Mathematics-I

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Lecture – 5
Continuous Functions

In today's lecture we are going to discuss about continuous function and its properties. So first of all, we have to understand analytically what it means to say a function is continuous. Intuitively all of us know what it means, a continuous function.

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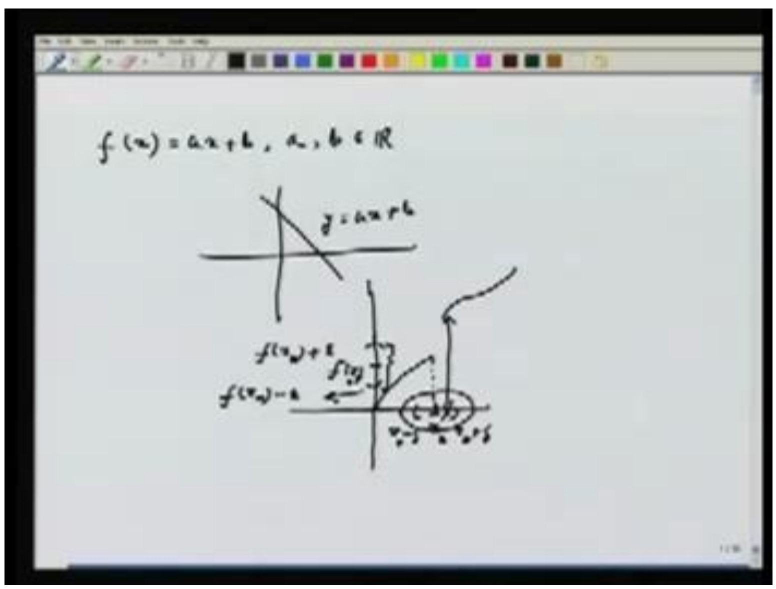

For example, suppose I look at a function whose definition is, where a and b, these are 2 real numbers. Depending on a and b, if I want to draw the graph of the function f, it will look something like this. Well, this looks really continuous; there is no gap in the function. You know, one can draw it continuously. This is the intuitive feeling which we want to make analytic, in terms of the language of mathematics. But first again let us try to examine what does it mean to say a function is not continuous. Again, intuitively we know what it means. Let us try to draw a function whose graph is not continuous.

It should look something like this, that suppose a point here which I call x naught and the graph of the function looks something like this up to x naught and just after x naught, it starts from here. You see, just at x naught there is a big jump. Well, this is certainly example of a function which is not continuous. But again, in terms of analysis let us try to see what it means.

Well, this height, if I draw it here, this is f of x naught here. Then what does it say? It says that if I look at this kind of neighboring points, let me call this point f x naught plus epsilon and let us call this point f of x naught minus epsilon. Then if I look at neighboring points of x naught, whatever neighboring points I look at, let me call them x naught minus delta, x naught plus delta. I see that if I take points from this region, it is not true that f of x lies in this band.

You see what is happening here? If I take a point here, then the height of the function is this, which is going out of f x naught plus epsilon. This is the property which you are going to exploit in defining what is a continuous function. So let us just try to write down what we have got for this example of function, which is discontinuous according to our intuition.

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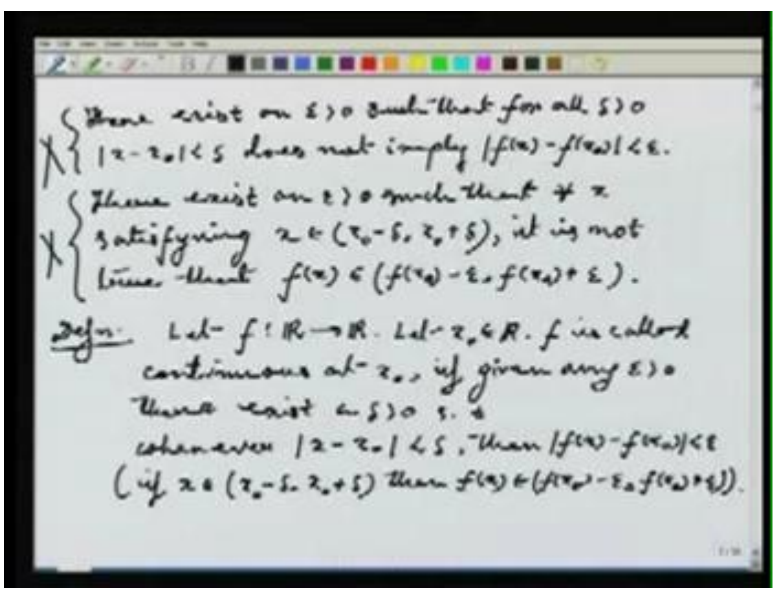

What we got is as follows: that there exist an epsilon bigger than 0 such that for all delta bigger than 0, mod of x minus x naught less than delta does not imply mod of f x minus f x naught is less than epsilon. Written in other language, there exist an epsilon bigger than 0 such that for all x satisfying x in the interval x naught minus delta x naught plus delta, it is not true that f x belongs to f x naught minus epsilon f x naught plus epsilon.

According to our pictorial analysis, it just says that there is some gap in the graph of f and when I say a function is continuous at a point x naught, this is precisely the thing which I do not want to happen. Then what should be the definition of continuity of a function, if I do not want this or this to happen? I do not want this to happen. So I define continuity of a function at a point x naught in the following way.

Let f be a function from R to R. Let x naught be a real number. f is called continuous at x naught if given any epsilon bigger than 0 there exist a delta bigger than 0 such that whenever modulus of x minus x naught is less than delta, then modulus of f x minus f x naught is less than epsilon. In other words, if x is in the interval x naught minus delta and x naught plus delta, then f x belongs to f x naught minus epsilon and f x naught plus epsilon.

Notice who depends on what. I start with epsilon. Always remember the example of discontinuous function which I said. There, what has been said is, I can always find an epsilon for which no delta works. If I do not want this to happen, I have to look at the opposite statement of that: which should mean that whatever epsilon you take, there is at least one delta which works, which works means, in this language, that is, x is in this interval then f x belongs to this interval.

This is definition of continuity, but if I want to claim that this is definition of continuity what should do is, the functions which I intuitively know are continuous functions, I should able to verify those functions, are also continuous with respect to definition. Any mathematical definition should match with intuitive feeling we have, right? So let us check now. So I start back with the function which I started with.

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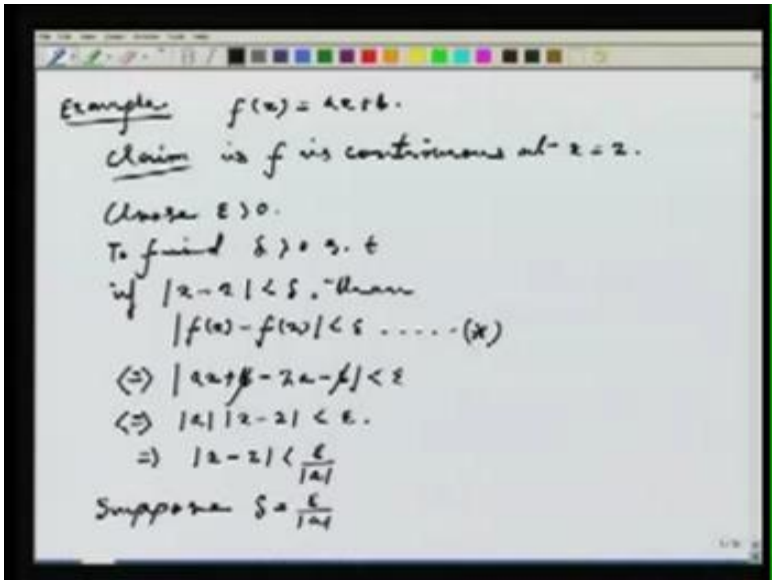

Let us take the function f x equal to a x plus b and let us try to check the continuity of this function at the point, let us say 2. So my claim is f is continuous at x equals to 2 and once I do the proof, we will be able to see that the same proof works for any other value of x. 2 is just a particular case. There is nothing holy about 2.

What I have to do? I have to choose an epsilon. Can I take epsilon equal to 1 and find a delta and show the condition of continuity is verified. Will it work? I say no because the definition says it has to work for any arbitrary epsilon. That means I cannot assign a value to epsilon. I just have to start with an arbitrary epsilon and out of that epsilon I have to find out a delta.

So choose epsilon bigger than 0. I cannot specify the value of epsilon because epsilon is arbitrary. Now I have to find a delta. So to find delta bigger than 0 such that, if mod of x minus 2 is less delta then mod of f x minus f 2 is less epsilon. Now what I know is, I know the explicit value of f. I know what f is. So I will put the value of f in this equation which I call star. Well, it is not quite an equation, sorry, it is an inequality. So this then looks like modulus of a x plus b minus 2 a minus b. So I have to prove that this is less than epsilon.

Let us try to estimate, look at this quantity. First of all, what you get is that this b cancels each other. So what you are left with is: mod a times mod x minus 2 is less than epsilon. Still my job is waiting me. I have to find a delta. What is delta? Well then, this implies that mod x minus 2 is less than epsilon by mod a. Now just check. Suppose I choose, delta is equal to epsilon by mod a. Suppose I choose delta to be this. What happens?

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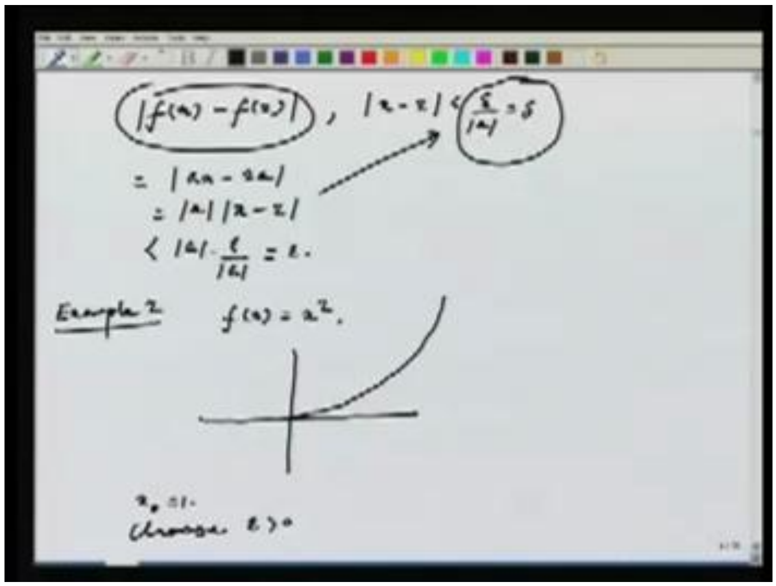

Well, I again look at mod of f x minus f 2, the condition I am taking is, modulus of x minus 2 is less than epsilon by mod a because this is my delta and now I want to check whether modulus of f x f 2 is less than epsilon. This is what I want to estimate. If I again write it down, we will see just, I have to work backward, the way I got delta, now I will try to go in the reverse direction, which is easy because this quantity is just modulus of x minus 2 a which is mod a times mod of x minus 2. But mod of x minus 2, I already know that this is less than epsilon by mod a. So this quantity now is less than mod a, epsilon mod a which is equal to epsilon and this is precisely what I wanted to prove. I got a delta, my value of delta is epsilon by mod a and if mod x minus 2 is less than this delta, I can prove that mod of f x minus f 2 is less than epsilon.

Let us look at another example which slightly more complicated than this. Let us look at the function f x equal to x square. If you think about the graph of this function, many of you might have seen the graph of this function. It looks like this on the positive axis. At 0, obviously the function is 0 and as x grows, x increases, the value of x square increases. It gets larger and larger. The graph of the function then will look like. Well, there is something involved here. You might actually ask me that how do I know that the graph of the function cross the way I have written? It is not the other way. Well, that you have to wait a bit.

But believe me for the time being that the graph of the function on the positive axis actually looks like this. It looks certainly continuous, because I do not see any gap. So I will like to check analytically again through my definition that how does continuity gets verified? Let us say, for x equal to 1. So let us choose the point x naught equal to 1 and I have to check continuity of the function at the point x naught equals to 1. So again, what I have to do? I have to start with epsilon bigger than 0 and then, I have to find a delta bigger than 0. So choose epsilon bigger than 0 and then I look at the quotient, the difference f of x minus f of 1.

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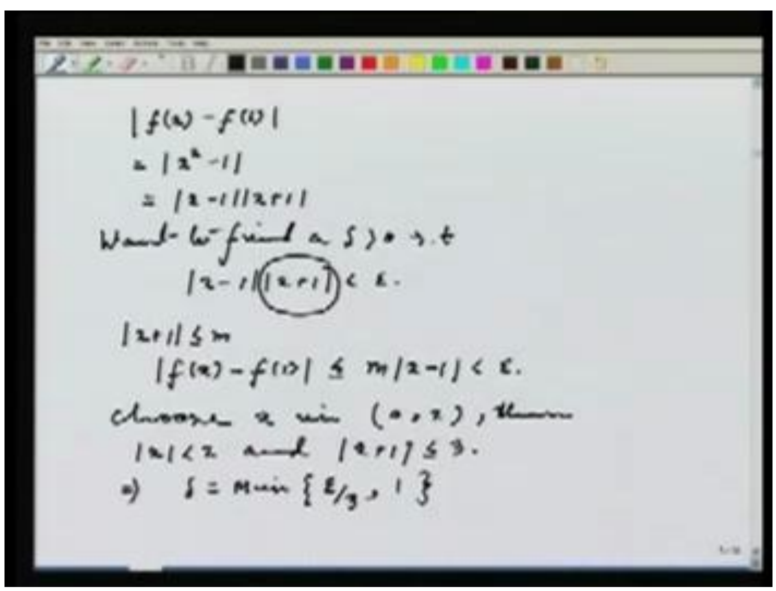

How does this look like? This is then modulus of x square minus 1. So this is then modulus of x minus 1 into modulus of x plus 1 and now I want that this quantity should be less than epsilon. So want to find a delta bigger than 0 such that. Notice that delta is in my hand. I will choose it according to my will. Now which delta should work? I say, first I need to concentrate on this quantity. If I can somehow choose x such that mod x plus 1 is less or equals to M, I choose only those x s. Then it will turn out that modulus of f x minus f 1 is less or equals to M times mod x minus 1 and I want to make this less than epsilon.

Now let me make this M specific. What I will do is, I will choose x in 0, 2. Then mod x is anyway less than 2 and mod of x plus 1 is less than or equal to 3. This implies, if I choose delta to be equal to minimum of epsilon by 3 and 1. Let us see, whether it works, this choice of delta.

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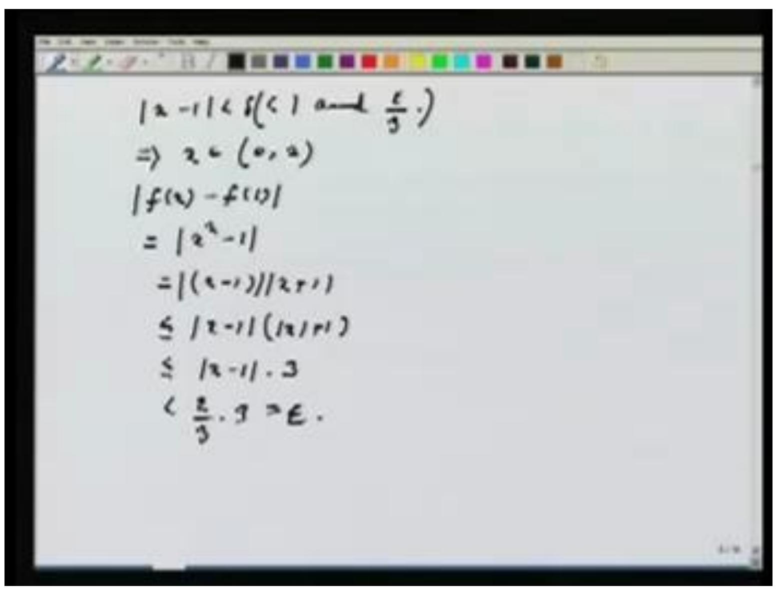

Let us take x such that mod of x minus 1 is less than delta, which is less than 1. Delta is less than both these quantities, fantastic. Now, if mod of x minus 1 is less than 1, that means the distance between x and 1 is at the most 1. This implies, in particular x belongs to the interval 0, correct? Now let us try to estimate f x minus f of 1, which is by definition, x squared minus 1. I write it as x minus 1, x plus 1, which is less or equals to mod x minus 1 into mod x plus 1. But mod x is anyway less than or equals to 2. So this is lesser equal to mod x minus 1 into 3. But mod x minus 1 is less than epsilon by 3 also. So this is less than epsilon by 3 into 3, which is equal to epsilon.

So you see that this delta works. I am not claiming that this is only delta works. You can actually choose, see, from this example, very easily that if you choose any other delta 1, which is less than this delta which I have chosen, again you will able to prove that mod of f x minus f 1 is less than epsilon. But what is important to notice here is that, the definition of the, description of delta that explicitly depends on the epsilon which you are starting with and it also has something to do with the value of the function also, the way the function has been defined.

You see, this epsilon by 3 has come because of the factorization which I am using. At the same time, the value of delta is also depending on epsilon. That means, if you change epsilon, delta will also change. But there is nothing wrong in that because according to the definition of continuity, we just said that given an epsilon, there is a delta. Nobody prohibits delta from depending on epsilon. Now, although the definition of epsilon and delta is mathematically extremely rigorous, but for many practical purpose, given the nature of the function, sometimes it becomes difficult to apply this epsilon delta definition to check continuity of the function at a point.

For that reason, we want to manufacture some other procedure, by which, perhaps we can check continuity of function much more easily, and this actually brings us back to something which we have already discussed, the sequences. So what I am going to describe now is, to connect the concept of continuity with convergence of sequences. This is how we start and what I want to prove is as follows. So let me write it as a proposition.

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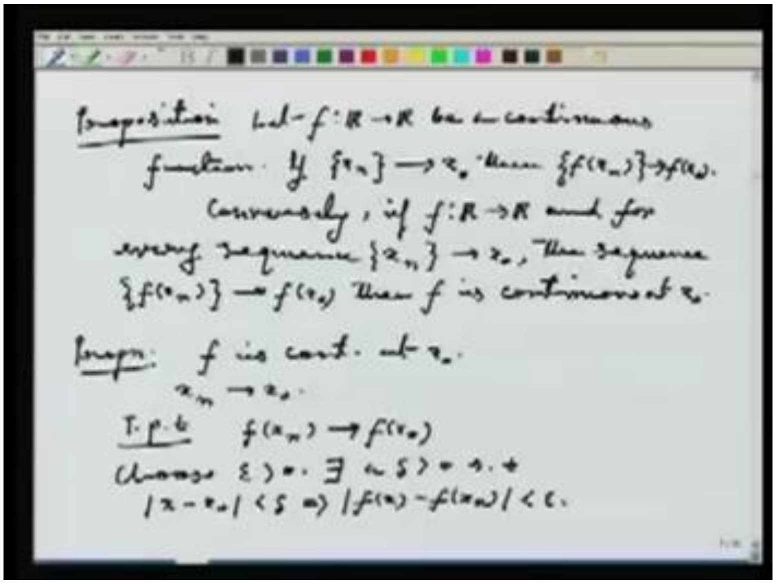

Let f from R to R be a continuous function. If x n, which is a sequence of real numbers converges to x naught, then if I look at the sequence f of x n which makes sense, because x n s are just points. I apply f on those points, I get another sequence of numbers, which is f x n. Then this f x n converges to the number, f of x naught. Conversely, if f is a function from R to R and for every sequence x n converging to x naught, the sequence f x n converges to f x naught. Then f is continuous at x naught.

This actually clears the concept that if you want to check continuity of a function at a point, let us say x naught, all you have to do take any arbitrary sequence x n converging to x naught, just check whether f of x n converges to f of x naught. What is the advantage of this first of all? We have seen mechanisms of finding limits of sequences, sandwich theorem and many such things. You can use those weapons to prove convergence of f of x n and once you know f, you already know what is f of x naught. So using the concept of limit of a sequence, you can actually check continuity of a function at a point. That is, in some way you are bypassing the concept of epsilon and delta.

But once I prove this proposition that will actually connect these two are equivalent. That is why the theorem is in both directions. First, let us try to prove the first part. So I assume that f is continuous at x naught. I choose a sequence x n which converges to x naught. To prove that the sequence f of x n converges to f of x naught, for that you know, what I have to do? Again I have to start with an epsilon. I have to find a stage after which the distance between f of x naught and all f of x n s, after a stage, the distance is less than epsilon. So choose epsilon bigger than 0.

Now, since I have assumed that f is continuous at x naught, given, this epsilon, there is a delta. So there exist a delta bigger than 0, such that mod of x minus x naught less delta implies, mod f x minus f x naught is less than epsilon. Notice my assumption is x n is a sequence which converges to x naught. That means, given this delta, there exist a stage after which modulus of x n minus x naught is less than delta, as x n is converging to x naught. So there exist a capital N such that for all n bigger than or equal to capital N , modulus of x n minus x naught is less than delta.

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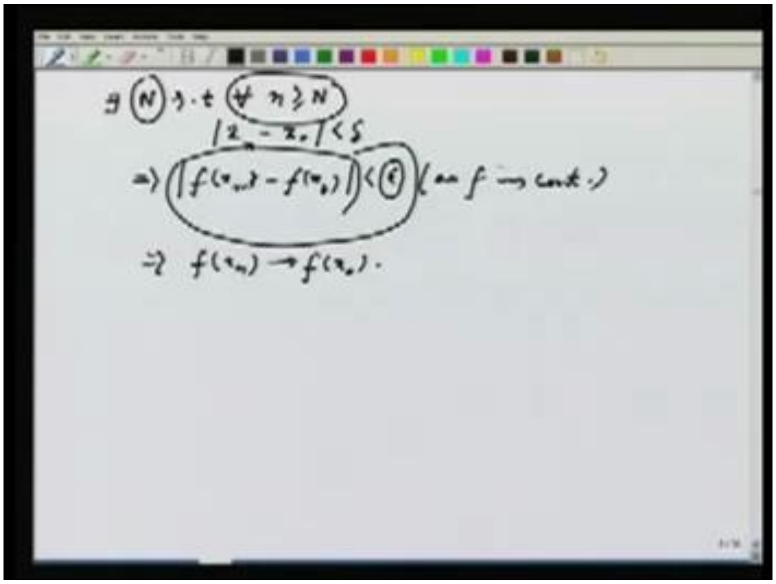

Now, again, first time I am using continuity of the function. This implies modulus of f of x n minus f of x naught is less than epsilon as f is continuous. But is not it, the convergence of the sequence f of x n to f of x naught? See given epsilon, this is the epsilon I started with. I found a stage capital N such that for all n bigger than or equals to capital N, this is happening and this is what I wanted to prove. This implies convergence of the sequence f of x n to f of x naught. Well now, it is other way. So what is my assumption? Now the converse part. What is my assumption now?

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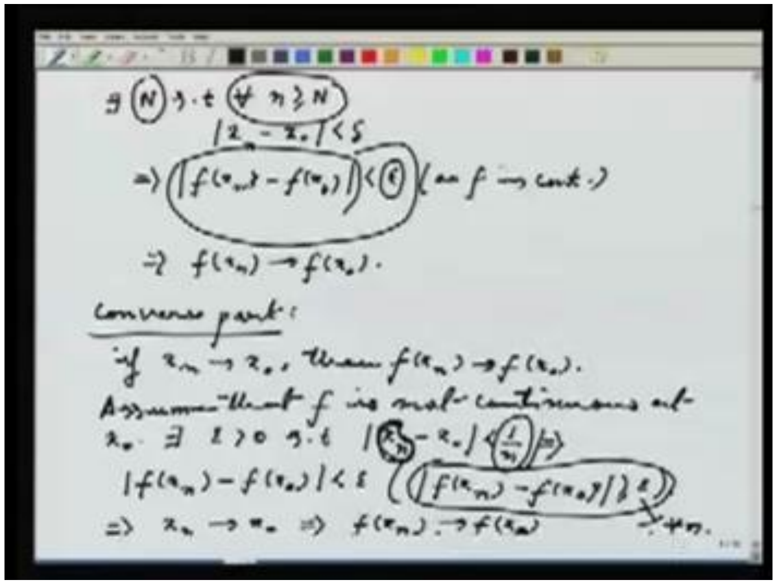

The assumption is, if x n converges to x naught then f of x n converges to f of x naught. I have to prove f is continuous. That means what? Given epsilon, I have to find a delta. Now, here, the complication starts. Directly, if you approach, it becomes difficult to prove it. What we do is, we use the contra positive argument which we have used earlier. You have noticed, what I do is, I will assume f is not continuous at x naught and that assumption, I will use to contradict the assumption which I have started with. That means, my assumption is actually wrong and hence it will follow that f must be continuous at x naught. Assume that f is not continuous at x naught. Well, if you go back to the first thing which have discussed in the lecture that, what does it mean to say a function is not continuous at a point. I am going to explore that.

It means there exists an epsilon for which no delta works. That means, whatever delta you choose, modulus of x minus x naught is less than delta but modulus of f x minus f x naught is bigger than epsilon. So there exist epsilon bigger than 0 such that mod x n minus x naught less than 1 by n does not imply, is less than epsilon. That is, in other words, how can I suddenly get this x n? Well, the definition says, there exist epsilon for which no delta works. What does mean to say no delta works? It means there exist x such that modulus of x minus x naught is less than delta but modulus of f x minus f x naught is bigger than epsilon. There exist an x; keep track of that x. I am actually calling x n because I am choosing delta to be equal to 1 by n. Corresponding to this delta, there exist an x which I am calling this x n . This is how I get a sequence at least and notice that the sequence x n converges to x naught.

This implies, x n converges to x naught but then by my assumption, this implies f of x n converges to f of x naught but notice that what I got here that f of x n minus f of x naught bigger than or equal to epsilon, then how come this happens that f of x n onverges to f of x naught because this is true for all n. If a sequence has to converge somewhere, then after some stage, the difference has to become as small as I wish, but that is not happening. It is always bigger than epsilon and this is a contradiction.

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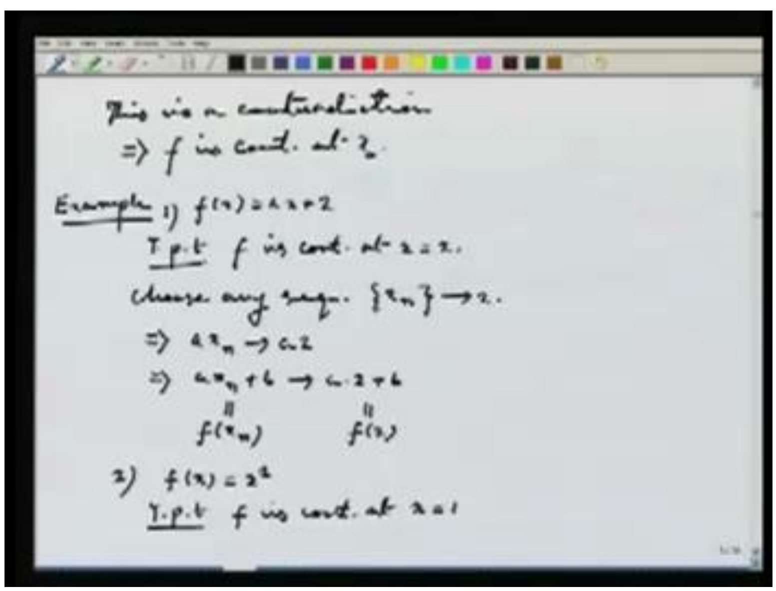

This is a contradiction and this implies the assumption I started with, that f is not continuous at x naught is actually wrong. This implies f is continuous at x naught and this is what we wanted to prove.

Now, let us try to see how to exploit the sequential criteria of continuity to show certain functions are continuous. I will start with the functions which I have already dealt with. What I will do is, I will use this sequential criteria to again show you that those functions are really continuous. Let us start with this example and we will see, in this case, proof is much easier. We start with the function, f x is equal to a x plus b. I want to prove that f is continuous at x equals to 2.

How do I proceed through, this sequential criteria? I just choose any sequence x n which converges to 2, if you remember our lessons on sequence, if a sequence x n converges to l, then any scalar times x n converges to scalar times l and if x n converges to l, y n converges to m, then x n plus y n converges to l plus m, so on and so forth. I am going to use all those properties.

Now I say that this implies a times x n converges to a times 2. This again implies that a times x n plus b converges to a times 2 plus b. Here, I am taking b as constant sequence. If you notice clearly that this quantity is nothing but f of x n and this quantity nothing but f of 2. So I have shown that f of x n converges to f of 2. That means, f is continuous at x equal to 2.

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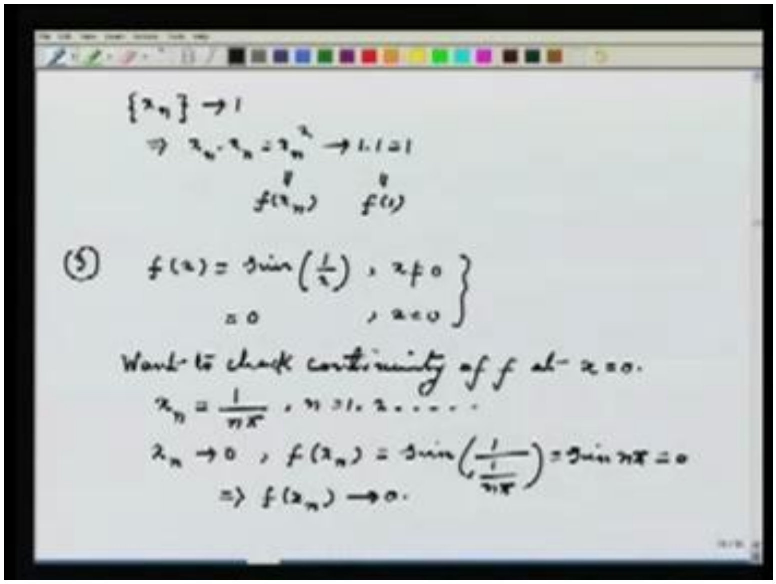

Now let us look at the second example, say f x equal to x square and I am checking continuity at x equal to 1. So how should I proceed? Start with sequence x n which converges to 1. But then this implies that x n times x n which is x n square converges to 1 times 1, which is 1 and then this quantity is nothing but f of x n and this is nothing but f of 1 and that is what I wanted to prove. If x n converges to 1, f of x n converging to f of 1. This implies, f is continuous at 1. See how simple at the proofs are.

Now I will show you certain other functions which usually look more complicated in terms of epsilon and delta, can actually be dealt with sequences quite easily. For example, let us look at this function: f of x equal to sine of 1 over x, where I choose x to be bigger than 0 and I define it to equal to be 0, if x is equal to 0. So I define the function f x equal to sine of 1 by x, when x is not equal to 0 and I define it to be 0, if x is equal to 0 and I want to check the continuity property of the function at x is equal to 0. So want to check continuity of f at x equal to 0. So what I do is, I have to choose sequences. Let me choose this sequence first x n equal to 1 by n pi; n equal to 1, 2.

Notice that x n converges to 0, but then what happens to f of x n? That turns out to be sine of 1 by 1 by n pi, which is sine n pi, which is 0. So this implies f of x n converges to 0.

Does this prove the continuity of the function at x equal to 0? I say it does not because what I have shown is, there is a particular choice of a sequence x n for which f of x n converging to f of 0 but the definition says for any arbitrary sequence x n, f of x n should converge to f of 0.

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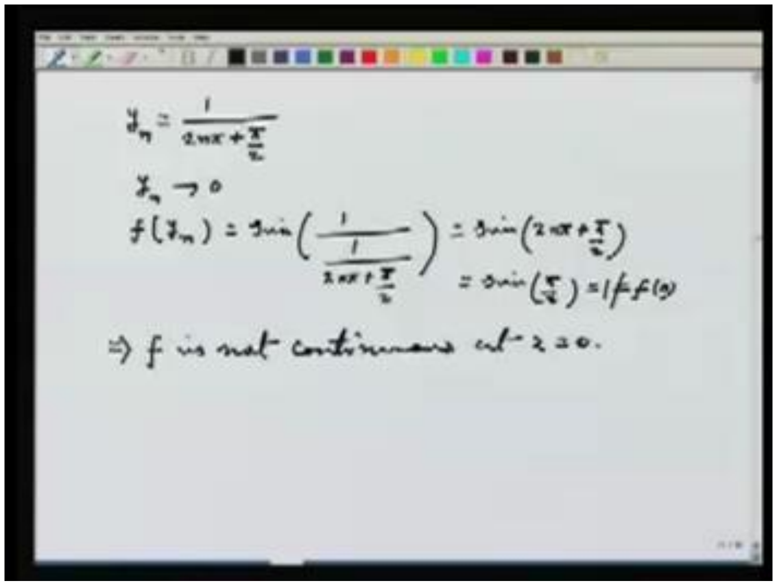

Now look at some other sequence and see what happens. Let me choose this sequence which I am going to call y n. This is 1 by twice n pi plus pi by 2. Notice that y n converges to 0 as n goes to infinity. But then what is f of y n? That is, sine of 1 over 1 over 2 n pi plus pi by 2, which is just sine of twice n pi plus pi by 2. But all of us know sine of 2 n plus theta is sine theta. So this is sine pi by 2, which is equal to 1 and this is not f of 0. That means, the function is not continuous at 0 because if it were then for any sequence x n converging to 0, f of x n should converge to f of 0. That is not happening. I have found a sequence y n, which converges to 0 but f of y n does not converge to f of 0.

You might say, that is my fault, I have defined f 0 to be equal to 0. That is why it is not happening. Can I define f at 0 in such a way that it will become continuous? Well, choice of sequence actually tells that is not possible. Whatever value of f 0 you give these two sequences are always there for which f of x n will converge to either 1 or 0. That means, you cannot choose a value of f 0 where all sequence f of x n will converge to that value. It is not possible. This implies, f is not continuous at x equal to 0.

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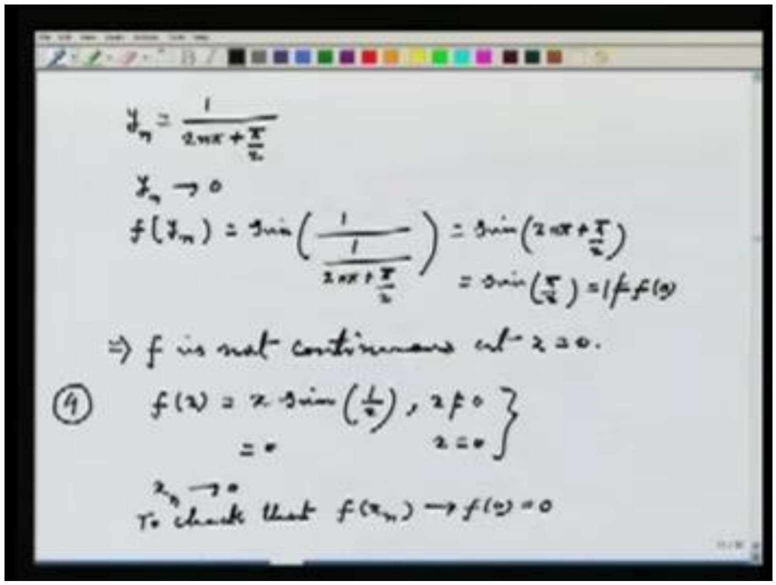

Let us look at the fourth example, where I am going to do a little bit of plastic surgery with the function which I have already taken. Let me define now, f of x sine 1 over x, when x is not equal to 0. I define it to be equal to 0 when x is equal to 0 and I want to check whether this function is continuous at x equal to 0. So I take a sequence x n which converges to 0 to check that f of x n converges to f of 0, which is 0. I have to show that this is true or false. If it is true, it is continuous. If it is not, then f is not continuous.

Now notice that as x n converges to 0 and sine of 1 by x n is actually a bounded function. So I want to prove that x n sine 1 by x n actually converges to 0.

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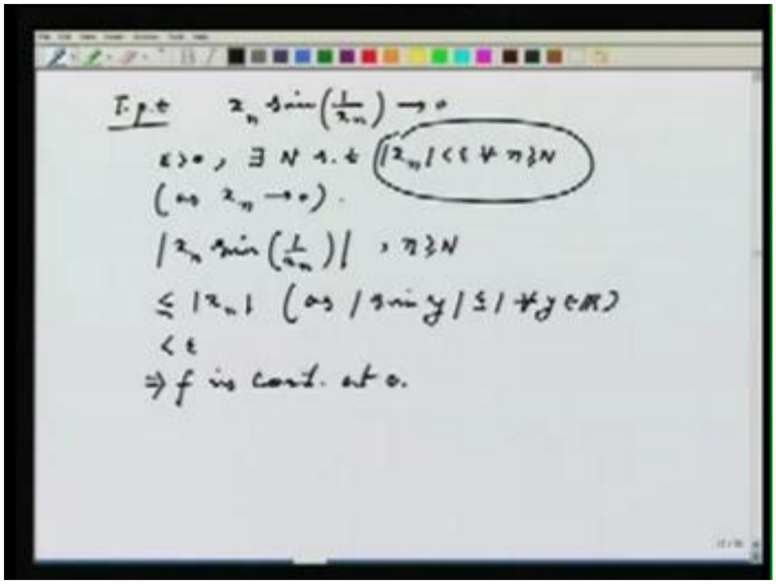

So to prove that x n sine 1 over x n converges to 0. Once I prove this, this would imply f is continuous at x is equal to 0. Now to prove this, I just apply the definition of sequence if I like. Start with epsilon bigger than 0, then there exist N such that mod of x n is less than epsilon, for all n bigger than or equal to N, because I have assumed x n converges to 0. Then what do I know about x n sine 1 by x n where n is bigger than or equal to capital N? I know that this quantity less than equal to modulus of x n as mod sine y is lesser equal to 1 for all y in R and modulus of x n is anyway less than epsilon. That is what I have written here.

This implies f is continuous at 0. This is how actually one uses the sequential criteria to check continuity of a function or even discontinuity of a function. Now using these sequential criteria, it becomes possible to construct more and more examples of continuous functions. This is how we proceed. I just write it as a theorem. If you have gone through all the results of the sequence which I taught you, you will be able to prove all these results quite easily.

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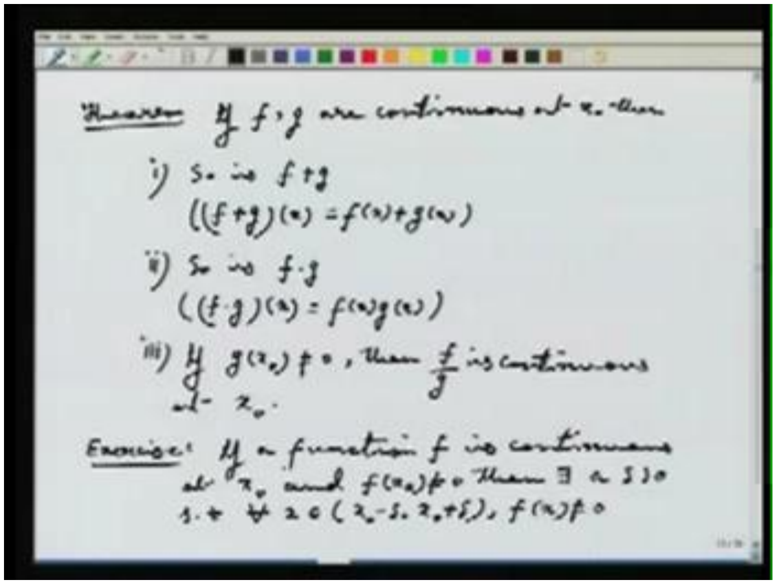

So the first result is if f g are continuous at x naught, then so is f plus g. What do I mean by f plus g? It is another function whose definition is f plus g at x is f x plus g x. I can also prove if f and g are continuous at x n naught, then so is f dot g. What is f dot g? f dot g is another function whose definition at x is f x times g x. Similarly, if g of x naught is not equal to 0, then f divided by g is continuous at x naught.

But here, in this third problem, there is something which bothers us that can it happen that g is continuous at x naught g is non 0 at x naught but in every neighborhood of x naught g is 0. Can that happen intuitively? You feel it cannot happen, because then there is a jump g of x naught is either positive or negative then in every neighborhood if g is 0, then we are in trouble. So to understand that 3 is really true, if you want to fill it, you should solve the following exercise. It follows just by following the epsilon delta definition of continuity.

If a function f is continuous at 0 then, sorry, let me change it, if it is continuous at a point x naught let us say, and f of x naught is not equal to 0, then there exist a delta bigger than 0 such that, for all x in an interval around x naught of length 2 delta, which I will write as x naught minus delta x naught plus delta, it happens that f of x is also not equal to 0.

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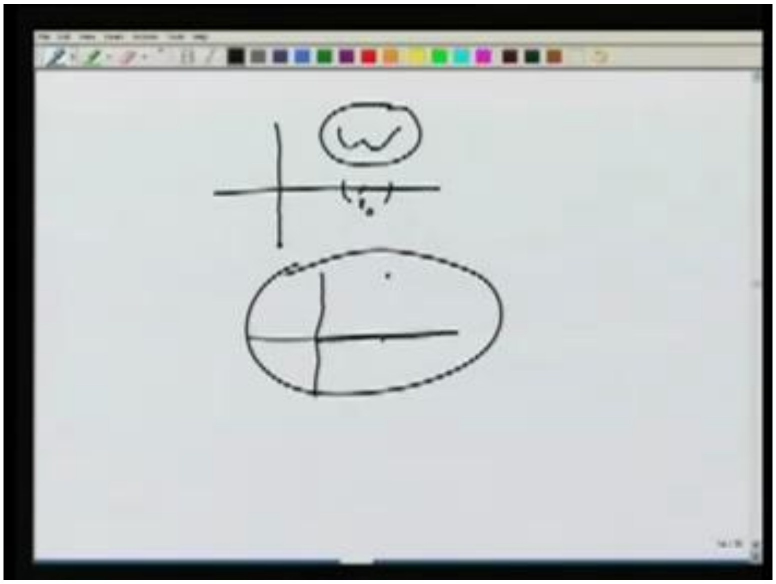

What actually it illustrates is, the following picture that this is your point x naught and f is non 0 at x naught, means, it is either here or it is below. Suppose it is here. Then the graph of the function has to look like. It cannot happen that the graph of the function looks like this and it is 0 here. That is quite obvious because this is the jump which we wanted to definition of continuity. You see, what I am asking you, is to prove the using the definition of continuity that this kind of situation does not happen.

Now, just let us test our understanding. We ask the following question. If we look at this picture which I have drawn, at this point x naught, f is continuous. But you see, in the picture, what is happening is, in the neighboring points of x naught also, f is continuous because the graph is continuous. Is it generally true that if a function is continuous at a point, then it is continuous in some neighborhood of that point also? So that is the question I want to ask now.

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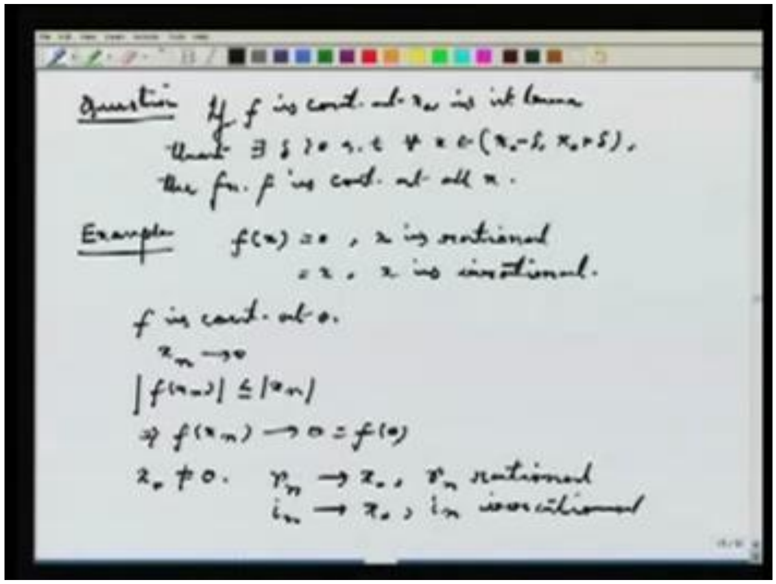

So question is, if f is continuous x naught, is it true that there exist delta bigger than 0 such that for all x in x naught minus delta x naught plus delta, the function f is continuous at all x? That means, is it true f is continuous in all neighboring points. If you think little bit, you will see, it is very difficult to imagine such kind of function whose graph you can really draw, but if you test analytically, I am going to show you that some functions exists where it is continuous exactly at 1 point and nowhere else.

So the example is this. Let us define this function f x equal to 0 if x is rational and equal to x if x is irrational. First, let us check where f is continuous. I say f is continuous at 0. How? Well, I choose a sequence x n which converges to 0 and notice that f of x n is always less or equals to mod of x n. This implies, f of x n converges to 0. This is nothing but f of 0. That means, f is continuous at 0 at least but what happens to other points? Let us take a point x naught, which is not equal to 0 and then I take a sequence of rational r n s converging to x naught, where r n are rationals and I take a sequence I n which are irrationals, which converge to x naught.

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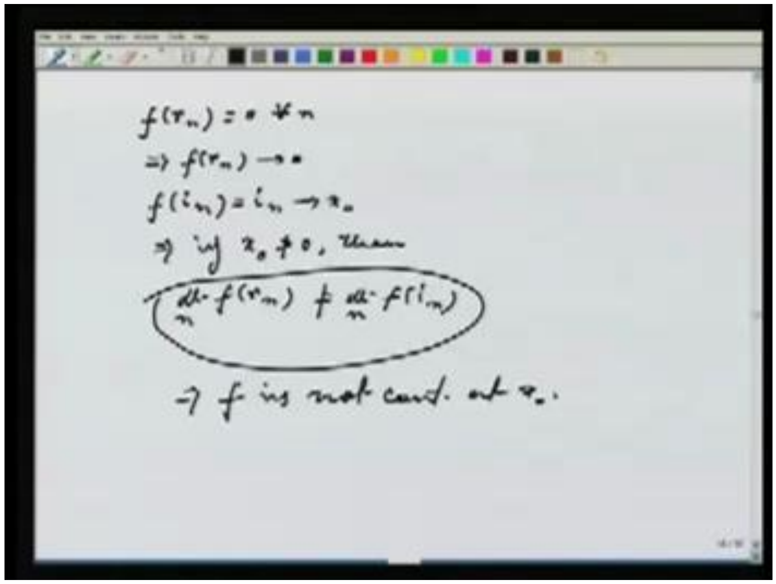

Then what happens to f of r n? By definition, f of r n is equal to 0, for all n. This implies that f of r n converges to 0. But what about f of In? By definition, this is I n which converge to x naught. This implies, if x naught is not equal to 0, then f of r n, the limit is not equal to, but this cannot happen if f is continuous. If f is continuous at x naught and x n converges to x naught and f of x n converges to f of x naught. That means in particular, it is meant, whatever sequence x n you take, which converges to x naught, all the f of x n s has the same limit, which is not the case here. I get a difference, implies, f is not continuous at x naught.

What is interesting about this function is you might have thought why we are for this analytical definition of continuity? Because intuitively, we can feel what continuity means, that you can draw the graph of the function without taking the pencil out of the page. The point is, for every function, you may not be able to draw the graph of the function and this is one function, whose example I have given, you cannot actually draw the graph of the function. You need some other tools to describe continuity of function and that is what is analytical description of the continuity which we will discuss. In the next lecture, we will discuss more about the deeper property of continuous functions.