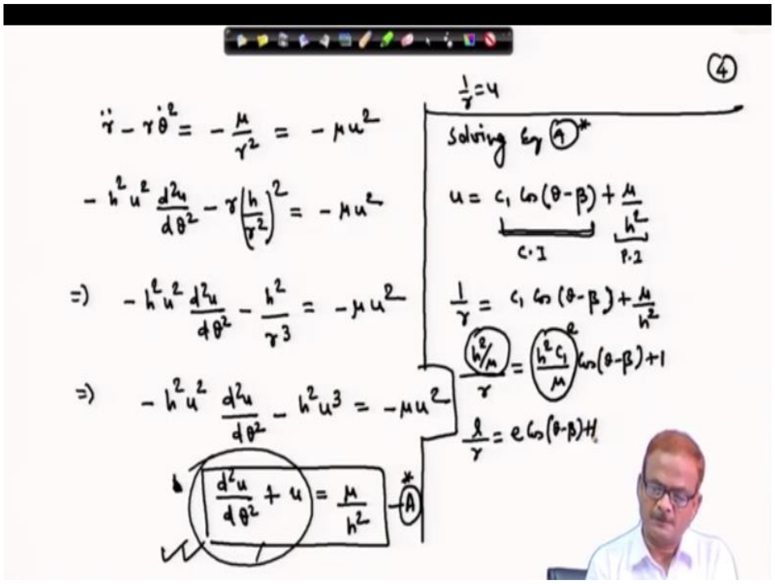

So

$$\ddot{r} - r\dot{\theta}^2 = - \frac{\mu}{r^2}$$

So right hand side, we can replace by $\mu\ u^2$ writing

$$\frac{1}{r} = u$$

and the left hand side, we have to accordingly utilize the relations from here. This is your $\ddot{r}$. So we will use this first

$$-h^2u^2\frac{d^2u}{d\theta^2} - r\left(\frac{h}{r^2}\right)^2 = -\mu u^2$$

We need $\dot{\theta}$. So first we will convert this, then we will replace.

So r times $\dot{\theta}$is h /r^2. $\dot{\theta}$ is h /r^2. We go according to this relationship

$$.\ \dot{\theta} = \frac{\mathrm{h}}{\mathrm{r}^2}$$

So here this will be

$$-h^2u^2 \frac{d^2u}{d\theta^2} - \frac{h^2}{r^3} = -\mu u^2$$

Divide throughout by u^2 and this minus and minus sign gets canceled out. So h^2 also we can take it on the right hand side divide by h^2 u^2.

So this becomes

$$\frac{d^2u}{d\theta^2} + u = \frac{\mu}{h^2}$$

Now what we can see that this side is in the simple harmonic motion form and therefore, it is very easy to solve. So the exercise we have done of reducing, here it was in a nonlinear form, no conducive to integration very easily and by those substitution, it has been rendered in this format, which is easily integrable. Other ways are also there. So sometimes if I get time, I will introduce that.

And moreover remember that during the course, I will be giving hard copy of all these things. If I am explaining it later on, because this is a mix of the course, this is both elementary and the advanced part will be present in this one. So some part of this the hard copies are available. So I will give you the printed hard copy, the typed one and the other parts may be the hand written part, I will supply. The solution to this, now it is very easy and we can work it out.

This equation we will number as, we have not numbered any equation, so let us leave it. Now let us say this is A here. Solving equation A and I will put here a star, so this gives you a relation of the form. The part we have written this is called complimentary integral and this part we call as a particular integral. If you remove this and just insert this part here $u = \frac{\mu}{h^2}$ so you can see that this part will be 0 and both sides will be satisfied.

You know from the simple harmonic motion that this is the solution. So if now we write it as

$$\frac{1}{r} = c_1 \cos(\theta - \beta) + \frac{\mu}{h^2}$$

and rewrite it

$$\frac{(h^2/\mu)}{r} = \left(\frac{h^2}{\mu}\right) c_1 \cos(\theta - \beta) + 1$$

. We go to the next page or either here I can conclude this part. This particular quantity I will write as l. So l/r and this quantity I will write as

$$\frac{l}{r} = e\cos(\theta - \beta) + 1$$

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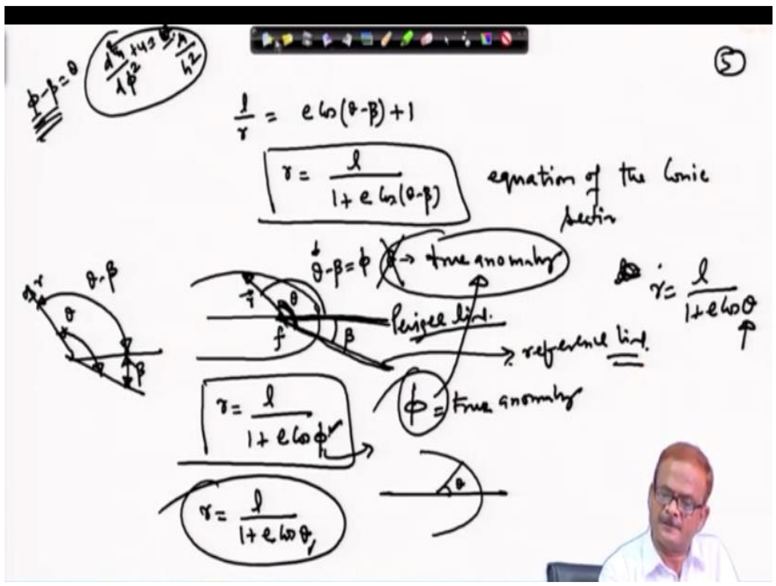

or if I rewrite it this becomes

$$r = \frac{l}{1 + e\cos(\theta - \beta)}$$

and if you remember the conic section equation, in the beginning of the few lectures, we have worked out. So this is the equation of the conic section. Suppose this is the focus and from here, *r* is always measured from the focus, remember. This angle is, usually we have written this as a true anomaly theta as we have indicated as the true anomaly.

But here in this case, this is not being measured from this line. Suppose this is being measured from this line, from here. If I write this angle as β, you can see that this part will be $\theta - \beta$ and let us indicate this as φ. So r will be

$$r = \frac{l}{1 + e\cos\varphi}$$

So here φ, this is part rather than calling this as the true anomaly, phi we will call as the true anomaly, because θ is being measured from this difference, from this place, from here to here.

This is your *r* vector and from here to here we are measuring theta and this is your β. So this angle from here to here, then this becomes $\theta - \beta$. Now what I am going to do, that instead of using this phi, I am always going to use this particular equation e cos θ assuming that we are measuring θ from this place. Remember, this is a very simple matter. This symbol I am just replacing with some other symbol. This symbol has been replaced by this symbol.

You can work with this, but it is a customary to use θ throughout. So for that, I require that instead of expressing everywhere the angles as the θ like here, the angles we have used everywhere as θ, all the places. So instead of that, I could have written there φ. So if I write in terms of φ, this will appear here is $\varphi - \beta$ and then I can write equal to θ. So I can write

$$\varphi - \beta = \theta$$

if I express everywhere the equation like $\frac{d^2u}{d\varphi^2}$.

If I would have written in terms of this, h^2, this equation $\frac{\mu}{h^2}$. So this is $\frac{\mu}{h^2}$. If I write in terms of this, so the angle, this will appear. So it is a customary to write the expression of the conic section as 1 + e cos θ. It is expressed in θ where θ is called the true anomaly. True anomaly is measured from the perigee position. So we will come to that, what is the perigee position and other things.

So here in this case, I will write this as a perigee, perigee line or periapsis and this is some reference line. So you can see that using this very simple format, we have been able to work out the conic section equation.

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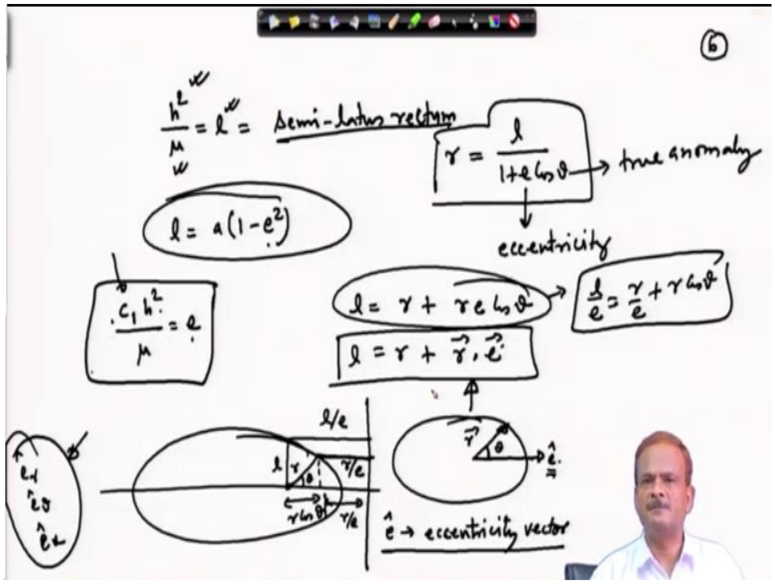

So the quantity

$$\frac{h^2}{\mu} = l$$

this is nothing but your semi-latus rectum. So if you know h; for a particular particle or maybe a planet and if you know μ, which is a planetary gravitational constant. So immediately you can calculate this l. As you know,

$$l = a(1 - e^2)$$

Another one, already we have written here this quantity, so if we know this constant h is known, if c_1 is also known, so e can be calculated. So we can insert the value of e here in this place.

We will see over a period of time how to tackle all these problems. For the time being, it is suffices that we always remember that our equation of conic section is given by

$$r = \frac{l}{1 + e\cos\theta}$$

where e is called the eccentricity and θ is the true anomaly and l is the semi-latus rectum, which is given by this expression. Now if we rewrite this part, so

$$l = r + re\cos\theta$$

and if you remember in our conic section part, once we were discussing about the ellipse.

This is your r, this is θ and then we had directrix here. This was l, this was l/e and this was r/e. So see the same format we are getting or not. If we take this part and rewrite this as l/e, so, this becomes

$$\frac{l}{e} = \frac{r}{e} + r\cos\theta$$

Check this; this is r/e, this part and r cos $\dot{\theta}$will be nothing but this particular part. This is r cos $\dot{\theta}$. So the conic section, whatever we have described earlier, it is visible directly from this place. That means, the gravitational force motion can be described by this conic section equation. Also, you can see from this place, if I again draw this ellipse r is a vector along this direction and this is $\dot{\theta}$. So

$$l = r + \vec{r}.\overrightarrow{e}$$

so e is a vector along this direction along the periapsis. Then only, you get $re\ cos\theta$. So obviously, this is not a unit vector.

This is eccentricity vector, $\widehat{e}$ is called the eccentricity vector and do not confuse it with what we are using as $\widehat{e_r},\ \widehat{e_\theta}\ and\ \widehat{e_k}$. Never confuse with this. These are the unit vectors and this is the eccentricity vector. This is not a unit vector. The eccentricity vector is directed along the periapsis and therefore $\vec{r}.\overrightarrow{e}$gives you $re\ cos\theta$. So right now, I am stating like this, but later on we will prove this directly using the equation of motion under the gravitational force.

So today, we conclude with this lecture. Thank you very much for listening. We will develop this step by step and I hope you find it useful and interesting also. Thank you very much and I will be supporting all these lectures with the hard copy of the material and also the hand written part, so you will not face any problem during the course. Thank you.