Space Flight Mechanics
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Lecture-60
General Orbit Perturbation Theory (Contd.)

Ok welcome to lecture 60. We have been working with the Lagrange planetary equation and we were trying to reduce it in terms of F_r, F_θ and F_A.

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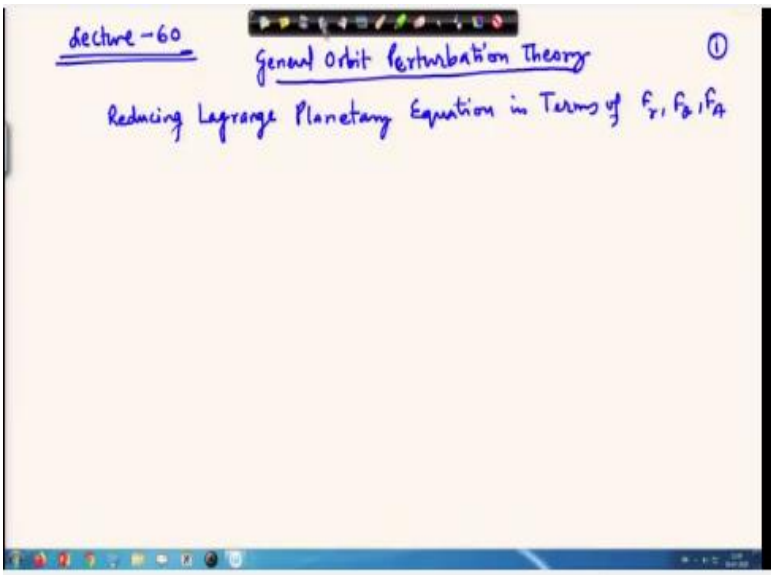

So, in that context if you have if you go back.

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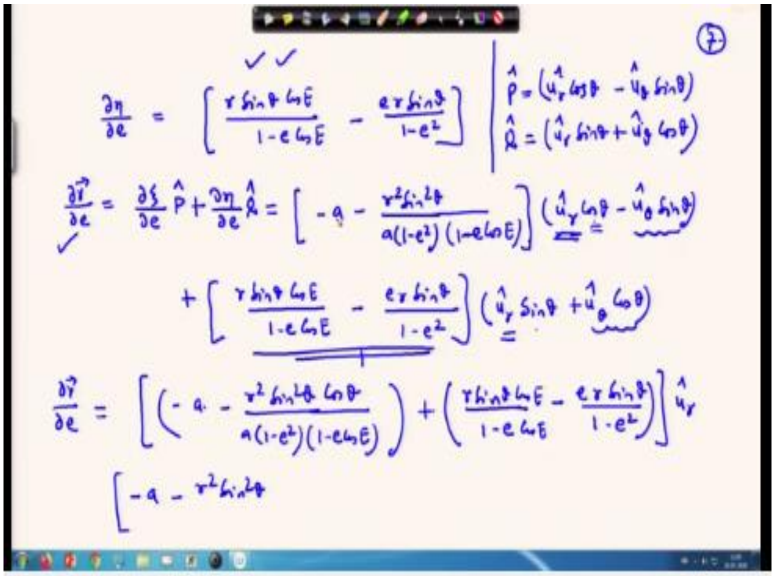

So, this is the equation for the $\frac{\partial \vec{r}}{\partial e}$ we have worked out. So, if we now expand this equation and combine the terms of u_r and u_θ together. So, let us combine it here in this place itself. So, $\frac{\partial \vec{r}}{\partial e}$ times this can be written as the term which u_r first we combine, that becomes - u a - a times $r^2\sin^2\theta$, cos θ, this cos θ gets multiplied with this and then from this part quantity in the bracket has to be multiplied by sin θ and added.

So, we add to this r sin θ cos E 1 - cos E and this multiplied by $\hat{u}_r$. And thereafter the θ term we pick up and write here, so all these equations are pretty lengthy to work with. In this case ok here one more term is there once we multiply it with cos θ, so because cos θ we have taken inside.

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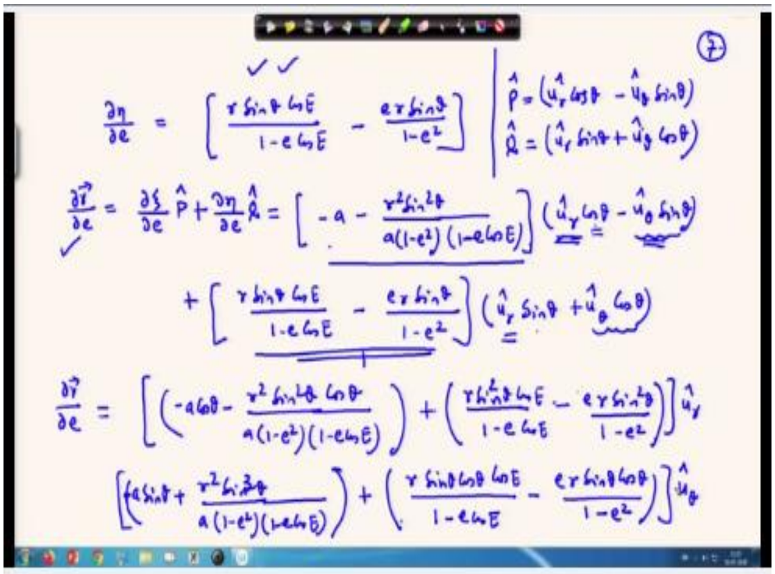
$$\frac{\partial \vec{r}}{\partial e} = \left[-a\cos\theta - \frac{r^2\sin^2\theta\cos\theta}{a(1-e^2)(1-ecosE)}\right] + \left[\frac{r^2\sin^2\theta\cos E}{1-ecosE} - \frac{ersin^2\theta}{1-e^2}\right](\hat{u}_r) + \left[a\sin\theta\frac{r^2\sin^3\theta}{a(1-e^2)(1-ecosE)}\right] + \left[\frac{rsin\theta\cos\theta cosE}{1-ecosE} - \frac{ersin\theta cosE}{1-e^2}\right]\hat{u}_\theta$$



So, we should write here - a cos θ and similarly here inside we have not multiplied, so we need to multiply that also. So, we need to complete that part, r sin θ, cos E times sin θ, u_r part is getting multiplied. So, this will be sin a^2 θ cos E and then - e r $\sin^2$θ divided by 1 - e^2 and times $\hat{u}_r$. Similarly here in this place, once this term we are taking this term, so we have to multiply and this term will be - a minus, minus that gets minus sign will absorb inside.

$$\frac{\partial \vec{r}}{\partial e} = \left[-a\cos\theta - \frac{r^2\sin^2\theta\cos\theta}{a(1-e^2)(1-ecosE)}\{rcos\theta - a\cos E\ (1-e^2) + ae\ (1-ecosE)\}\right]\hat{u}_r +$$

$$\left[a\sin\theta + \frac{r^2 sin\theta}{a(1-e^2)(1-ecosE)}\{rsin^{2\theta} + acos\theta cosE(1-e^2) - ae\ cos\theta(1-ecosE\}\right]\hat{u}_\theta$$

So that minus, minus that gets plus a sin θ ok. And then the next one we have to multiply, so r square and that minus, minus this also gets reduced to plus and r square term it remains there, this term becomes Q, so sin Q times a 1 - a^2 1 - e cos E. So, this is the term we have picked up and written here. And thereafter we pick up this term and write it here, so this will be plus sign r sin θ into cos θ cos E divided by minus and this is $\hat{u}_\theta$. Now whatever the terms we can cancel, we can cancel it.

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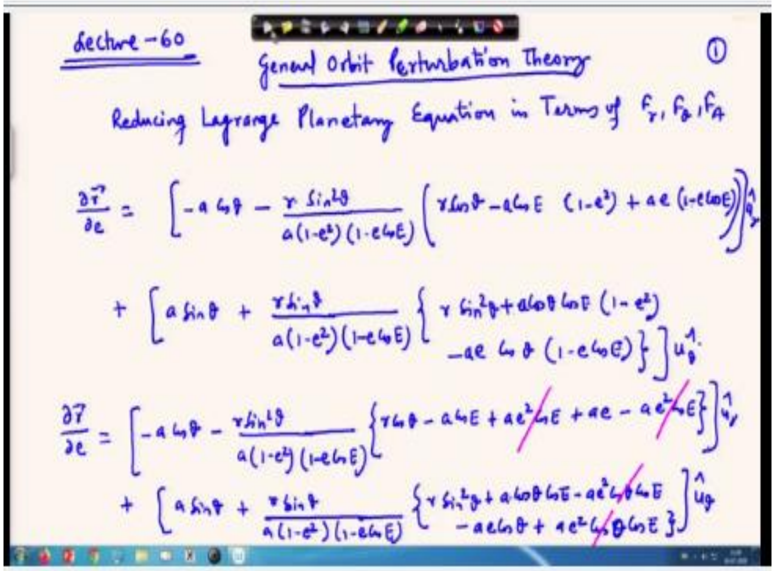

So, in next page we go, - a cos θ the first term we are picking it here, from here $r^2\sin^2\theta$ a times $(1-e^2)(1-ecosE)$, we are taking it outside. As you will see that some of the terms they cancel out, the first term then from this place, once we take it here the cos θ is remaining, r we have taken outside. So, 1 r will be there, so r cos θ and the same way the other terms can be written cos E times.

Here we can write a 1 - e^2ok, next we are taking up this term. So, because we have taken this term outside the bracket, so accordingly this term appears here. And then + a e times 1 - e cos E and this bracket closed here and this is $\hat{u}_r$. The same way the next term we can write as related to u_θ r sin θ a cos θ cos E times 1 – e^2 a times e cos θ 1 - e cos E $\hat{u}_\theta$.

Now whatever the terms can be dropped out, we will cancel it. Here we have 1 - e cos E, in this place we have 1 - e cos E dropping in a term it will cause a lot of problem. So, here if you look in the next step I will write for clarity in the next step, r cos θ and we are expanding it a cos E - a times e^2 minus, minus that gets plus a times e^2 cos E + a e - e^2 cos E × $\hat{u}_r$.

$$\frac{\partial\vec{r}}{\partial e}=\left[-a\cos\theta-\frac{r^2\sin^2\theta}{a(1-e^2)(1-ecosE)}\{rcos\theta-acosE+ae^2cosE+ae-ae^2cosE\}\right]\hat{u}_r+$$

$$\left[a\sin\theta+\frac{r^2sin\theta}{a(1-e^2)(1-ecosE)}\{rsin^2\theta+acos\theta cosE-ae^2cos\theta cosE-ae\,cos\theta-aecosE\}\right]\hat{u}_\theta$$

And plus this term here a sin θ only we expand it here r times sin square θ + a cos θ cos E - a e^2 cos θ cos E. And then from this place - a e cos θ + a e^2 cos θ cos E bracket closed times $\hat{u}_\theta$. Now whatever the terms is can be dropped out we will drop it, this term and this term cancels.

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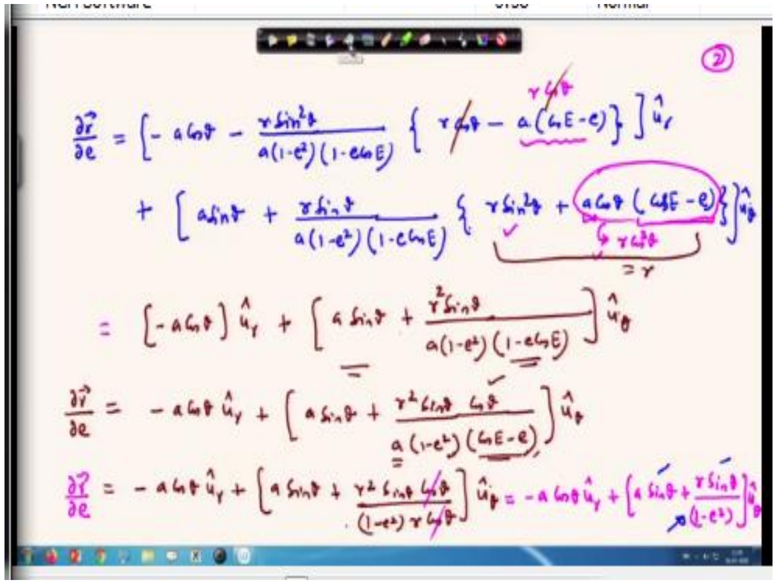

So, the above equation then gets reduced to r $\sin^2\theta$, so we have here r cos θ and then this term and this term cos E - e ok. So, if one step I will save here by writing this quantity, quantity a times cos E - e is nothing but r cos θ. So, this whole term then gets reduced to r times $\cos^2\theta$, so, this term is nothing but r × $\cos^2\theta$, so this plus this that makes it r, so this term will the second term then gets reduced to a simple format.

$$\frac{\partial \vec{r}}{\partial e} = \left[-a\cos\theta - \frac{r^2\sin^2\theta}{a(1-e^2)(1-e\cos E)}\{r\cos\theta - a(\cos E - e)\}\right]\hat{u}_r + \left[a\sin\theta + \frac{r^2\sin\theta}{a(1-e^2)(1-e\cos E)}\{r\sin^2\theta + a\cos\theta(\cos E - e)\}\right]\hat{u}_\theta$$

Similarly here in this place this term is r cos θ ok, so therefore we can immediately see that this term and this term this cancels out ok leaving out with - a cos θ times $\hat{u}_r$ and + a sin θ. And from here we will have r sin θ divided by a times 1 – e^2 1 - e cos E and then r coming from this place, so this is equal to r, so we make it r square $\hat{u}_\theta$ ok, already we have written here, this part.

$$\frac{\partial \vec{r}}{\partial e} = [-a\cos\theta](\hat{u}_r) + \left[a\sin\theta + \frac{r^2\sin\theta}{a(1-e^2)(1-e\cos E)}\right]\hat{u}_\theta$$

$$\frac{\partial \vec{r}}{\partial e} = -a\cos\theta\,\hat{u}_r + \left[a\sin\theta + \frac{r^2\sin\theta\cos\theta}{a(1-e^2)(\cos E-e)}\right]\hat{u}_\theta$$

Now, we have to look into can we simplify it a little bit more, so whatever the relations we are aware of, so you will utilize those relation and insert here in this place to simplify it. So, this quantity is then can be written as r sin θ a sin θ + r^2 sin θ cos E - e this times $\hat{u}_\theta$. So, we are utilizing those relationships which we have derived earlier in our earlier lectures, here we have replaced this quantity by r cos θ.

$$\frac{\partial \vec{r}}{\partial e} = -a\cos\theta\hat{u}_r + \left[a\sin\theta + \frac{r^2 sin\theta\cos\theta}{(1-e^2)rcos\theta}\right]\hat{u}_\theta$$

So, if the part the substitution we have done here in this place, this is 1 - e cos E we have replaced by the quantity cos θ divided by cos E - e. So, look back into the lectures there you will find all these derivations. So, we have got to this point, now cos θ, cos θ here cancels out this part and this part it cancels out ok and r, r also will cancel out. So, this will get simplified to - a cos θ $\hat{u}_r$ + a sin θ, r sin θ divided by 1 - e^2 times $\hat{u}_\theta$ ok with this substitution now we have got the $\frac{\partial \vec{r}}{\partial e}$.

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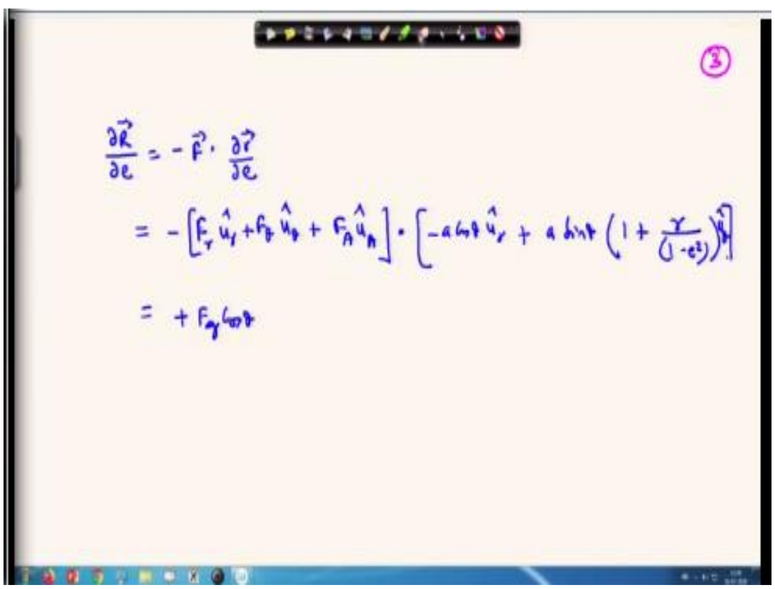

And therefore $\frac{\partial \vec{r}}{\partial e}$ can be written as – F. F_θ times $\hat{u}_\theta$ and F_A times $\hat{u}_A$ d$\frac{\partial \vec{r}}{\partial e}$ which is - a cos θ $\hat{u}_r$+ a sin θ. Here sin θ we are taking it outside the bracket and writing it this way. This has been taken outside the bracket 1 + r by 1 - e^2 u_θ and this is the dot product here. Therefore this gets reduced to minus, minus that gets plus, F_r.

$$\frac{\partial \vec{R}}{\partial e} = -\vec{F}.\frac{\partial \vec{r}}{\partial e}$$

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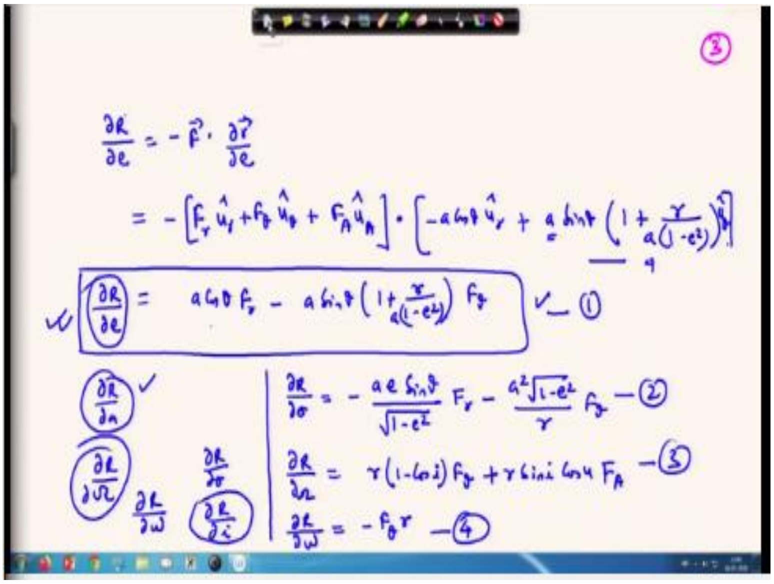

I will write it this way a cos θ times F_r that gets minus, minus plus and thereafter the next one - a sin θ 1 + r by 1 – e^2 F_θ. So, this is a $\frac{\partial R}{r}$ is not a vector, it is a potential function we have chosen, so, we should write it that way, so it is a scalar. So, therefore we have got $\frac{\partial \vec{R}}{\partial e}$, a cos θ F_r - a sin θ 1 + r this is ok**.** Here once we take it outside, so a will be coming here see if in this equation a was already taken off.

$$\frac{\partial \vec{R}}{\partial e}= [F_r\hat{u}_r + F_\theta\hat{u}_\theta + F_A\hat{u}_A].\left(-a\cos\theta\hat{u}_r + \left[a\sin\theta + \frac{r}{a(1-e^2)}\right]\hat{u}_\theta\right)$$

$$\frac{\partial \vec{R}}{\partial e} = acos\theta F_r - asin\theta\left(1 + \frac{r}{a(1-e^2)}\right)F_\theta \qquad \text{.... (1)}$$

So, here there is no a present ok, so once we take a outside as we have done here in this place, we have taken a outside and therefore the a will appear in the denominator. So, in this place we have multiplied by a. So, till now, what we have got we have got $\frac{\partial \vec{R}}{\partial a}$ and $\frac{\partial \vec{R}}{\partial e}$. And what other things are required those are $\frac{\partial \vec{R}}{\partial \Omega}$, $\frac{\partial \vec{R}}{\partial \omega}$ ok and $\frac{\partial \vec{R}}{\partial i}$, so I will conclude with this one.

So, you can see that the derivation maybe a little longer but if you follow a systematic trend then you will get the result ok. So, I will conclude with this I will not derive everything here in this place. So, concluding the result at this place $\frac{\partial r}{\partial \sigma}$, $\frac{\partial r}{\partial a}$ already we have done. So, $\frac{\partial r}{\partial \sigma}$was a remaining $\frac{\partial r}{\partial \sigma}$also we have to get.

So, this quantity can be written as - a e sin θ divided by ok this is one equation then $\frac{\partial \vec{R}}{\partial \Omega}$ where u equal to ω + θ, this 3 are here and the $\frac{\partial r}{\partial e}$ already is written here. So, let us say this is 1, this is 2, 3, 4 $\frac{\partial r}{\partial i}$we need to write here and $\frac{\partial \vec{R}}{\partial \Omega}$ we need to write.

$$\frac{\partial R}{\partial \sigma} = -\frac{aesin\theta}{\sqrt{1-e^2}}F_r - \frac{a^2\sqrt{1-e^2}}{r}F_\theta \qquad \ldots (2)$$

$$\frac{\partial R}{\partial \Omega} = r\,(1 - cos\theta)F_\theta + r\;sini\;cosu\;F_A \qquad \ldots (3)$$

$$\frac{\partial R}{\partial \omega} = -\,F_\theta r \qquad \ldots (4)$$

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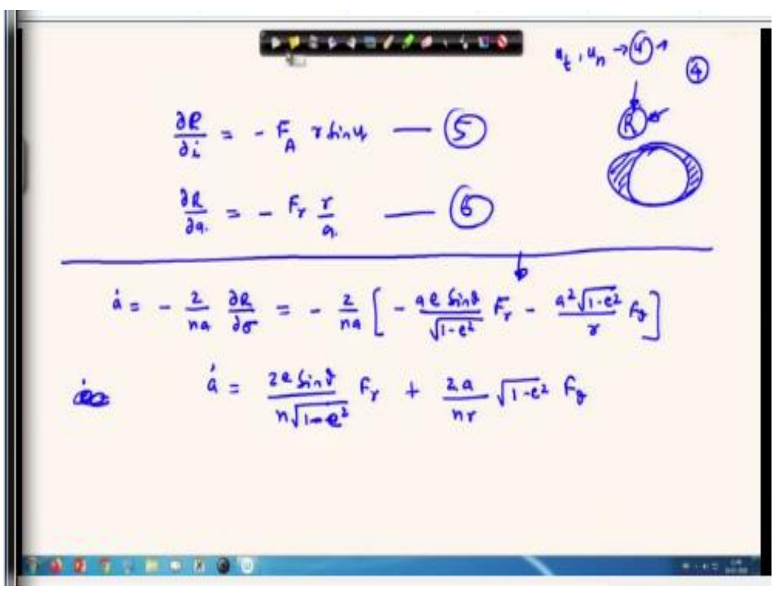

$\frac{\partial \vec{R}}{\partial \Omega}$ Already we have written $\frac{\partial r}{\partial a}$ we need to write. So, previously we have derived, so again we will summarize here in this place $\frac{\partial r}{\partial i}$ and $\frac{\partial r}{\partial a}$ we have written as F_r times - F_r times say by r +. So, ∂r this is - F_r times r by a, so we have 1, 2, 3, 4, this is 5 and 6. So, once we have got this, now we need to take the Lagrange planetary equation and insert all these things in that equation to get the final format.

$$\frac{\partial R}{\partial i} = -F_A\, r\, sinu \qquad ...(5)$$

$$\frac{\partial R}{\partial a} = -F_r \frac{r}{a} \qquad ...(6)$$

So, therefore a dot then becomes - 2 by na $\frac{\partial r}{\partial \sigma}$. So, $\frac{\partial r}{\partial \sigma}$ we have written here, so this quantity can be inserted. So, the Lagrange planetary equation once we convert in terms of F_r , F_θ and F_A, so this is the way it looks like. So, you can see that R was not known but here the F_r generally the potential will be required this potential can be from the perturbation of the say in the case of the earth and it is satellite.

So, it maybe perturbation due to the non spherical earth, it maybe perturbation due to the moon, ok if moon, sun, all these things will be taken into account. So, in the case of the aerodynamic drag once it is a present, so we have to go by this not by this one. So, u_t and u_n and perpendicular to this vector, which I do not remember by which symbol we have denoted it. So, we need to utilize this but this has also got it is own application ok.

$$\dot{a} = -\frac{2}{na}\frac{\partial r}{\partial \sigma} = \frac{2esin\theta}{n\sqrt{1-e^2}}F_r + \frac{2a}{nr}\sqrt{1-e^2}F_\theta$$

So, this is the way we convert from one format to another format whenever it is required. And of course but it is a difficult to remember these equations, it is not possible to remember all these things. But for the exam purpose, we will be more analytical and with the understanding of the subject rather than doing the just the mathematics or remembering it. Similarly, we have $\dot{e}$ already I have concluded all these equations.

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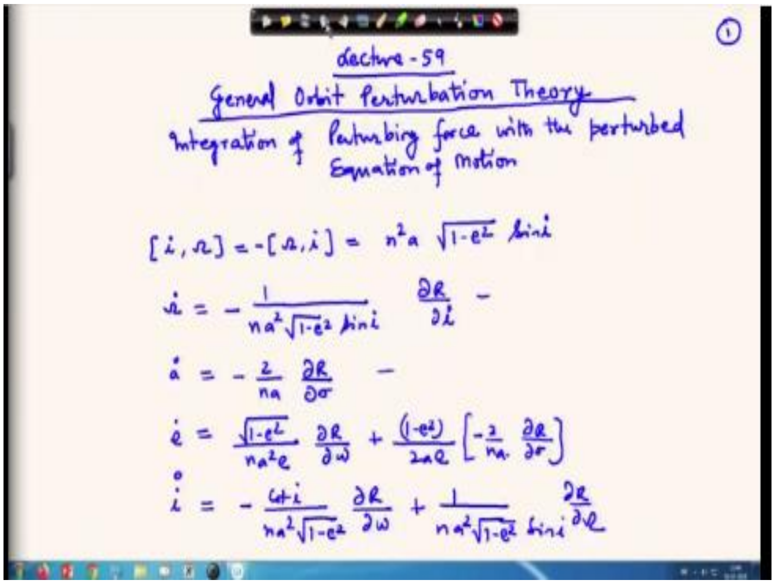

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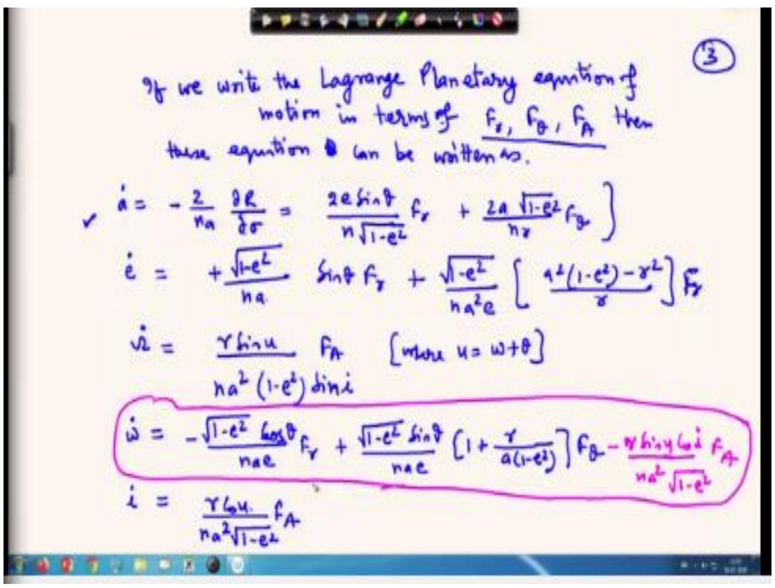

If we go back and here if you see, I have already concluded, this part already I have written here 2 e sin θ n times 1 – e^2 ok. So, the same thing we have arrived at by following sequential way of working. So, once we let us expand it and write it in a proper way, so this gets reduced to 2 e sin θ divided by n times 1 – e^2 under root F_r then this gets plus sign 2a divided by nr 1 - e^2 F_θ. So, we will quickly we will wind this topic now so e dot.

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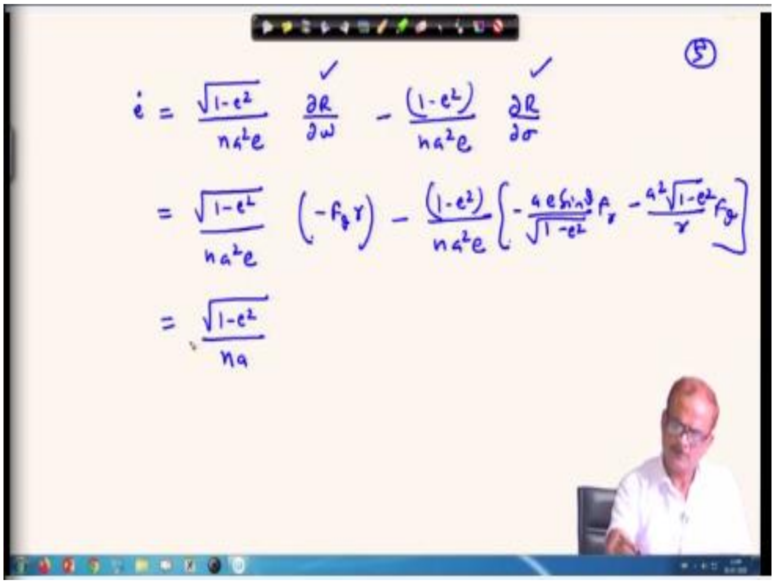

$$\dot{e} = \frac{\sqrt{1-e^2}}{na^2e}\frac{\partial R}{\partial \omega} - \frac{(1-e^2)}{na^2e}\frac{\partial R}{\partial \sigma}$$

Now we can see that $\dot{e}$ will be $1 - e^2$ under root divided by $\partial\sigma$. So insert in this equation, these quantities we have already written and $\frac{\partial r}{\partial \sigma}$ we have to insert. And once I simplify it, this will get reduced to $1 - e^2$, the rest of the steps you can carry out, I am skipping some of the steps here minus ok.

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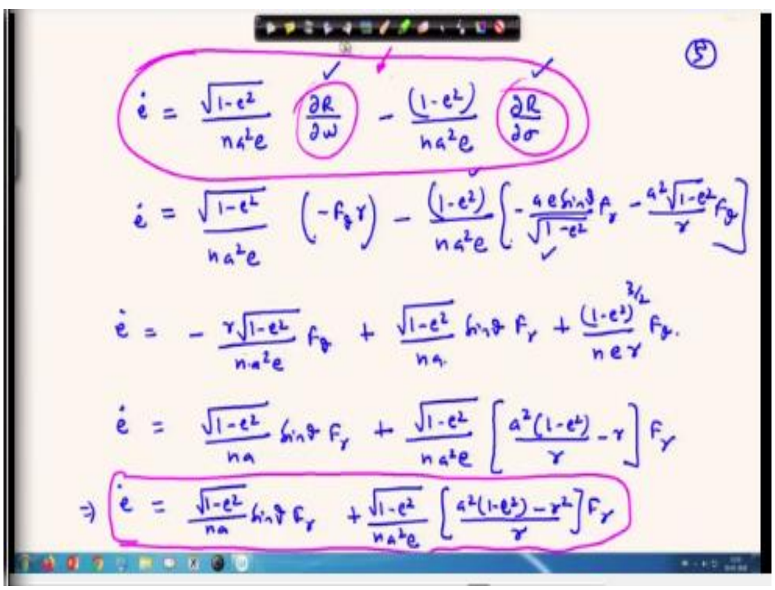

Here this and this we simplify write it this way, a will cancel out e cancels out on this the denominator we get is na sin θ F_r. And then plus $1 - e^2$ to the power 3 by 2 a square cancels out, this is n e r F_θ. So, F_θ term can be combined together and this can be written as $\dot{e}$ equal to 1 – e^2 under root divided by na sin θ times F_r +, so this is a question related to $\dot{a}$ ok.

$$\dot{e} = \frac{\sqrt{1-e^2}}{na} sin\theta F_r + + \frac{\sqrt{1-e^2}}{na^2 e}\left[\frac{a^2\,(1-e^2)-r^2}{r}\right] F_r$$

So, same way we can pick up the Lagrange planetary equation which will be appearing in this format and then we need to replace the terms like this.

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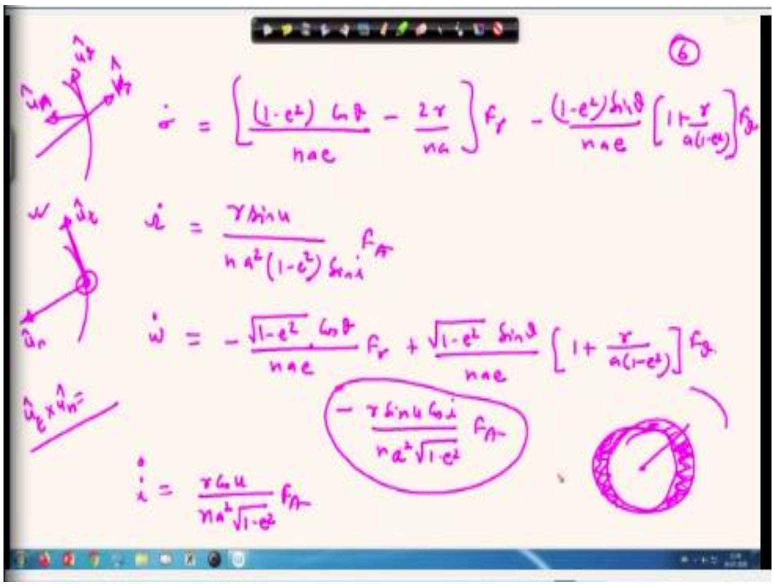

So, if we do that we get all the other terms, then the $\dot{\sigma}$ will be 1 – e^2cos θ ok. If this thing I am leaving to you to work it out because once you know the basic things, so it is just a matter of inserting and simplifying the resulting equation ok. So, this way we have worked out all the necessary terms ok. And here while writing this one term we have missed out which we have written here, this term was here r sin u divided by times cos i.

$$\dot{\sigma} \;=\; \left[\frac{(1-e^2)cos\theta}{nae} - \frac{2r}{na}\right] F_r - \frac{(1-e^2)sin\theta}{nae}\left[1 + \frac{r}{a(1-e^2)}\right] F_\theta$$

$$\dot{\Omega} = \frac{rsinu}{na^2(1-e^2)sini} F_A$$

$$\dot{\omega} = \frac{-\sqrt{1-e^2}\;cos\theta}{nae} F_r + \frac{\sqrt{1-e^2}\;sin\theta}{nae}\left[1 + \frac{r}{a(1-e^2)}\right] F_\theta - \frac{r\;sinu\;cosi}{na^2\sqrt{1-e^2}} F_\theta$$

$$\dot{i} = \frac{rcosu}{na^2\sqrt{1-e^2}} F_A$$

This is the correction here in this point, this in the lecture 59 ok. So, this way we conclude this topic. Now but still our job is not done. Say if once we have got this planetary equation, so until unless we put this F_θ and F_r all these things and then integrate this equation, we would not be able to know what the value of e will be in the future. So, we were looking for a general solution, so that step remains.

But we do not have that much of lecture, so I will provide some of the supplementary materials to support all these things. And for today, we stop here ok with note that here in this point, we missed this term in the previous lecture, we have inserted here ok. So, r sin you cos i, this is a because 1 small part is remaining which I wanted to do that in terms of this part we have done for the orbit is there.

Then the F_r $\hat{u}_r$is acting here in this direction perpendicular to this is the $\hat{u}_\theta$ and then we have $\hat{u}_A$. The other case that we have to consider in that the velocity vector will be tangent to the curve which will be u_t. And normal to this, we will have the $\hat{u}_n$ and perpendicular to this then we will have the $\hat{u}_t$ cross $\hat{u}_n$, so a vector perpendicular to this unit vector.

Now this part is still we have to do, so what I propose that this can be covered in the next lecture and thereafter we will wind up this topic. And then we will go into finding the some of the terms like in the case of the gravitational potential, how the in the case of the non spherical earth. Suppose if we have a non spherical earth it is not a sphere but rather we have some extra material present here.

So, because of this the satellite will get perturbed, so how to compute all these terms, ok. So, I will discuss this in the next 2 lectures I will try to cover this particular part and rest of the things I will provide a supplementary material. Because it is already 60 lectures we have covered, ok, so thank you very much, we will continue in the next.