Space Flight Mechanics
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Module No # 17
Lecture No # 84
Trajectory Transfer (Contd.)

Okay so last time we have discussed about the Hohmann transfer this time we will complete the bi-elliptic transfer and bi-elliptic transfer as state earlier this is nothing combination of 2 Hohmann transfer.

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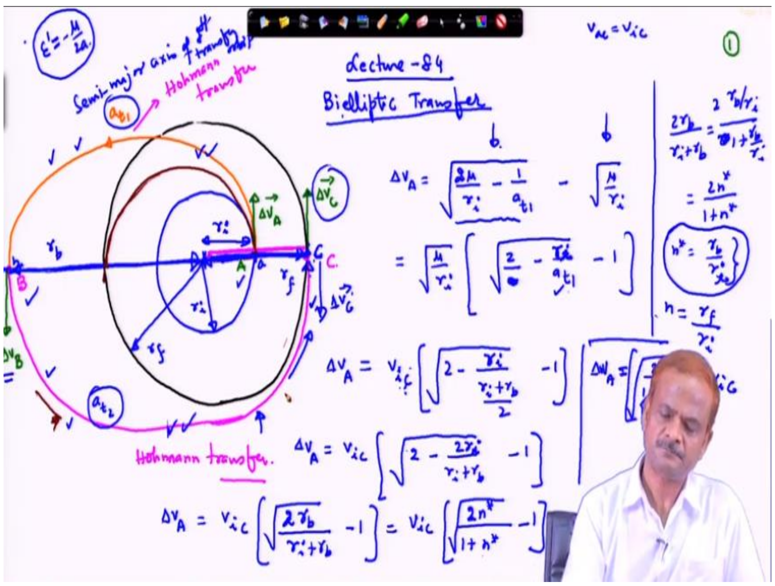

Let us go and work out the corresponding problem so we have one initial orbit here and this is the final orbit initially the satellite is located at a this is the center of attraction then to get into this orbit sometimes if you look energy wise. So another kind of manipulation may be better and this we represent as this is your transfer orbit. We write here semi major axis of this as a let us write this as a_1 or a_{t1}transfer orbit.

Then to get into this orbit we apply another impulse so this is also the Hohmann transfer and this is also Hohmann transfer. So first impulse is applied at A the next impulse will be applied at B and the last impulse will be applied at C. So here this is Δv_A Δv_B and here in this point Δv_C direction is not fixed. Δv C in the opposite direction what we show it like this so working principle is the same as we have done for the Hohmann transfer nothing different.

$$\Delta v_A = \sqrt{\frac{2\mu}{r_i} - \frac{1}{a_{t1}}} - \sqrt{\frac{\mu}{r_i}}$$
$$= \sqrt{\frac{\mu}{r_i}} \left[\sqrt{2 - \frac{r_i}{a_{t1}}} - 1\right]$$

So Δv_A so I write in short because already I have worked out all the details here in short I will write 2 μ by an initial radius obviously is r_i and this radius is r_f why the name is bi-elliptic because you have the first ellipse half of that and the second ellipse half of that l. So 2 /μ r_i– 1 / a_{t1}–μ/ r_i this is the velocity in the circular orbit this is the velocity in the elliptical orbit at a. So this is the impulse required and from here μ/ r_i we take it as common this gets reduced to 2 / 2 – 1 / a_{t1} under root – 1.

$$\Delta v_A = v_{ic} \left[\sqrt{2 - \frac{r_i}{\frac{(r_i + r_b)}{2}}} - 1\right]$$
$$\Delta v_A = v_{ic} \left[\sqrt{2 - \frac{2r_i}{r_i + r_b}} - 1\right]$$
$$\Delta v_A = v_{ic} \left[\sqrt{\frac{2n^*}{1+n^*}} - 1\right]$$

We have written as v_{ic} so we follow that notation v_{ic} of let us write as v ac to make it more convenient rather than writing r_i this is a and here v notation will be using a small a, b, and c. So v_{ac} velocity at a in the circular orbit or v_{ac} is nothing but your v_{ic}. May be let us forget this v_{ic} write this as the v_{ic} this times 2 – 1 / a_{t1}. So what is the semi major axis of this orbit so this will depend on the r_i and this r_b.

$$\frac{2r_b}{r_i + r_b} = \frac{\frac{2r_b}{r_i}}{1 + \frac{r_b}{r_i}} = \frac{2n^*}{1+n^*}$$
$$n^* = \frac{r_b}{r_i}$$

So this we can write as 2 – 1 / r_i+ r_b divided by 2 this whole under root – 1 okay r_i we have taken outside so we should also put here r_i here in this part. So this part is missing r_i and here 2r_i so 2r_i we write here 2r_b divided by 2r_b 2r_i cancels out. So r_b divided by r_i+ r_b under root -1 2 times n^* divided by 1 + n^* under root -1 what we have done here. So we 2r_b r_i+ r_b is that so we have divided numerator and denominators by r_i. So r_b/ r_i and this is we gets 1 + r_b/ r_i so we get here 2n^* /1 + n^* where n^* we have written as r_b/ r_i.

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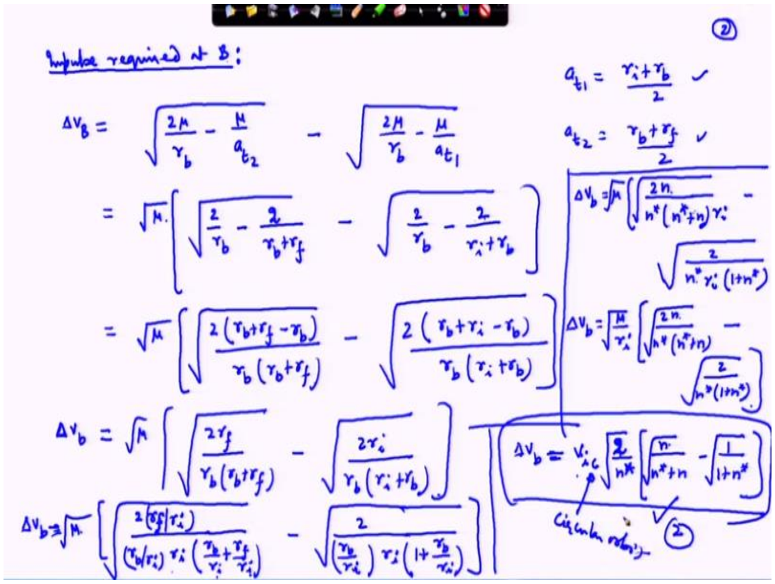

So here we can name this as equation 1 v_{ic} 2n* divided by 1 + n* -1 now the impulse required at B. This is very easy part only thing the concept you have to understand this is not more than 12 + mathematics. But the actual practice once you apply it of there you are considering transfer from 1 planet to another planet so the case gets complex because of the presence of the planet the fear of an influence is there.

So you are not taking the just as the point so first we will do one problem using just a point mass of considering one as to write this is massive extract but it is just like a point. So that the problem gets easy to work out and there we neglect the fear of influence okay and based on that then we work. So I will come to those all the points so due course of if I solve some problems so you will get to know all the concepts.

$$\Delta v_B = \sqrt{\frac{2\mu}{r_b} - \frac{\mu}{a_{t2}}} - \sqrt{\frac{2\mu}{r_b} - \frac{\mu}{a_{t1}}}$$

So impulse required at B this will be equal to so Δv_B so velocity at point B in the second elliptical orbit 1/μ times 1 . Here we are giving this impulse so Δv_b we are looking for so what is the velocity initially at this point and what is the velocity finally at this point? So this orbit is different this is a_{t2} it is a simple semi-major axis. So this is the semi major axis of first transfer orbit and this is semi major axis of this and transfer orbit.

$$a_{t1} = \frac{(r_i + r_b)}{2}$$

$$a_{t2} = \frac{r_b+r_f}{2}$$

So we use the same equation as we have done earlier used here this part 2μ / r_i gets replaced by r_b okay. So after providing the impulse this is going in the second order so a not it as found into the second orbit $a_{t2}$and $-2\mu/ r_b$ the radius has not changed before impulse this is related to a_{t1}. And then we need to simplify it a_{t1} we know this quantity is $r_i+ r_b$ divided by 2. And a_{t2} is $r_b+ r_f$ divided by 2 as it is evident from this figure.

$$= \mu \left[\sqrt{\frac{2(r_b+r_f-r_b)}{r_b(r_b+r_f)}} - \sqrt{\frac{2(r_b+r_i+r_b)}{r_b(r_i+r_b)}} \right]$$

This is your r_b from here to here and this is r_f so half of that this constitutes semi major axis of this orbit. Okay similarly we have to calculate for this orbit so we can use this 2 quantities here and insert here in this place so this we get as μ you can take it outside and this becomes to divided by r_b-1 a_{t2} is $r_b+ r_f$ divided by 2. So 2 comes in denominator μ times now here the way this is appearing so what we do see r_i in nowhere r_i is manifest.

$$\Delta v_B = \mu \left[\sqrt{\frac{2(r_f)}{r_b(r_b+r_f)}} - \sqrt{\frac{2(r_i)}{r_b(r_i+r_b)}} \right]$$

So accordingly we have to divide it and write it in a proper way so n^* is the quantity we have defined as r_b/ r_i the quantity n we have defined as r_f/ r_i. These are the only 2 distances which we are concerned with this from here to here this is your r_b from this place to this place. And this distance from here to here this turns out to be r_f and the distance from here to here this is your ri. Okay so we need to reside this little it so that it gets into a proper format so first we will write herein this place try to accommodate here.

$$\Delta v_B = \mu \left[\sqrt{\frac{2\left(\frac{r_f}{r_i}\right)}{\frac{r_b}{r_i} r_i \left(\frac{r_b}{r_i} + \frac{r_f}{r_i}\right)}} - \sqrt{\frac{2}{\frac{r_b}{r_i} r_i \left(1+\frac{r_b}{r_i}\right)}} \right]$$

Okay we divide the numerator and denominator by r_i so this is 2 r_f / r_i and r_b/ r_i this gets and here $r_b+ r_f$ is there. So we have to do it in a proper way so what will do that we will write here r_i and $r_b/ r_i+ r_f/ r_i$ in this under root $-r_i$ is present here. So we can rewrite this as 2 divided by r_b/ r_i okay so this gets reduced in the form $2r_f/ r_i$ this is r_f so 2n divided by r_b/ r_i is n^* okay. And then r_b/ r_i here this becomes $n + n^*$ r_f/ r_i is r_b/ r_i is n^* and r_f/ r_i is n_i should reverse it this is $n^* + n$.

$$\Delta v_B = \sqrt{\mu} \left[\sqrt{\frac{2n}{n^*(n^*+n)r_i}} - \sqrt{\frac{2}{n^* r_i (1+n^*)}} \right]$$

$$\Delta v_B = \frac{\sqrt{\mu}}{\sqrt{r_i}}\left[\sqrt{\frac{2n}{n^*(n^*+n)}} - \sqrt{\frac{2}{n^*(1+n^*)}}\right]$$

Okay and the r_i is also there r_i then minus this square root is also present then square root 2 divided by r_b/ r_i is n^* okay then r_i and times 1 + n^*. And here once this taken the μ under root so, μ is going without under root fine I have written so this is the correction here and μ under root once we take it outside r_i. So this μ/ r_i under root and then 2n divided by n^* times n^* + n under root – 2 divided by n^* 1 + n.

$$\Delta v_B = v_{ic}\sqrt{\frac{2}{n^*}}\left[\sqrt{\frac{n}{(n^*+n)}} - \sqrt{\frac{1}{(1+n^*)}}\right]$$

More compactly this can be written as Δv_b=μ/r_i is nothing but v_{ic} and if we take 2 / n^* outside so this gets reduced to n divided by n^* + n under root – 1 / 1 + n^* under root. This is equation 1 we have written in terms of the v_{ic} this is we have written in terms of v_{ic} again this is equation 2. Now, you will see the advantage of written in this way once you solve these problems.

So your effort in the calculation this gets reduced okay even if you have to computer programming so there also it is very convenient. So this is Δv and this should not be understood as the point C this is circular orbit standing for circular orbit okay. The last point is the C at C once the satellite reaches here in this orbit by coasting throughout from this place to this place. So at this point impulse is required 3 impulse here in energy is higher in which one you will see that as we have written the energy E′ =μ/ 2a.

So the semi major axis for the this pink orbit the a_{t2}okay this is larger that r_f okay and therefore energy in this orbit will be more as it is evident from this place – μ / 2a. Okay we go on the next page and then we explore this.

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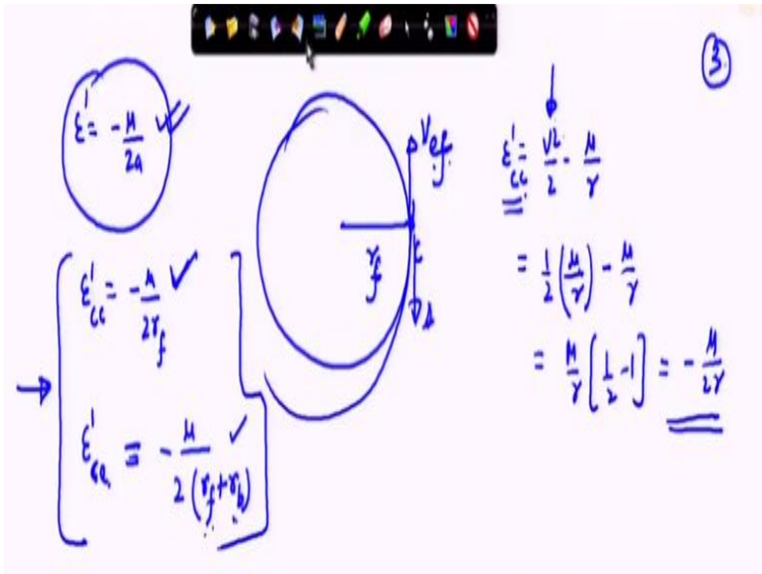

See E = -μ/ 2a once we are writing okay so if it is circular orbit so a gets replace by r. So in that this is μ/ 2r E′. So at the point C this is the energy in the orbit while and this we can write as r c or we have written as the r_f okay so E′ r_f in the circular orbit. And E′ at c in the elliptic orbit this is cc we can write circular orbit and this is in the elliptic orbit. So this is $-\mu$ / 2 (r_f+ r_b)so r_b if it is much larger than r_f so you can see that or say r_b is greater than r_f say you can calculate what will be the energy?

This is valid for both the elliptic and the circular orbit why? Because E′= v^2/ 2 –μ/ r and v in the circular orbit is μ/ r under root so this gets μ/ r under μ/ r and once you subtract it immediately you can check that this quantity will be μ i r can be taken common 1 / 2 – 1 so this –μ/ 2r so this is in the circular orbit. So this result is valid for both the circular orbit and the elliptic orbit provided you properly insert the value for the v.

So here in the circular orbit this is the energy and the elliptic orbit this is the energy so that means you need reduce the velocity at this point. So the velocity the actual velocity after coming from this place to this place that will be in the elliptical orbit v_e at c v in the elliptical orbit at C v_{ef} whatever we want to write okay. And then one impulse needs to be applied in this direction I have shown here in this direction this is customary to show it like this but will the opposite one okay.

So you have to basically apply impulse here in this direction Δv so that the velocity is reduced and this orbit is regularized. So following this notation whatever we have done earlier so same notation we can work for the impulse required at C.

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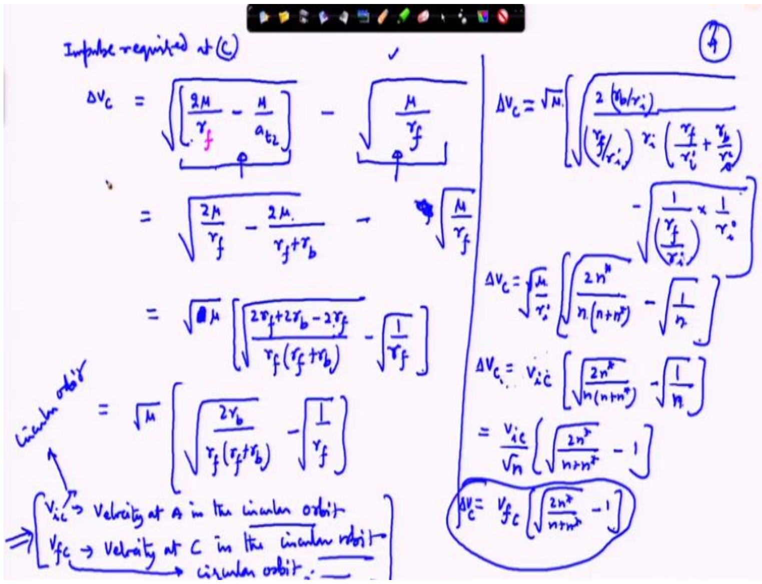

$$\Delta v_C = \sqrt{\frac{2\mu}{r_f} - \frac{\mu}{a_{t2}}} - \sqrt{\frac{\mu}{r_f}}$$

So Δv_c the impulse required so final velocity in the velocity at point c in the elliptic orbit the μ2 μ / r. Okay for this we need to r_f we have used here notation at this point so from here to here this point to this point this distance we have to take. Okay so 2 μ / r_f–μ/ a_{t2}this under root this is the velocity at point C in the transfer orbit minus velocity in the circular orbit at point 2. And that will be μ/ r_f a_{t2} is nothing by r_f+ r_b divided by 2.

$$= \sqrt{\frac{2\mu}{r_f} - \frac{2\mu}{r_f + r_b}} - \sqrt{\frac{\mu}{r_f}}$$

$$= \sqrt{\mu}\left[\sqrt{\frac{2r_f + 2r_b - 2r_f}{r_f(r_f + r_b)}} - \sqrt{\frac{1}{r_f}}\right]$$

$$= \sqrt{\mu}\left[\sqrt{\frac{2r_b}{r_f(r_f + r_b)}} - \sqrt{\frac{1}{r_f}}\right]$$

$$\Delta v_C = \sqrt{\mu}\left[\sqrt{\frac{2\left(\frac{r_b}{r_i}\right)}{\frac{r_f}{r_i} r_i \left(\frac{r_b}{r_i} + \frac{r_f}{r_i}\right)}} - \sqrt{\frac{1}{\frac{r_f}{r_i}} \left(\frac{1}{r_i}\right)}\right]$$

So this goes in the numerator or μ under root we take here and this is 2 r_f+ 2r_b– r_f divided by we divide numerator and denominator by r_i. This gets reduced to 2r_b / r_i r_f/ r_i then multiply here by r_i and similarly write here r_f/ r_i+ r_b/ r_i under root -1 / 2 r_b/r_i we have written as n^* and r_f/ $r_i$as n and r_i

we can take it outside from both the places. And if we take under root n from this place this gets resist to $2n^*$ n + n^* under root -1 and this is nothing but our quantity v_f in circular orbit.

$$\Delta v_C = \frac{\sqrt{\mu}}{\sqrt{r_i}}\left[\sqrt{\frac{2n^*}{n(n^*+n)}} - \sqrt{\frac{1}{n}}\right]$$

$$\Delta v_C = v_{ic}\left[\sqrt{\frac{2n^*}{n(n^*+n)}} - \sqrt{\frac{1}{n}}\right]$$

$$= \frac{v_{ic}}{\sqrt{n}}\left[\sqrt{\frac{2n^*}{(n^*+n)}} - 1\right]$$

$$\Delta v_C = v_{fc}\left[\sqrt{\frac{2n^*}{(n^*+n)}} - 1\right]$$

This circular orbit does C with notation we have used this is wanted to avoid v a v_{ic} rather than using or properly tied it okay. Now v_{fc} I will write here so that you draw get confused so this v_{fc} $2n^*$ n + n^* under root -1 v_{ic} is the velocity at A in the circular orbit this is very important to note because the notation I have used this 2 have got mixed up. Velocity at A in this circular orbit and v_{fc} is the velocity at C in the circular orbit so this c stands for circular orbit this stands for circular orbit.

So we do not have to confuse this or rather thus indicating v_i and v_f and we can delete the c this is the only way we can do. So this c either I drop here then we do not confuse with this point c rather I rename this point C these are the 2 options of doing this. So abc into okay I am leaving it as it is v_{ic} indicating by clarification given by this 2 statements okay. Maybe in your once you note it at that time you can make it d instead of C so that we do not confuse with the circular orbit C and the point C.

Okay now we have got one more thing we have to check here we have subtracted from here in this place this is the velocity in the elliptic orbit and this is velocity in the circular which is the final orbit. We have not subtracted from the final orbit velocity the electric orbit velocity and we need to check that indeed this quantity turns out to be positive this is important to check. Otherwise you have to take the magnitude of this quantity.

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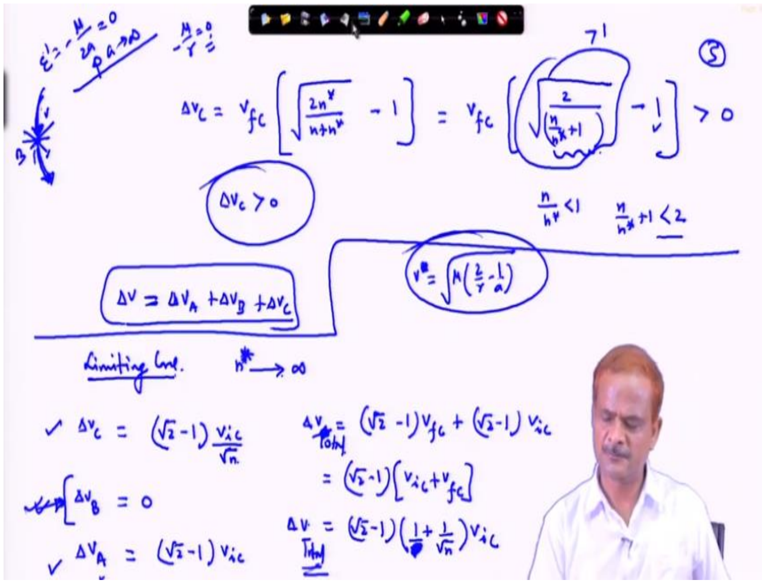

$$\Delta v_C = v_{fc}\left[\sqrt{\frac{2n^*}{(n^*+n)}} - 1\right]$$

So $\Delta v_C = v_{fc}$ and then $2n^*$ divided by $n + n^*$ -1 we write it in this way. So what we can see in this place n / n^* is less than 1 okay and therefore the denominator will be less than 2. So $n / n^* + 1$ this will be less than 2 and therefore this quantity will be greater than 1 so this quantity is greater than 1 and from there if you subtract 1 this quantity Δv_C turns out to be greater than 0.

$$\Delta v = \Delta v_A + \Delta v_B + \Delta v_C$$

$$\Delta v_C = \frac{(\sqrt{2} - 1)v_{ic}}{\sqrt{n}}$$

And now you can add all of them $\Delta v = \Delta v_A + \Delta v_B + \Delta v_C$ so this is the total impulse required. Limiting case n^* which is reflecting the intermediate orbit tends to infinity if that happens then your Δv_C this gets reduced to we can check from this place this under root 2 – 1 times v_{ic} divided by 2 root n. Okay similarly Δv_B will get reduced to Δv_B we have worked out here this is as n^* tends to infinity.

$$\Delta v_B = 0$$

$$\Delta v_A = (\sqrt{2} - 1)v_{ic}$$

So you can see that this quantity will tend to 0 this quantity will become 0 other quantities we accordingly we can write here this in the denominator n^*. So simply this will tend to infinity 0 this also goes to 0 so we get here $\Delta v = 0$ Δv_B this quantity will be 0 and Δv_A we have calculated here. If n^* tends to 0 so in that case this is Δv_A we put divide by n^* in the numerator denominator so this gets added to 1 divided by n^* + 1 under root – 1 times v_{ic}.

So n^* tends to infinity this gets reduced to 2 – root 2 -1 and this is root 2 -1 times v_{ic} therefore total impulse required Δv_c = root 2 – 1 v_{fc} + root 2 – 1 times v_{ic} or rather both of them writing in the same v_{ic} then in that case this will appear as 1 + 1 / root n times v_{ic} Δv total. Okay so this conclusion what we have drawn here this is important what we can see that at Δv at B we do not require any impulse.

$$\Delta v_{total} = \left(\sqrt{2}-1\right)v_{fc} + \left(\sqrt{2}-1\right)v_{ic}$$
$$= \left(\sqrt{2}-1\right)\left[v_{ic} + v_{fc}\right]$$
$$\Delta v_{total} = \left(\sqrt{2}-1\right)\left[1 + \frac{1}{\sqrt{n}}\right]v_{ic}$$

At C we require impulse and at A we require impulse why because this is very evident that your E' this equal to $-\mu/2a$ as a tends to infinity. So E' this becomes equals to 0 okay a tends to infinity this quantity is 0 so approaching from the orbit at B okay and then again starting from this place. So here the total energy is 0 and therefore starting also and approaching also both the energy are 0 that is no change velocity okay because the position in this not changing here.

Okay position is not changing means potential energy is not changing which in this case μ/r this equal to also 0 with minus sign here. So there is no change in velocity here and therefore at B we do not require any impulse and this is what is we obtain from this equation. So at Δv at A and B only thing that we need to give the impulse okay so using this technique. If you have to send the object in the parabolic orbit you can work it out the satellite you have to send in the parabolic orbit or either multiple orbit different kinds of orbit can also be mixed up.

And what is the simplicity that you have to use this equation v_A^2 or v =μ times 2 / r – 1 / a which we have earlier derived as this viva integral. Okay by this we have finished the bi-electric transfer now it comes to whether the bi-electric transfer is beneficial or the Hohmann transfer is beneficial which one should be used. So it is a very simple here to see that in the bi-electric transfer because it is going in a larger orbit so the amount of travel the time to travel from A to B from this point to this point.

And then again from B to C it will be much larger if we directly go from the inner orbit to say from this place from here if I directly go to this point. So time will be much smaller okay immediately we can achieve in this if I have to come to this. So only this half of this time extra is required so you require here the extra time to travel to this point and again extra time to travel along this one okay. So in a bi-electric transfer it may be better than the Hohmann transfer in terms of energy consumed okay.

And it depends on the condition which we need to look into okay but the time of travel it drastically increases and therefore this kind of orbit can be used if and only if you are not worried about that time. If you are worried about the time whether time is mandatory to be maintained so at that we have to resort to the home and transfer or either if you have you record the faster least time you have produce.

So you have to go for some other orbit okay the shortest will be the straight line going from a one point to another point this infinite velocity but this is not practical you cannot get infinite velocity. And therefore the shortest possible time you get always in the hyperbolic orbit okay next it goes in the parabolic orbit and then the electric orbit. So accordingly we choose our orbit and it will also depend on the automatically you will be forced to use that kind of configuration.

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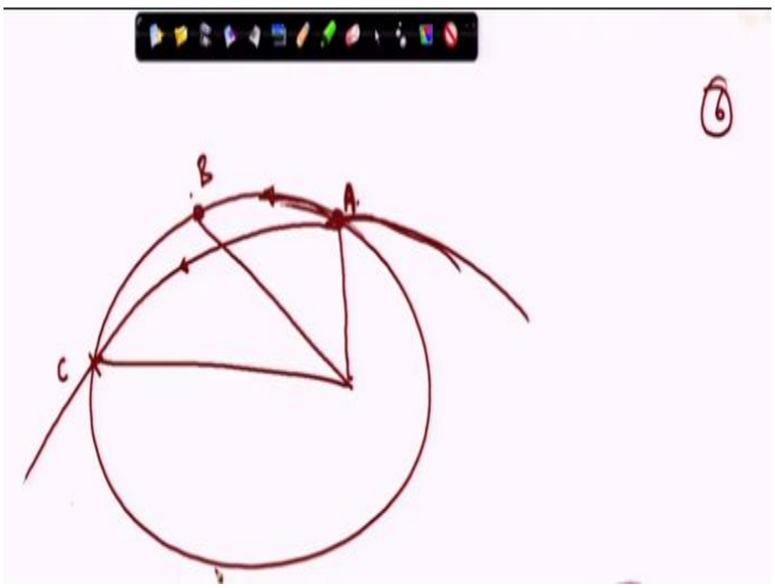

Say here in this case as we were discussing earlier that if one satellite is at A okay another satellite is at B and you have to catch satellite A okay you have to catch satellite B means you have to dark satellite A with B. And that is to be done at point C okay in that case you have to send it in a faster orbit. So mostly once you calculate this faster orbit it will turn out to be a hyperbolic orbit okay. This we need to work out okay so I will do this problem also later to this will come in the (()) (42:34) problem.

So one change I am going to do that the twelfth lecture I will shift to the eleventh week this is the tenth week we are running with okay. And element week lecture I will shift to the twelfth week so that some continuity is maintained. I have got the feedback that the lectures are becoming very lengthy and we do not have enough time to study okay. In 1 week sometimes or perhaps for third or fourth the 7.5 hours has been covered instead of 2.5 hours what should have been okay.

So students are not able to give that much of time but getting into the concepts okay we once I need to explain all this things so that takes time okay. Writing few equations or just I bring the slides and present the equations so it can be very fast okay I can finish in short time. But if I try the concepts properly and once I write it so the concepts get developed properly. If I do not write one by one so you will not know from where what is coming as usual is done in the book.

Few equations are given and then the chapter is closed so I have derived all the equations sequentially and it is a record for your proper standing and even if you want to use for professional purpose all this things can be used. So we close this lecture here and we will continue in the next lecture thank you very much.