Mathematics-II
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Lecture - 11
Linear Transformation Part – 1

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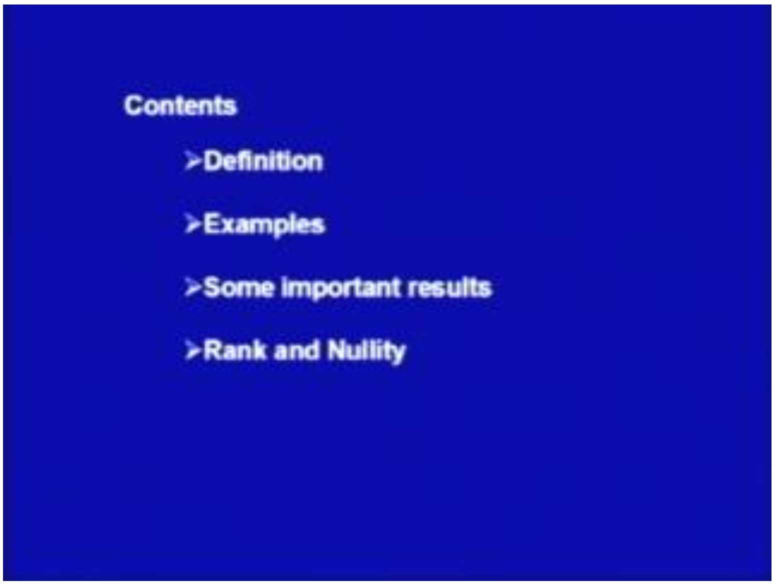

This lecture includes the definition, will give some examples of linear transformation. Then, we will discuss some important results regarding this. And then I will introduce the concept of rank and nullity. And finally, some results related to rank and nullity.

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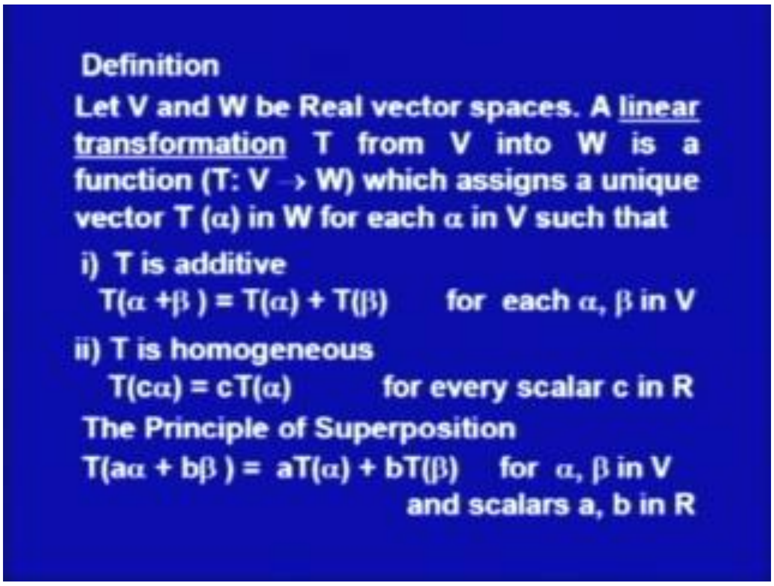

To start with I will first give the definition of linear transformation. For this, let us consider two real vector spaces V and W. And a linear transformation T from V into W is a function. I say T is from V to W, which assigns a unique vector T alpha in W for each alpha in V, such that it satisfies the properties. The first property is that T is additive. By this I mean to say, that when T operated on alpha plus beta, for each alpha and beta in V. Then, it will becomes transform to T alpha plus T beta. By this, we say that T is additive.

The second property is T is homogeneous. By this I mean to say, that when T is applied on c alpha, alpha is the vector. And c is the scalar in R, then it becomes c times T alpha. So, T c alpha is equal to c T alpha. And if these two properties are satisfied, then such a transformation T is called a linear transformation. I will introduce the principal of super position. By this I am mean to say, that if I have two vectors alpha beta in V. Then T of a alpha plus b beta is equal to a of T alpha plus b of T beta, where a and b are scalar in R.

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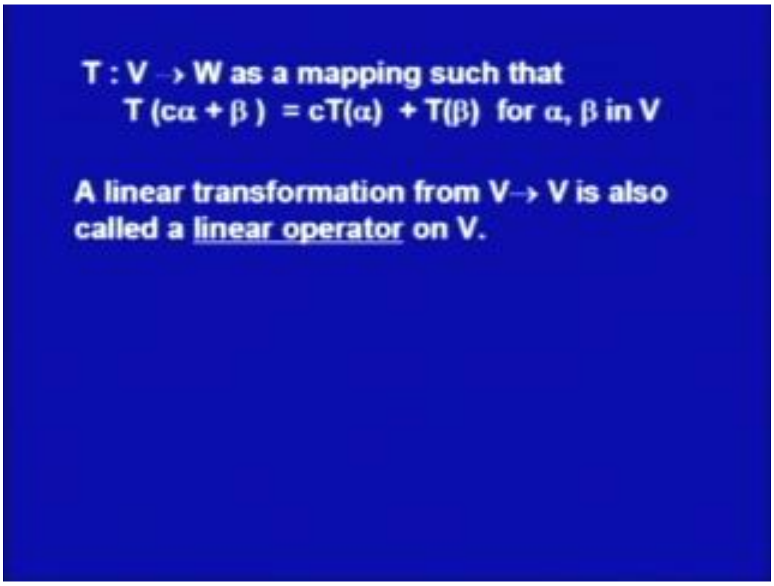

The way I have define this transformation, I can say that T from V into W is a mapping. Such that, T of c alpha plus beta is equal to c time T alpha plus T beta, for alpha beta in V. In fact, this will become an alternative definition for linear transformation. So, instead of satisfying two properties, linear properties and homogeneous property. The two properties are combine into one single property, defined in this manner. Now, a linear transformation from V to V is also called a linear operator on V.

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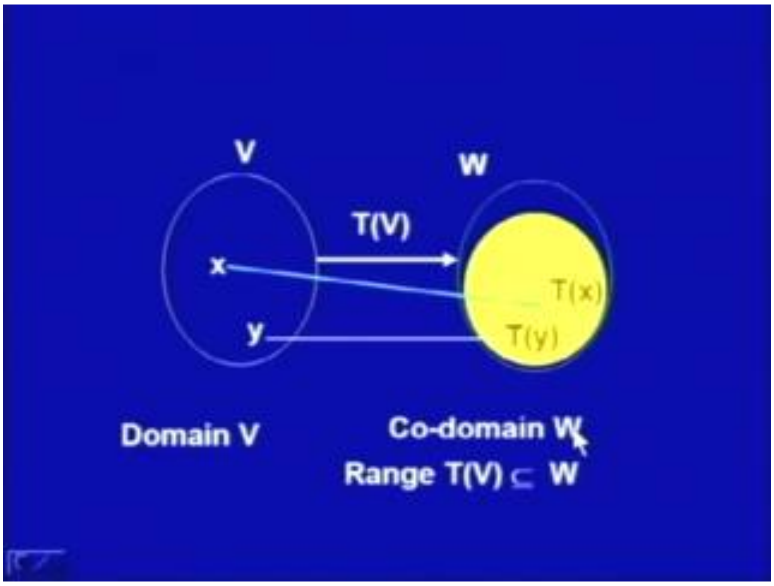

To illustrate what I have done, let us consider two vector space V and W. By this I am mean to say, V consist of a set of vectors two operators, one is addition of vectors, another scalar multiplication define in V as well as in W. And they satisfy certain properties. So, that way V and W are two vector spaces. Then, a linear transformation from V into W is denoted by T V provided a vector in x goes to T x in W. And a vector in y goes to T y in W.

And they satisfied the property the homogeneous property and the linear property, which I have defined earlier. Now, in this case I say V is the domain of the linear transformation, while W is co-domain for the transformation. The set, which consist of images of x and y in V, all x and y in V. That set is called range of T V. And definitely this is a subset of W. Later on, we will prove that it may be a vector space.

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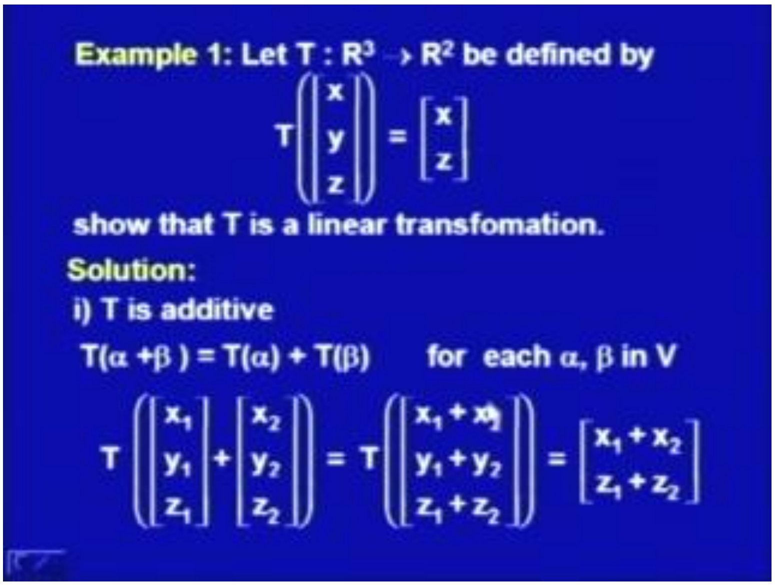

Now, let us take an example we have a transformation T from R cube to R 2, which is defined as T of x y z a vector goes to x z in R 2. So, this is a vector in R 3. And this is a vector in R 2. If we define this transformation in this particular manner, then will show that T is a linear transformation. If it is a linear transformation one has to prove that, T is additive by this I mean to say, that if I take alpha and beta two vectors in R 3. Then, T of alpha plus beta is T of alpha plus T of beta for each alpha beta in V.

So, let us consider alpha is x 1 y 1 z 1. And beta is x 2 y 2 z 2, so these are two vectors in R 3. Now, T of this is equal to T of this vector. Because, the sum of two vectors is x 1 plus x 2, this is y 1 plus y 2 and z 1 plus z 2. So, sum of these two vectors is equal to this. So, T of this is equal to this vector. Now, this transformation says that, the first component will become the first component of T alpha. And the third component will become the second component. So, that way T of x 1 plus x 2 y 1 plus y 2 z 1 plus z 2 will become x 1 x 2 and z 1 plus z 2.

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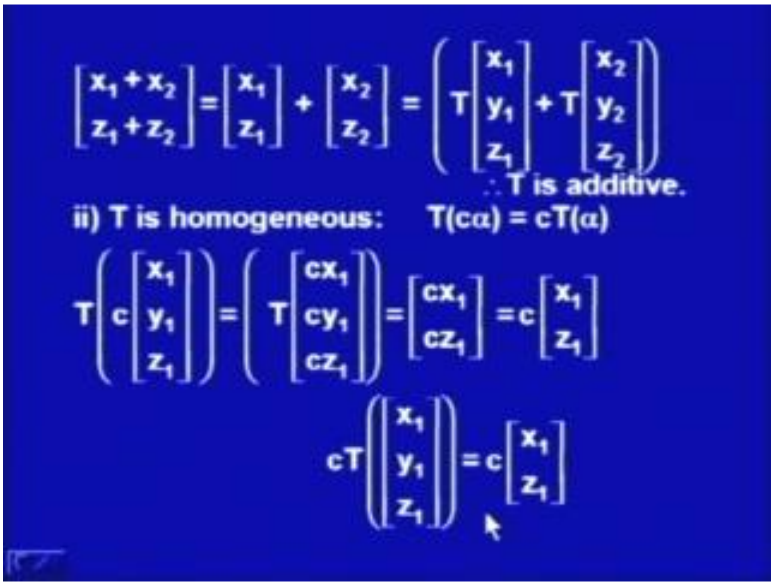

Now, this will simplify to x 1 z 1 plus x 2 z 2. And by definition of the transformation T x 1 z 1 is T of x 1 y 1 z 1 and x 2 z 2 is T of x 2 y 2 z 2. And that is how, we have prove that T of alpha plus beta is equal to T of alpha plus T of beta. And that proves the additive property. Now, we prove that T is homogeneous for this purpose, we have to show that T of c alpha is equal to c times T alpha c be in a scalar.

Now, to prove this let us consider alpha is x 1 y 1 z 1 in the scalar c. Then T of c times x 1 y 1 z 1 is equal to T of c x 1 c y 1 c z 1. What I have done is I have taken this scalar c inside this. So, this is equal to c of x 1 and c times z 1. So, T of this three dimensional vector is this two dimensional vector. And from this, we can take c outside. And what we have is c times x 1 z 1. And that means, c times T of this is equal to c times this vector. That means, this a vector in R 3 maps to a vector in R 2. So, c times T x 1 y 1 z 1 is equal to c times x 1 z 1 and that proves the homogenous property. And hence, it is a linear transformation.

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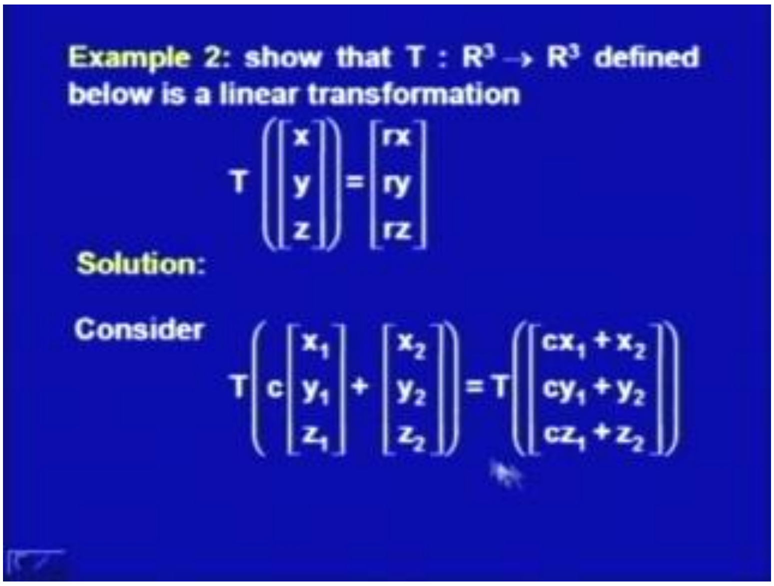

In this, we have again define a linear transformation from R 3 to R 3. And we will again show that it is a linear transformation. So, the transformation is T of x y z is r times r x r y and r z. Now to prove this are again we have to first show the linear property. So it is c times x 1 y 1 z 1 plus x 2 y 2 z 2. Now, I am combining the two prosperities, the homogenous as well as the additive property. So, I will say T time T of c x 1 y 1 z 1 plus x 2 y 2 z 2 is equal to T times c x 1 plus x 2, c y 1 plus y 2, c z 1 plus z 2, so basically I have combine these two vectors in this form.

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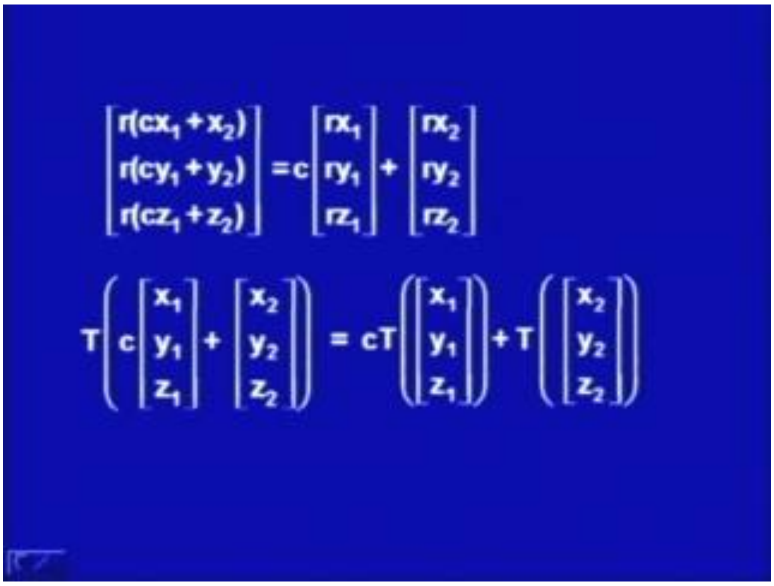

Now, using the definition for the linear transformation. This expression can be simplified to r times c x 1 plus x 2, r times c y 1 plus y 2, r times c z 1 plus z 2. That is the effect of the transformation is that each component is r times the original value. So, this left hand side is now c times r x 1 r y 1 r z 1 plus r x 2 r y 2 r z 2, it is a sum of two vectors. And that means, the transformation T applied an c times first vector plus the second vector is equal to c times the transformation applied on the first vector plus transformation applied on the second vector. This proves that T is a linear transformation.

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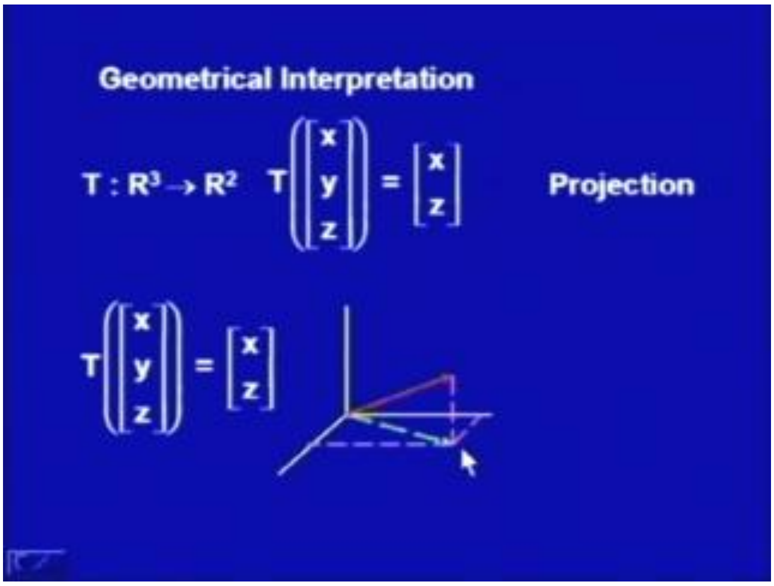

Now, I will give geometrical interpretation to the examples, which we have taken so far. The first example is the linear transformation T define from R cube into R 2 is T times x y z is equal to x z. You may call it a projection transformation. Let us, consider this vector, this is the vector three dimensional vector x y z it is this is x axis, y axis, z axis. This is the x component, this is the y component and this is the z component.

Now, when T this vector when we consider T of this vector, then what we have is x z. So, what is x z, this is x and this is y. So, this is the vector which is the projection of this vector. So, this transformation means, that this vector is projected this vector has a projection this. So, when we applied linear transformation on this vector, what we get is this projected vector.

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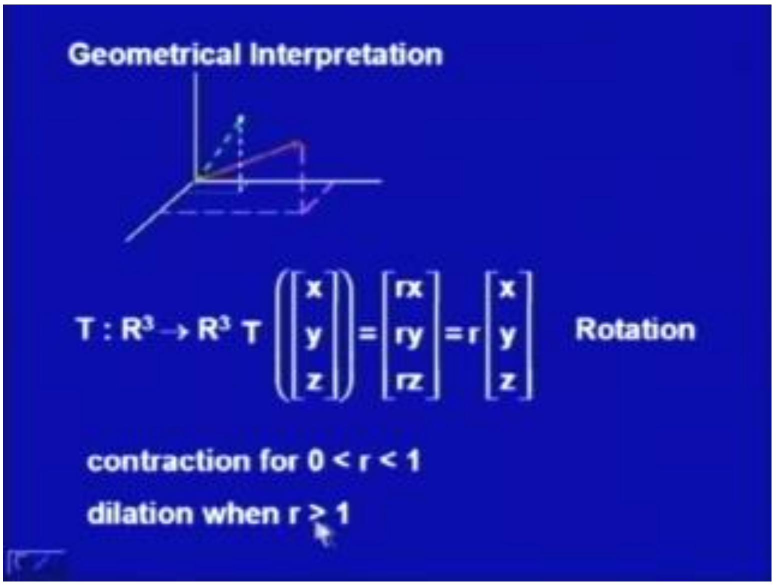

So, the second example we consider the vector alpha a R 3 as x y z. This is the vector x y z, it has three components x y and z. Now, when we apply linear transformation on it. Then, it becomes r x, so r x this is r x, this is r y and this r z. So each of these is r times these values. So, r x, r y, r z are these components and then the vector will be this vector. So, this vector will be transform to this vector.

And what we have seen that, this vector is rotated to this vector, when we apply this linear transformation. And that is why we call this transformation as rotation. Now, in the process if r is less than 1, it is a positive number lying between 0 and 1. Then we say this is contraction and when r is greater than 1 we say it is dilation.

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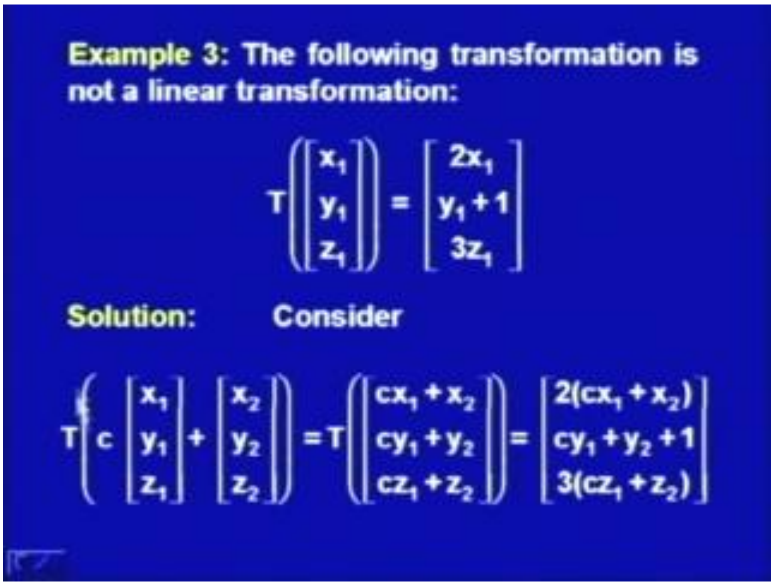

Then next example, T of x 1 y 1 z 1 is equal to 2 x 1 y 1 plus 1. And the third component is 3 z 1. So, the transformation from R 3 to R 3 this transformation is not a linear transformation. So, what we have to do is that this linear property is not satisfied for this transformation. So, let us consider T applied on c times first vector plus second vector in the domain set domain vector space. So, it is T times c x 1 plus x 2 if I combine the two. Second component is c y 1 plus y 2 and c z 1 plus z 2.

Now, then T operated on this vector then according to this definition, the first component is two times this. So, it is 2 times c x 1 plus x 2. The second component will map to y 1 plus 1. So c y 1 plus y 2 becomes c y 1 plus y 2 plus 1. And the third component c z 1 plus z 2 will becomes 3 times this component. So, c z 1 plus z 2 is 3 times c z 1 plus z 2, so T of this vector is this.

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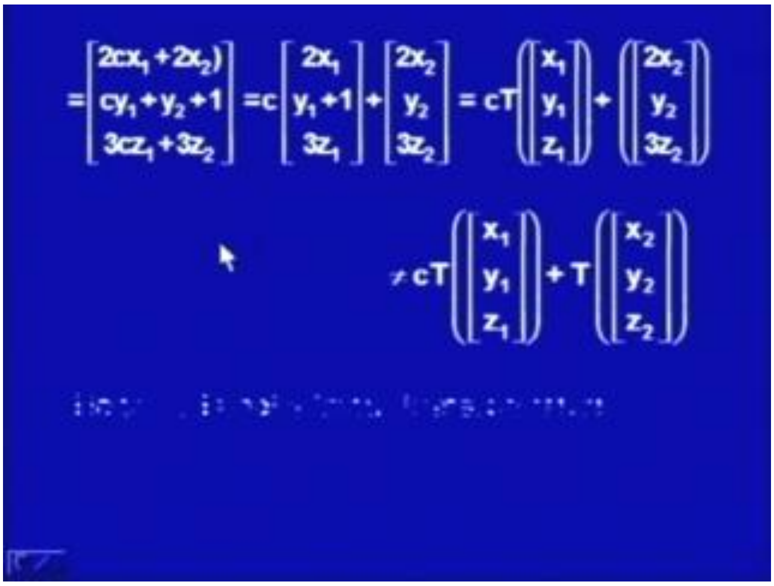

So, let us simplified this, so T 2 times c x 1 plus 2 times as x 2 is c times 2 x 1 plus 2 x 2, but when we come to this, this c y 1 plus y 2 plus 1. It is c y 1 plus y 1 plus 1 plus y 2, c y 1 plus 1 and y 2. And this is c can be taken out is 3 z 1 plus 3 z 2. So, this vector is nothing but c times T of x 1 y 1 z 1 plus this vector. But, this vector is not T of this vector, because plus 1 is missing from here. So, what we can say is a linear property is not satisfy or we can say that this is not equal to c T of x 1 y 1 plus z 1, that is T of x 2 y 2 z 2 to the this transformation is not a linear transformation.

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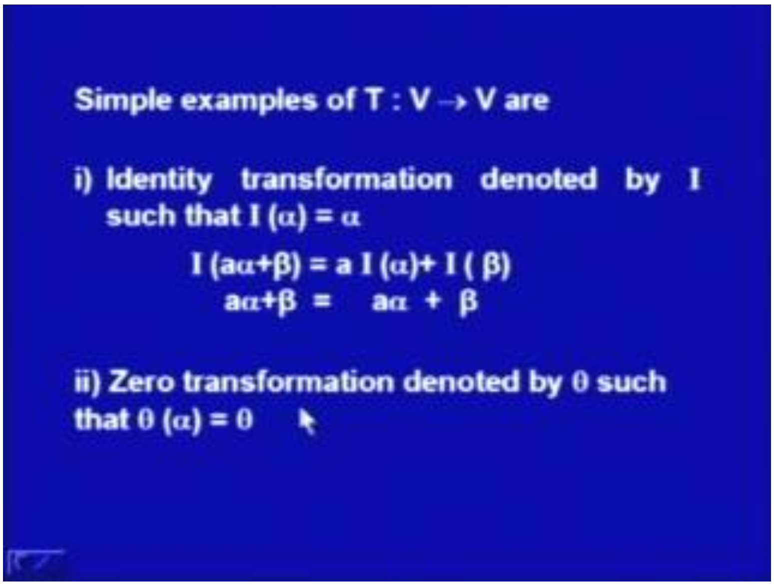

Apart from these example, there as some simple examples of give a transformation from V to V are the first is the identity transformation we denoted by I. Such that, every vector in alpha maps to itself. Since a transformation from V to V. So, every alpha vector alpha in V will not to itself such a transformation is identity transformation. And one can easily see, that it is actually satisfy this property. Like I a alpha plus beta is a times I alpha plus I beta.

And that means, I applied and this a alpha plus beta. And I applied on this is a alpha plus beta. So, both the sides are equal and one can say that, identity transformation is the linear transformation. The another simple example is the zero transformation be denoted by theta. And we define it as that theta of alpha is equal to 0. That means, every alpha in this vector space V will map to theta additive identity. So this is a 0 transformation as a theta every vector alpha will now up to 0. So, this also satisfied linear property that can be very easy to prove and is that linear transformation.

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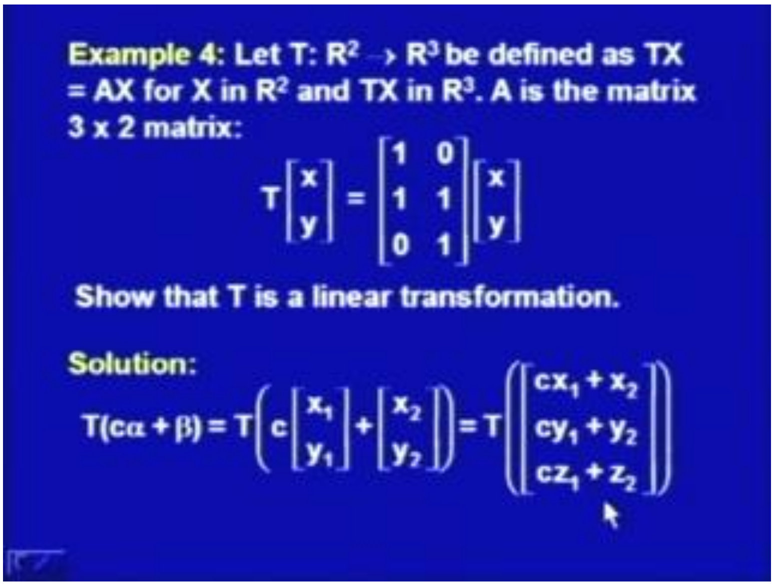

This is another example, here we define the linear transformation T from R 2 to R 3. And it is defined as T X is equal to A X for X in R 2 and T X in R 3, where A is the matrix of 3 by 2 order. So, let us define this transformation T as x and y ((Refer Time: 16:34)) in R 2 will map to this vector in R 3. Now, we show that T is a linear transformation. Now, again the method of proof is the same, we start with c alpha plus beta.

And the vector c alpha plus beta will be we operate T on this. And what we have is T times c x 1 y 1 plus x 2 y 2 set is a two dimensional vector. So, alpha is the x 1 y 1 and beta is x 2 y 2, when this is operated on T when T is operated on this vector. Then we should have T times c x 1 plus x 2, c y 1 plus y 2 c z 1 plus z 2.

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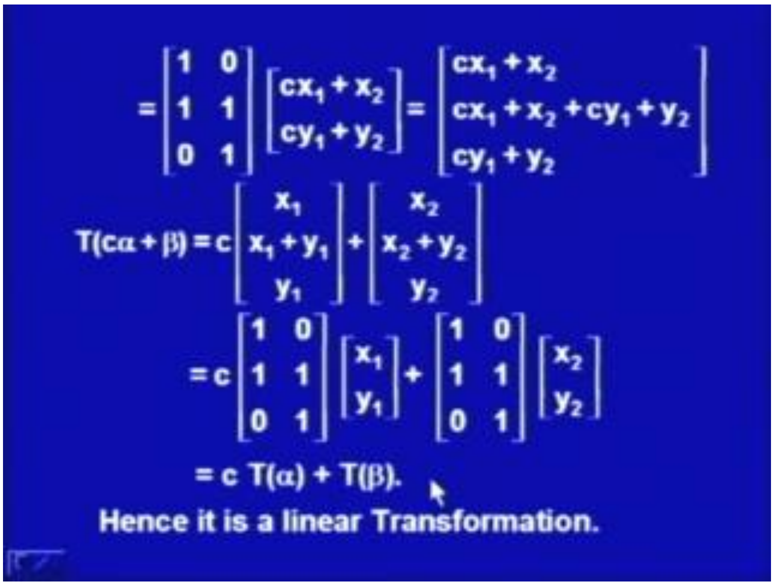

And that means, 1 0 1 1 0 1 this is the definition of this linear transformation. And when we multiply it is c x 1 plus x 2 this multiplied by this is c x 1 plus x 2 and plus c y 1 plus y 2 and then this is equal to c y 1 plus y 2. And then T c alpha plus beta one can see that this is equal to c times x 1 second component x 1 plus y 1 and third component is y 1 and plus from here, we can write down x 2 this x 2 plus y 2 the second component and what we have is y 2.

So, T c alpha plus beta is equal to c times 1 0 1 1 0 1 x 1 y 1 plus 1 0 1 1 0 1 x 2 y 2. So, if we use that this is actually equivalent to this and this means that this is c times T alpha plus T beta and that proves that it is a linear transformation.

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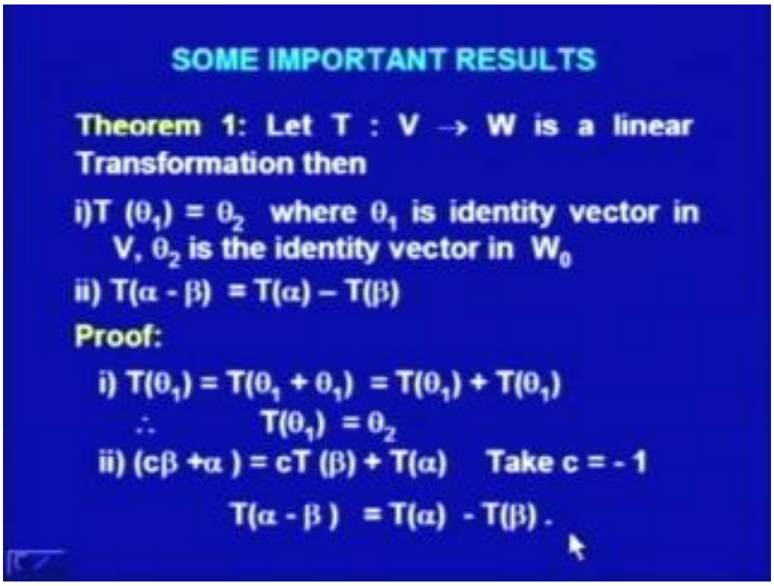

Now, we will discuss some important results in the form of theorems, theorem 1 says that if T is a linear transformation from the vector space V into W. Then, the first result is that T of theta 1 is equal to theta 2, where theta 1 is identity vector in V and theta 2 is the identity vector in W. The second result says that T of alpha minus beta is equal to T alpha minus T beta. Then, the first result says that the identity vector of V under this transformation will map to identity vector of W.

Now, and this says that this we have additive property, now this is the subtraction, now let us consider the poof for the first property. So, we can write down T theta 1, what is theta 1, theta 1 is identity vector in V. So, when identity vector is added into identity vector what we have is identity. So, theta 1 can be written as theta 1 plus theta 1, so T of theta 1 plus theta 1 since T is linear becomes T theta 1 plus T theta 1.

And that means, T theta 1 is nothing but the theta 2 because, T theta 1 is equal to theta 1 plus T theta 1, this is possible only 1 T theta 1 is equal to theta 2. And this shows that the identity of V will map to identity of W to prove the second property we consider T of c alpha plus beta is equal to c T beta plus T alpha. Because, T is a linear transformation, but what we can do is, we can simply take c is equal to minus 1. And that means T of alpha minus beta is equal to T alpha minus T beta, so proof of this second property is also simple.

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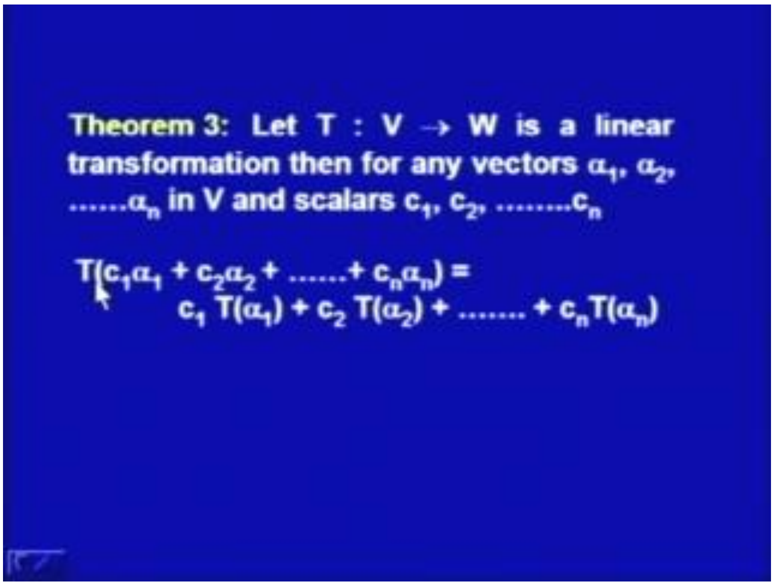

Now, some more results T is a linear transformation from V to W, then for any vector alpha 1, alpha 2, alpha n in V and the scalars c 1, c 2, c alpha, c n. Then, T of C 1 alpha 1 plus c 2 alpha 2 plus c n alpha n is equal to c 1 T alpha 1 plus c 2 T alpha 2 plus c n T alpha n, what we are going to do is, we have n vectors here we have n scalars what we have done is, we have taken the linear combination of and the vectors alpha 1, alpha 2, alpha n and the scalar c 1, c 2, c n.

Now, T of this vector now since we have V as a vector space. So, if alpha 1, alpha 2, alpha n, they belong to V. So, this linear combination will also belong to V. So, this is also vector in V and T is a linear transformation. Then, this T times this vector is equal to c 1 T alpha 1 plus c 2 T alpha 2 plus c n T alpha n. If we have only two vectors or we can say m is equal to 2, then T of c 1 alpha 1 plus c 2 alpha 2 is equal to c 1 T alpha 1 plus c 2 T alpha 2, this super position property, but actually these are n vectors.

So, to prove this what we can do is, we can consider c 1 alpha 1 plus c 2 alpha 2 plus c n alpha n, we can take this as one vector and we are c 1 alpha 1 one vector plus another vector linear property can applied. So, we will have c 1 T alpha 1 plus the second vector or the second vector the same thing can be applied with second vector repeatedly. And what will have a final result that T of c 1 alpha 1 plus c 2 alpha 2 plus c n alpha n is equal to c 1 T alpha 1 plus c 2 T alpha 2 plus c n T alpha 1. So, basically a generalization of what we had earlier, so for that is result was for two vectors this is for n vectors.

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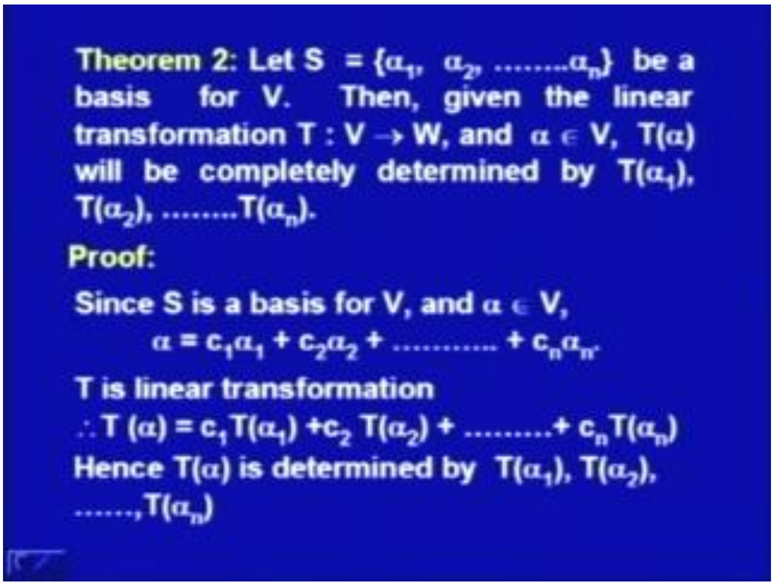

Now, since V is the vector space, so it may be having basis. So, let us consider S is a basis for the vector V. Then, given the linear transformation T and alpha belonging to V, then T alpha will be completely determine by it is ((Refer Time: 23:00)) T alpha 1 T alpha 2 T alpha n, the idea is that what will happen to these vectors alpha 1, alpha 2, alpha n.

So, we have these are basis this V is n dimensional vector space. Then, T alpha being the image of a vector at V, then the inner vector will be determined in terms of T alpha 1, T alpha 2, T alpha n. So, let us prove this, since S is a basis for V and alpha belong to V, then alpha which a linear combination of these vectors S being the basis can be written alpha is equal to c 1 c 1 alpha 1 plus c 2 alpha 2 plus c n alpha n, then if T is a linear transformation. So, at T of alpha is equal to c 1 T alpha 1 plus c 2 T alpha 2 plus c n T alpha n. Now, this is the result which we I have just established in my last theorem. Then, T alpha is determined by T alpha 1, T alpha 2, T alpha n, how see this vector alpha is determined by a the scalar c 1, c 2, c n and this alpha 1 maps to T alpha 1, alpha 2 maps to T alpha 2, alpha n maps to T alpha n. So, c 1, c 2, c 2, c n uniquely determine, so T alpha will also be uniquely determined and hence, we can say that T alpha is determined by T alpha 1, T alpha 2 and T alpha n.

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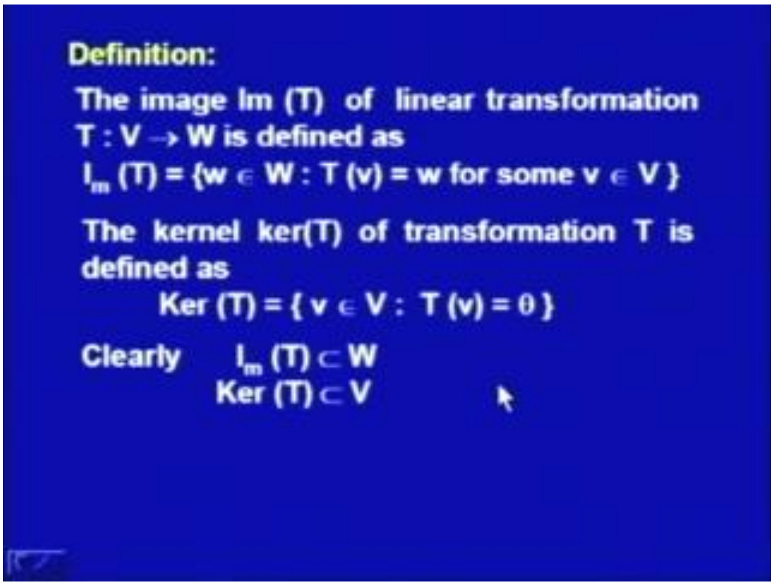

Now, we will introduce more concepts to start with the image T of linear transformation T from V to W is defined as the set consisting of all Ws. Such that T V is equal to W for some v belonging to V. Sometimes, this is also called as range of T, the kernel T of transformation T is defined as the set consisting of all vectors v in V. Such that, T v is equal to identity in W, that way we defined two sets image T of range T and kernel T.

So, these two sets are related with that linear transformation T, clearly image T is a subset of W and kernel T is a subset of V. In fact, it is prove that image T is sub space of W, W being the vector space, kernel T is the sub space of V.

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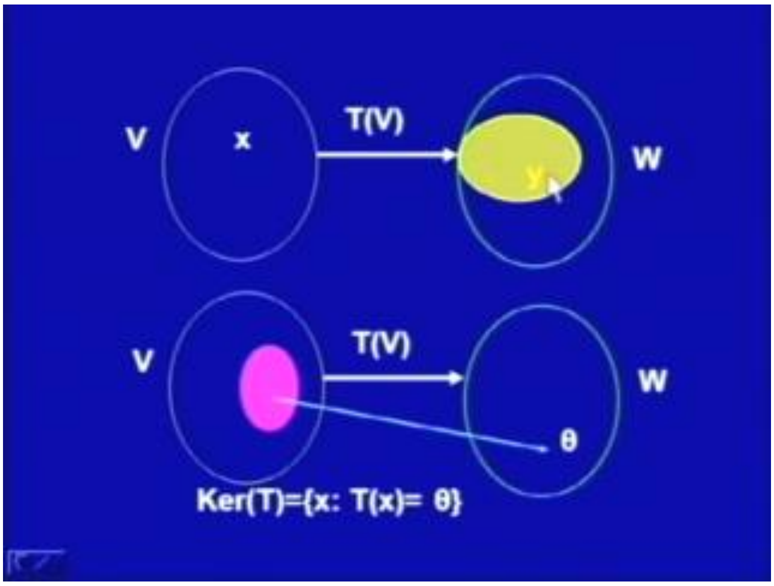

Let us show it pictorially we have two vector spaces V and W x belongs to V and under this transformation T this vector x will go to y. Now, this is the image set which is consisting of all y's which has some point x in the set V. So, y is an image of x all these y's include the range or image set which is contained in W.

Now, for the kernel T this is the domain, in which the vectors will map to the identity vector theta in W. So, all these vectors in this domain will map to theta. So, kernel T is the set of all x in V such that T x is equal to theta, so this is a subset of V and this is a subset of W.

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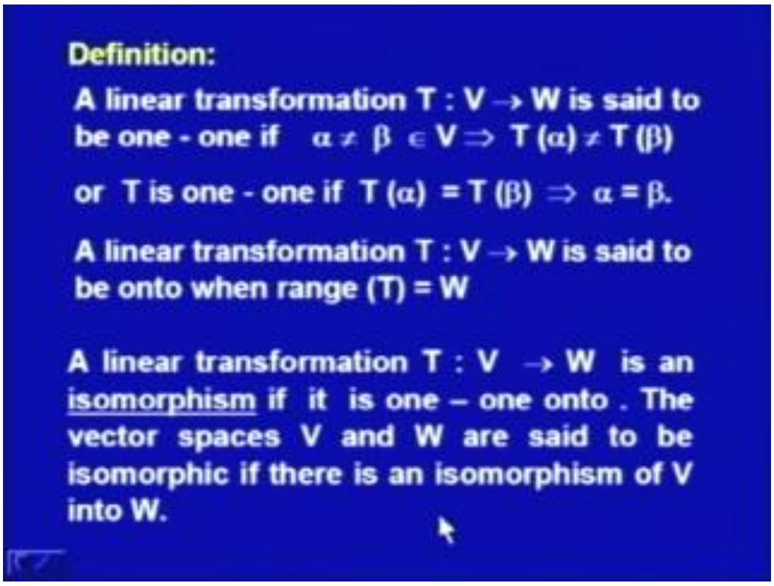

Now, we say that a linear transformation T is said to be one-one if alpha not equal to beta belonging to V, implies that T alpha and T beta they are not the same. That means, if we have two different elements in V, then there images cannot be the same they have to be different. That is, if T is one-one, then if T alpha is equal to T beta then alpha and beta have to be same, a linear transformation T is said to be onto when range of T is equal to W. And another definition if V T the linear transformation from V into W is an isomorphism if it is one-one onto. The vector space V and W are said to be isomorphic, if there is an isomorphism of V into W.

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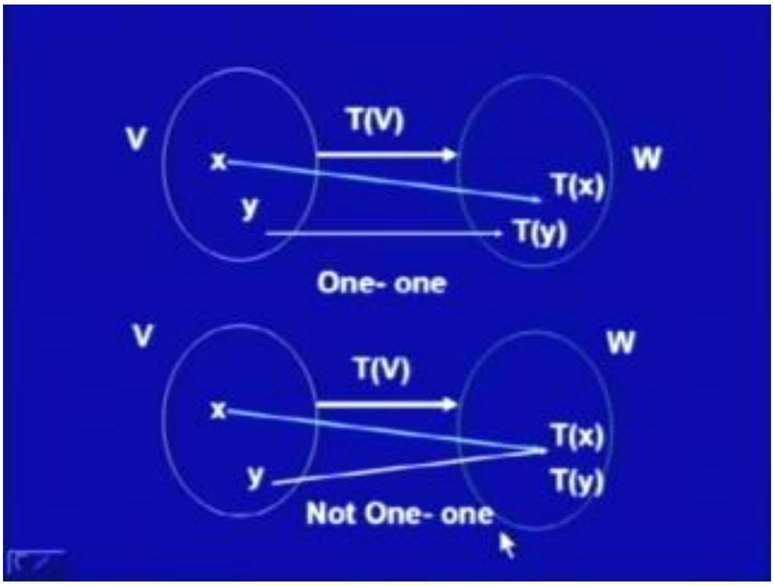

Again, we show it here we have two different elements, two different vectors in V the map to different vectors in W. So, x maps to T x and y maps to T y, this is true for any combination of x and y in V, then we say that the linear transformation is one-one. In another example, here we have 2 x and y in V, but both map to same element in W, that is x and y are different in V, but they are same in W and that means this transformation T is not one-one.

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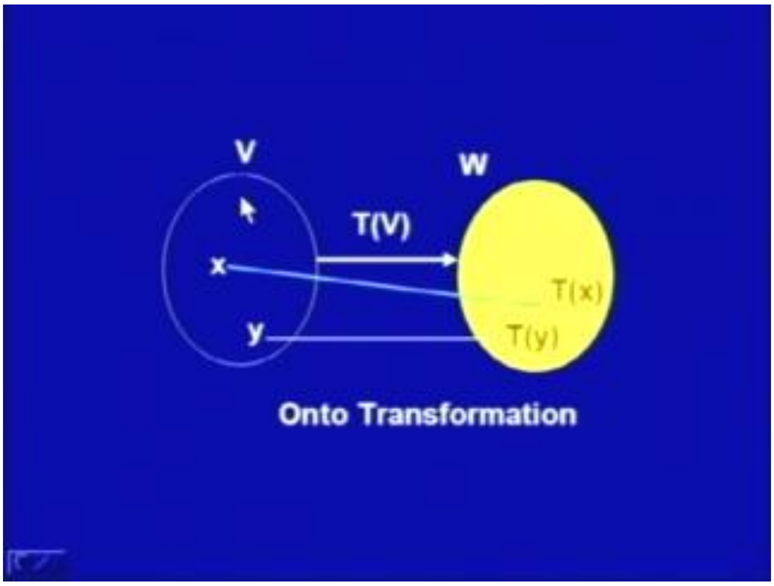

Now, in this case we have of the vector space V and a vector space W, T is a linear transformation x maps to this element T x in W y maps to this element T y in an W. And this is the set, in which we will have images of in which the elements in V will map to this set. That means, if this set and set W they have become the same, then it becomes an onto transformation. That means, there is no element in this W, which is not an image of this every element here is in image is image of some point in V that is onto transformation.

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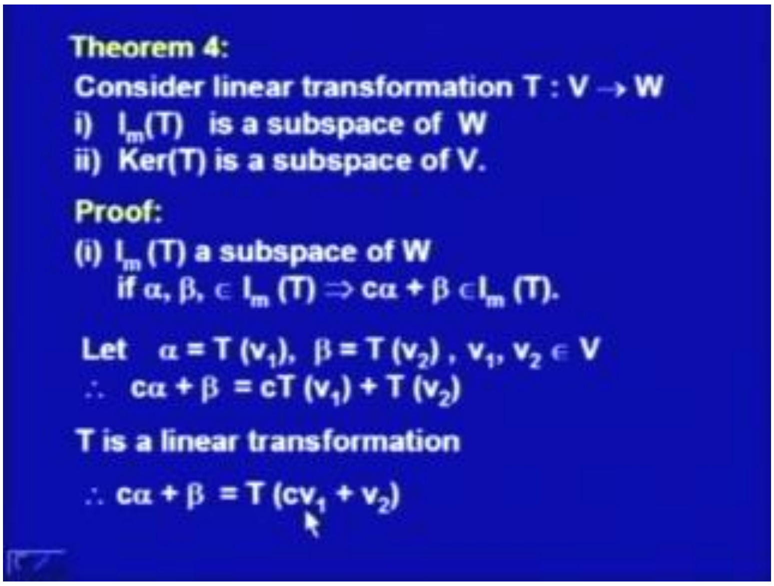

Now, we can prove these theorems that consider linear transformation T, then image T is a sub space of W and kernel T is a subspace of V. So, image T is a subspace of W and kernel T is a sub space of V. Now, first we prove the first result that image T is a subspace of W. So, to prove this we consider alpha beta belonging to image T. Then, if c alpha plus beta also belongs in image T, then image T is a subspace this is a very definition of subspace.

So, we consider alpha and beta belonging to image T, so if the belong image T. That means, there must be some v 1 in the set v, so that alpha is equal to T v 1 and some v 2 in v, so that beta is equal to T v 2. So, let us consider alpha and beta in this way. So, c alpha plus beta is equal to c of T v 1 plus T of v 2. And since, T is a linear transformation, so c alpha plus beta is equal to T of c v 1 plus v 2.

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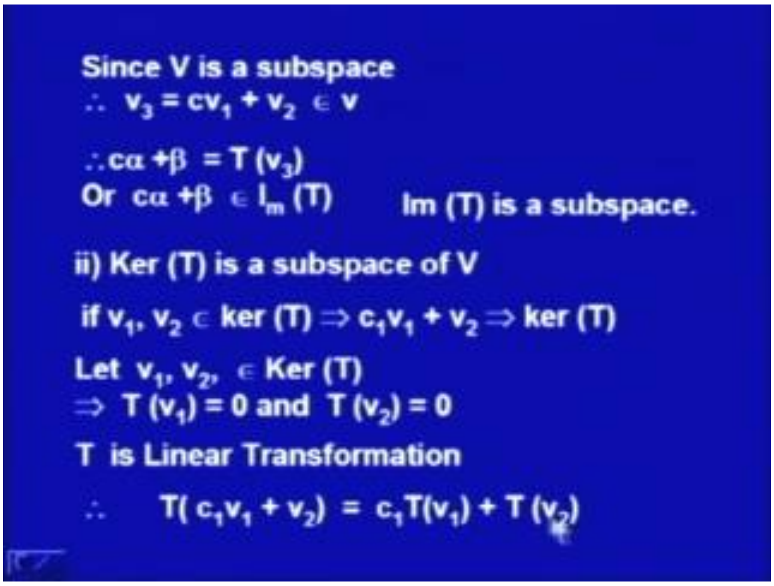

Now, V is a subspace therefore, c v 1 plus v 2 they also belongs to v, let us call this vector as v 3. Then, c alpha plus beta is equal to T of v 3. That means, c alpha plus b is an image of v 3 and; that means, c alpha plus beta also belong to image T and this proves that image T is a subspace. On the second part of the theorem is that kernel T is a subspace of v.

So, we consider two vectors in kernel T v 1 and v 2 and will prove that c 1 v 1 plus v 2 is also in kernel T that is the definition of subspace. So, we start with v 1, v 2 belonging to kernel T. So, if they belong to kernel T this implies that T v 1 is equal to identity theta and T v 2 is equal to theta. Now, since T is a linear transformation, then T of c 1 v 1 plus v 2 is equal to c 1 T v 1 plus T v 2.

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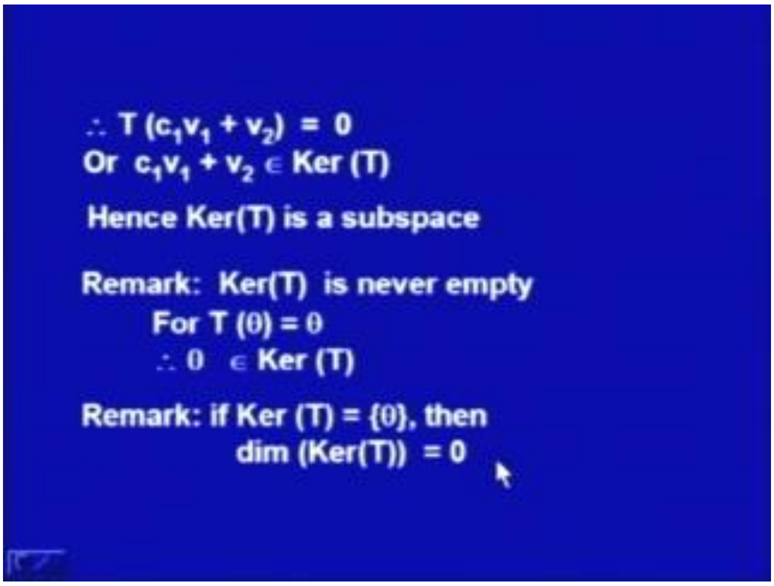

And that means, T of c 1 v 1 plus v 2 is also theta and this proves that c 1 v 1 plus v 2 also belongs to kernel T and that is kernel T is a subspace. Now, there is remark that kernel T is never empty because, T of theta is equal to theta. So, there is always a vector theta a kernel T, which will map theta itself. So, kernel T will never empty theta will always belong kernel T there may be more members in kernel T. But, this will theta will always B in kernel T and it is never empty. Then, secondary mark is, if kernel T is theta only then dimension of kernel T is equal to 0 this is another remark.

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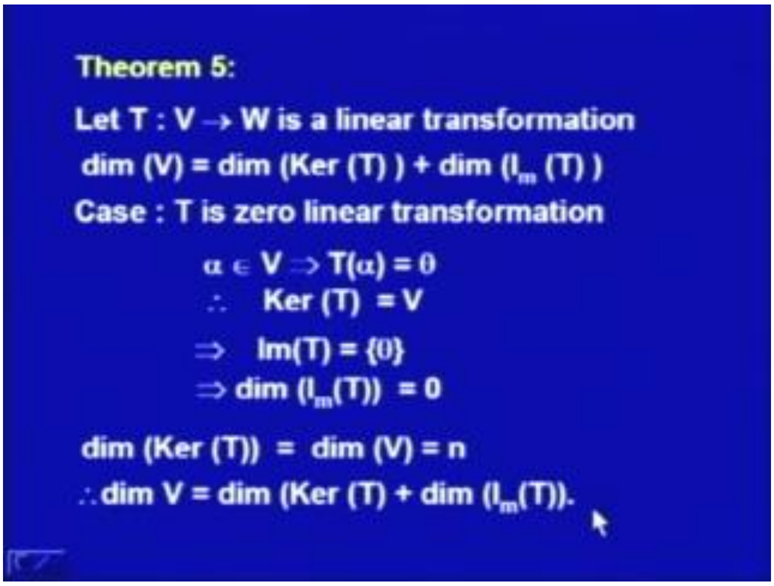

Then, we have and important result regarding dimension of kernel T and dimension of image T of range T. And it says that, if T is that linear transformation from V into W, then dimension of V is equal to dimension of kernel T plus dimension of image T. Now, to prove this result we prove it in two different cases, the first case is the T is the 0 transformation.

So, if T is a 0 transformation then alpha belongs to v implies that T alpha is equal to theta and kernel T in that case will be V and image T will be theta itself and dimension of image T in that case will be 0 or dimension of kernel T is equal to dimension of V is equal to n. So, dimension of V is equal to dimension of kernel T plus dimension of image T. So, that we have proved the first part, when T is an 0 linear transformation, the second case is the dimension of kernel T is k it is not 0.

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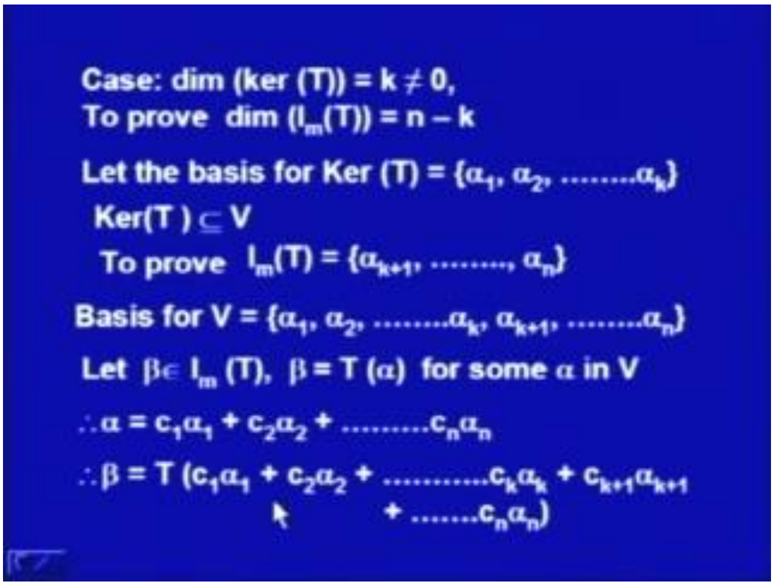

Now, to prove this we have to prove the dimension of image T is n minus k, so for this let us say that basis for kernel T is alpha 1, alpha 2, alpha k and kernel T being a subspace of V, we have to prove that image T is equal to alpha k plus 1 alpha n. So, this is what we are going to prove. So, image T is actually generated by this set, so basis for V is let us consider it to be alpha 1, alpha 2, alpha k, alpha k plus 1, alpha n.

So, we have n dimensional vector space V and the basis alpha 1, alpha 2, alpha k, this is the basis for kernel T, kernel T be in the subspace of V and we have some additional vectors. So, basis for V is this, let us consider beta belonging to image T, then there must be some alpha in V. So, that beta is equal T of alpha, so we write down this alpha as a linear combination of the vectors of basis of V. That is, alpha is equal to c 1 alpha 1 plus c 2 alpha 2 plus c n alpha n, then beta is equal to T of this vector.

So, I write down this vector as c 1 alpha 1 plus c 2 alpha 2 plus c k alpha k plus c k plus 1 alpha k plus 1 up to c n alpha n. So that means, I have divided this into two parts, this is the vector c 1 alpha 1 plus c 2 alpha 2 plus c k alpha k, the linear combination of vectors alpha 1 to alpha k; that means this will belong to kernel T and this is map to 0, so I will use the property of linear transformation.

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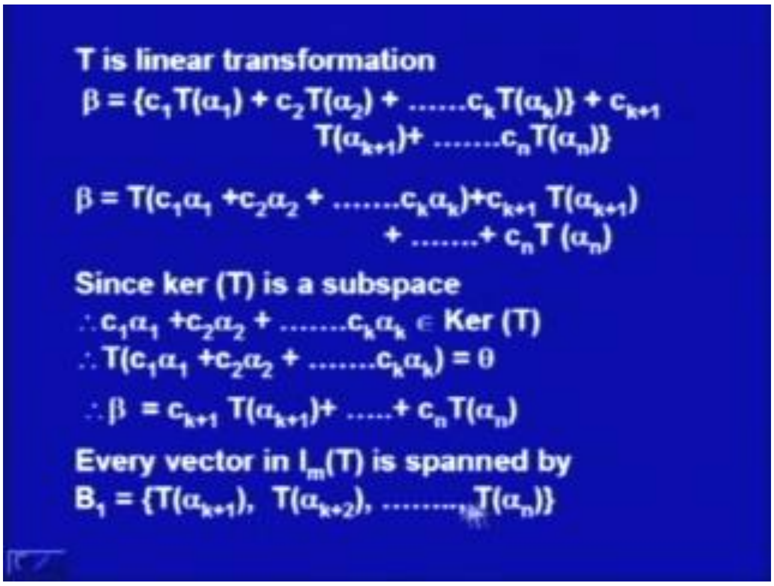

And I write down beta as c 1 T alpha 1 plus c 2 T alpha 2 plus c k T alpha k on one side and c k plus 1 T alpha k plus 1 plus up to c n T alpha n at the second term. So, I have two term I have represented beta as sum of two terms, this term and this term. And that means, beta is equal to T of c 1 alpha 1 plus c 2 alpha 2 c k alpha k plus the second term and this is a kernel T is a subspace. And therefore, c 1 alpha 1 plus c 2 alpha 2 plus c k alpha k belongs to kernel T and that is why T of this is equal to theta.

So, this we use beta is equal to simply this terms. So, beta is equal to c k plus 1 T k alpha k plus 1 plus the last term is c n T alpha n. Now, every vector in image T is spanned by this set. But, we have to say that if these vectors are linearly independent, then we have prove the result. So, we have just said that any vector beta is a linear combination of T alpha k plus 1 up to alpha n.

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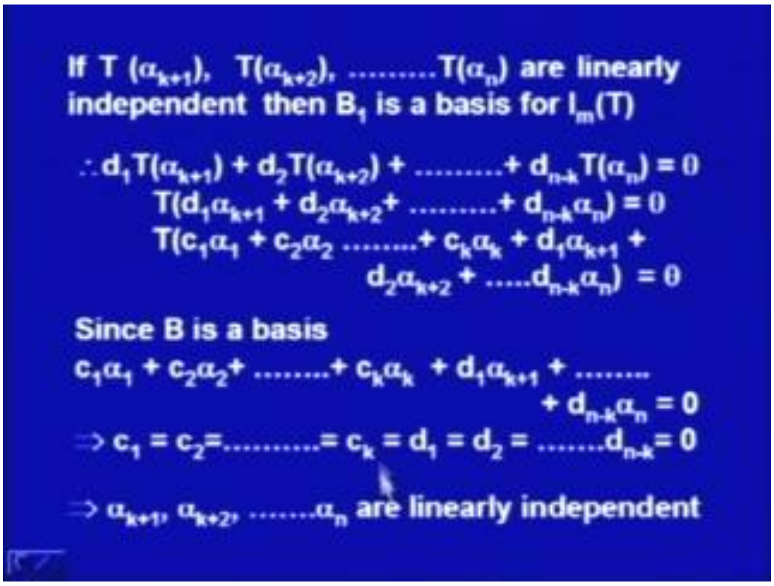

So, it is spanned by this set what we have to see that, they are linearly independent and in that case B 1 is a basis for image T. So, to prove this, let us consider d 1 T k plus 1 plus d 2 T of alpha k plus 2 d n minus k T alpha n is equal to theta. So, this linear combination of n minus k vectors is 0, now this is 0 then T of d 1 alpha k plus 1 is a linear property satisfied.

So, T of d 1 alpha k plus 1 d 2 alpha k plus 2 plus d n minus k alpha n is equal to 0 this is equal to T times I have added c 1 alpha 1 plus c 2 alpha 2 plus c k alpha k I have added this in this because, ((Refer Time: 38:44)) this is mapping to 0, so T of this is 0, so I have added this into this term.

So, since B is a basis, so c 1 alpha 1 plus c 2 alpha 2 plus c k alpha k plus d 1 alpha k plus 1 rest of the terms is equal to 0. And since it is a basis, so this linear combination is 0 means all the scalar terms c 1, c 2, c k, v 1, d 2, d etcetera are 0, so c 1, c 2 etcetera they are all identically 0. And; that means, alpha k plus 1 alpha k plus 2 alpha n are linearly independent because, these are 0, so these vectors are linearly independent.

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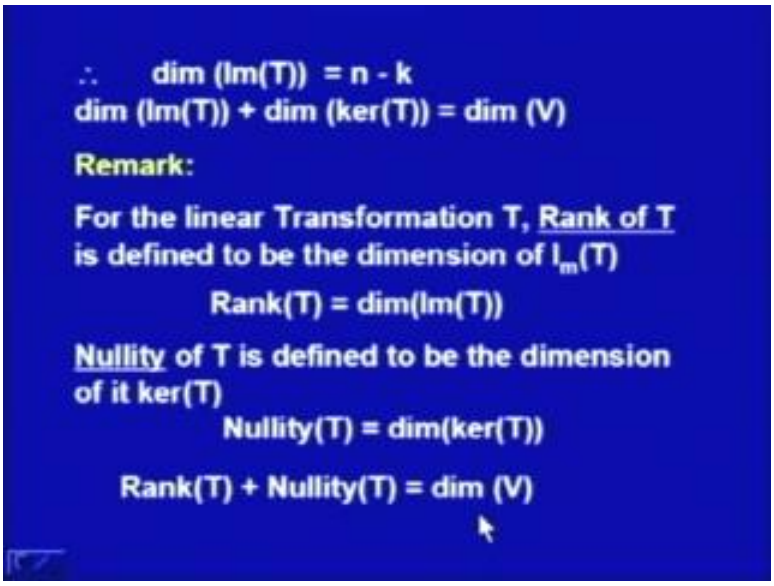

And that means, the dimension of image T is n minus k because, we are having n minus k vectors. So, what we have is dimension of image T plus dimension of kernel T is dimension of v. So, we have proved an important result for the linear transformation T rank of T is defined to be the dimension of image T. This is the definition that rank of T is defined to be the dimension of image T or rank of T is dimension image T, similarly nullity of T is define to be the dimension of kernel T. So, nullity T is dimension of kernel T and in this slide we can say that rank of T plus nullity of T is equal to dimension of V. So, this is an important result and will be using this at number of spaces.

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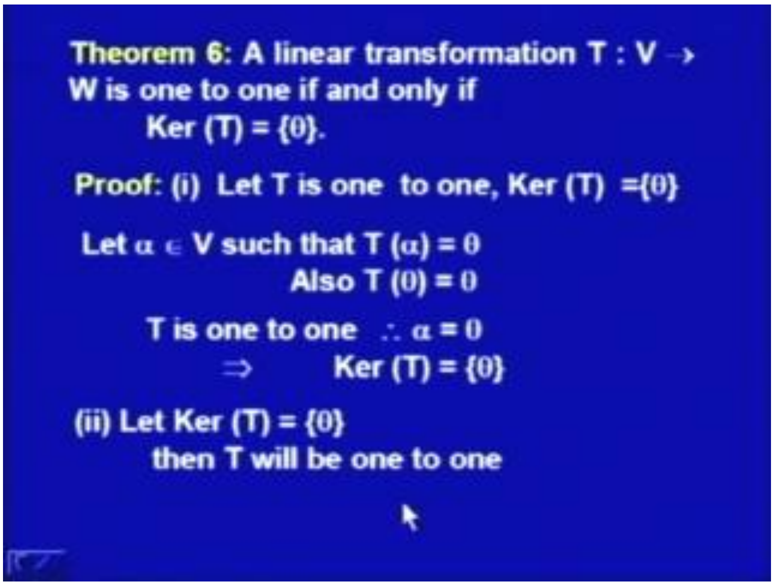

Now, another result a linear transformation is one to one if and only if kernel T is equal to identity, we have only one element in kernel T that is identity, such a transformation is one to one. Now, to prove this we have to prove two parts one is the T is one to one. Then, kernel T is equal to theta and the other part is if kernel T is equal to theta then T is one to one.

So, let alpha belongs to V such that T alpha is theta also T theta is theta. Because, theta will map to theta itself, this we have proved earlier. So, we have one more alpha which will take which this transformation T will take to theta, now this is one-one mapping. So, alpha has to be theta no two different elements will go to same element theta in W. So, kernel T has to be theta. So, by contradiction we have prove that kernel T is equal to theta. So, this part is proved, the second is if kernel T is equal to theta, then T will be one to one.

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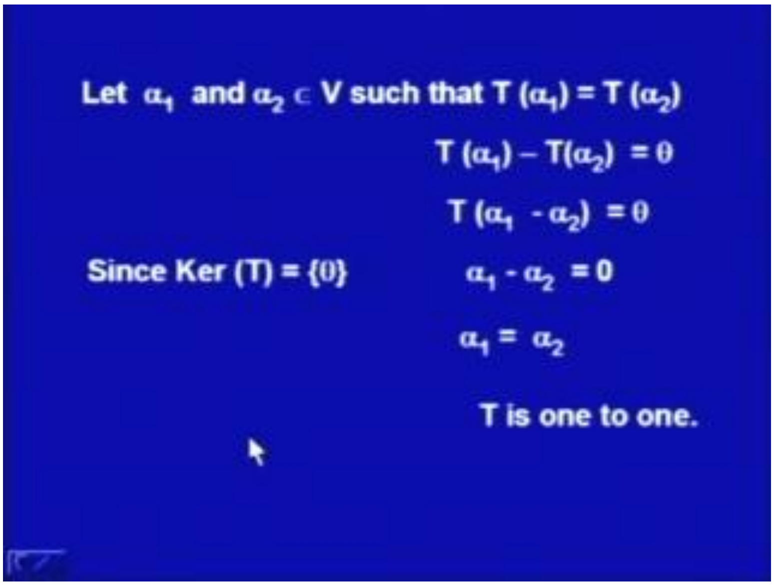

Now, to prove this result we consider alpha 1 and alpha 2 belonging to V such that T alpha 1 is equal to T of alpha 2. Then, T of alpha 1 minus T alpha 2 is equal to theta. And since, this is a linear transformation T of alpha 1 minus alpha 2 is equal to theta. But, alpha 1 minus alpha 2 will also belong to V, because a V is a subspace. So, there linear combination will also belong to V. So, T of alpha 1 minus alpha 2 is equal to theta, but we have said that only theta can map to this.

So, kernel T is equal to theta, so; that means, alpha 1 minus alpha 2 is equal to theta and that proves alpha 1 is equal to alpha 2. So, this means T is one-one, so even if we have started with two different values it comes out to be that these two vectors are the same and this proves that T is one to one.

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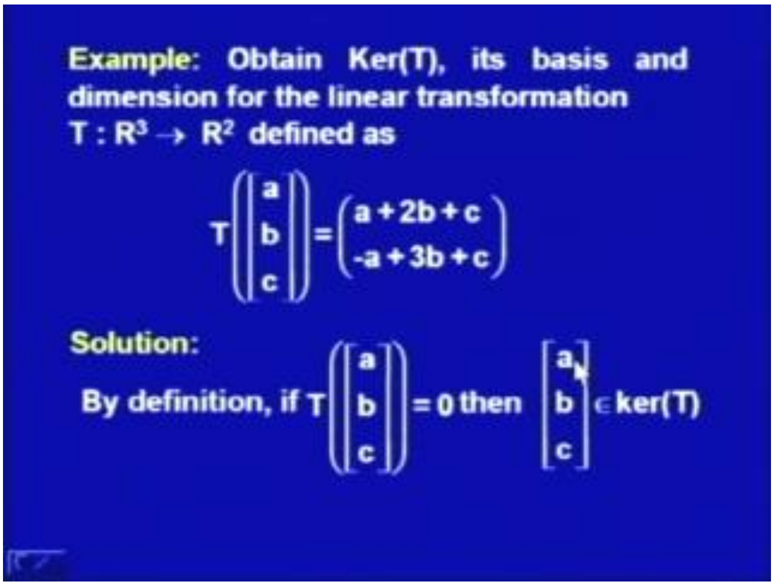

So, we have prove the result, now I will take some examples. The first example says that a transformation being given to us from R 3 to R 2 T a b c is equal to a vector in R 2 a plus 2 b plus c b in the first component and minus a plus 3 b plus c is the second component. This linear transformation is being given to us, we have to find kernel T and its basis and dimension. So, we will start with the definition of kernel T, if T a b c is equal to theta then a b c will belong to kernel T, this kernel T will be a member of R 3. So, that is why I have consider three dimension vector a b c. So, T of a b c will be 0, then a b c will belong to kernel T.

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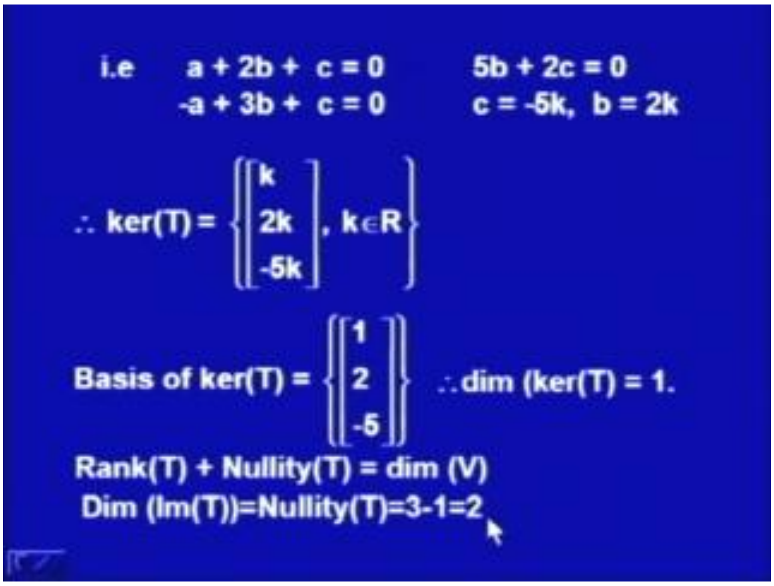

Now, to prove this we have to say that a plus 2 b plus c is equal to 0 minus a plus 3 b plus c equal to 0, this is to be there this is because a kernel T will be theta only. So, a the first component has to be 0 and the second component has to be 0. So, if you simplify then c is equal to minus 5 k and b is equal to 2 k, we can add the two and we will get this result.

That means, kernel T is equal to the a is k b is equal to 2 k and c is equal to minus 5 k k belonging to R. That means, any vector of this form will satisfy these equations. So, this vector will belong to kernel T. So, basis of kernel T will be 1 2 minus 5 any vector of this form can be generated from this. So, this is the basis for kernel T. So, kernel T will be having vectors of this form, which will be generated by this vector.

So, this forms a basis of kernel T and since kernel T involves a vector of this form. So, we say that this is the basis having only one vector, so the dimension of kernel T is 1. Now, we make use of rank T a nullity theorem, it says that rank T plus nullity T is equal to dimension V, dimension V is given to be 3 rank a nullity if given to be 1. So, rank T is equal to dimension image T is equal 3 minus 1 is equal to 2, so dimension of image T is 2.

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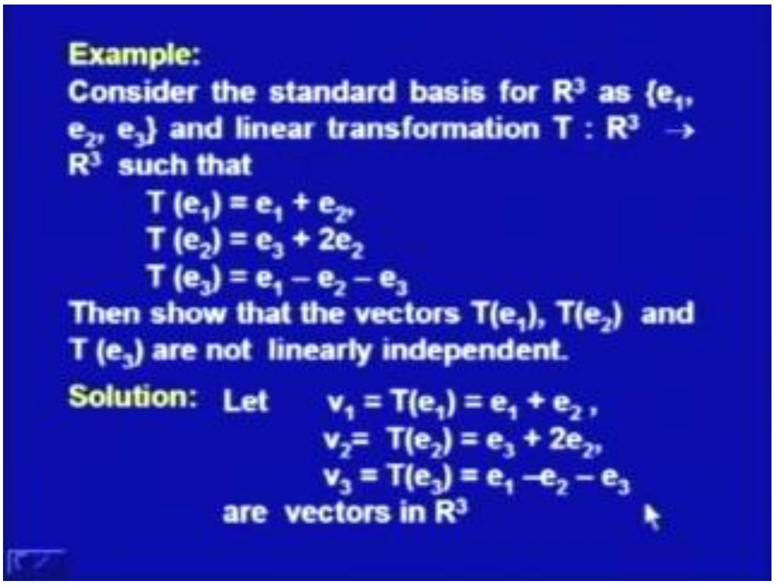

In the second example, we consider standard basis for R 3 being e 1, e 2, e 3 and we consider linear transformation from R 3 to R 3, define in this manner e 1 will map to e 1 plus e 2 T of e 2 will map to e 3 plus 2 e 2 and T of e 3 will map to e 1 minus e 2 minus e 3. Now, this is the map linear transformation is from R 3 to R 3, in my earlier result I have prove that the if you know how these elements are being mapped, you define the mapping and the mapping can easily be determined.

So, we define the mapping in terms of e mapping of e 1, e 2 and e 3, so this defines for mapping. Now, you have to show that the vectors T e 1 T e 2 and T e 3 are not linearly independent. So, let us consider v 1 as T of e 1 which is e 1 plus e 2 and v 2 is T e 2 which is e 3 plus 2 e 2 we will define here v 3 is T 3 as e 1 minus e 2 minus e 3, these are three vectors in R 3.

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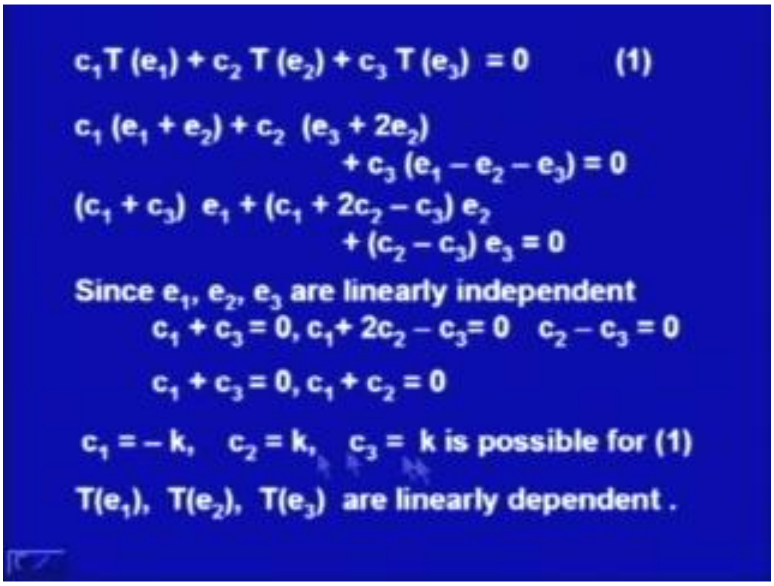

Let us, consider the linear combination of these three vectors that is c 1 T 1 plus c 2 T e 2 plus c 3 T e 3 as 0. If these three vectors are linearly independence c 1, c 2, c 3 will come out to be 0. Now, we use the definition of T e 1, T e 2, T e 3 been given to us. So, we write down c 1 e 1 plus e 2 as T e 1 plus c 2 T e 2 as e 3 plus 2 e 3 plus c 3 times T e 3 as e 1 minus e 2 minus e 3 0, we will combine the different terms we will have c 1 plus c 3 multiplied by e 1 plus c 1 plus 2 c 2 minus c 3 multiplied by e 2 plus c 2 minus c 3 multiplied by e 3 and this is equal to 0.

Since, e 1, e 2, e 3 are linearly independent, therefore c 1 plus c 3 is equal to 0 c 1 plus 2 c 2 minus c 3 equal to 0, this is the coefficient of e 2 and coefficient of e 3 if c 2 minus c 3 equal to 0. And we have three equations to solve c 1, c 2 and c 3 from the first equation c 1 is equal to minus c 3 and from second c 2 is equal to c 3. So, if you substitute c 1 plus c 3 is equal to 0 here, then will have c 1 plus c 2 equal to 0.

And that means, c 1 is equal to minus k c 2 is equal to k and c 3 is equal to k is the possible solution for this and k need not be 0. So, we have obtain a solution for this equation, which is nonzero solution and that means, T e 1, T e 2, T e 3 are linearly dependent, they cannot be linearly independent, because we have got this nonzero solution.

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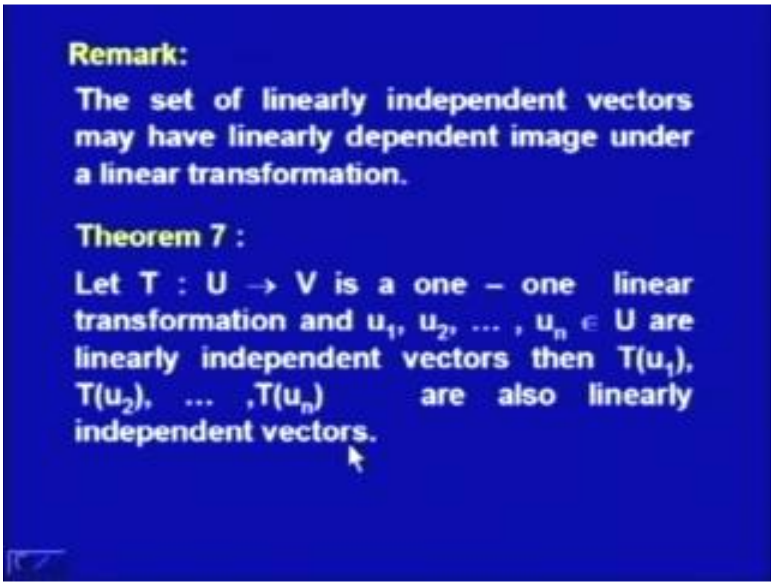

Now, we can put a remark here. The set of linearly independent vectors may have linearly dependent images under a linear transformation. So, this is what has happen in the earlier example, we have started with the linearly independent set of vectors e 1, e 2, e 3, we have a linear transformation T. But, what we have T of e 1, T of e 2, T of e 3 they are not linearly independent, but they are linearly dependent.

And that is the set of linearly independent vectors may have linearly dependent image, under a linear transformation. Now, this is a theorem we says that, if we have a linear transformation U into V, if it is one-one linear transformation. Then, the set of vectors u 1, u 2, u n belonging to U are linearly independent vectors. Then, it is images T u 1, T u 2, T u n are also linearly independent vectors. So, the basically this theorem provides a condition under which linearly independent vectors will map to linearly independent vectors. And the result is the that transformation has to be one-one.

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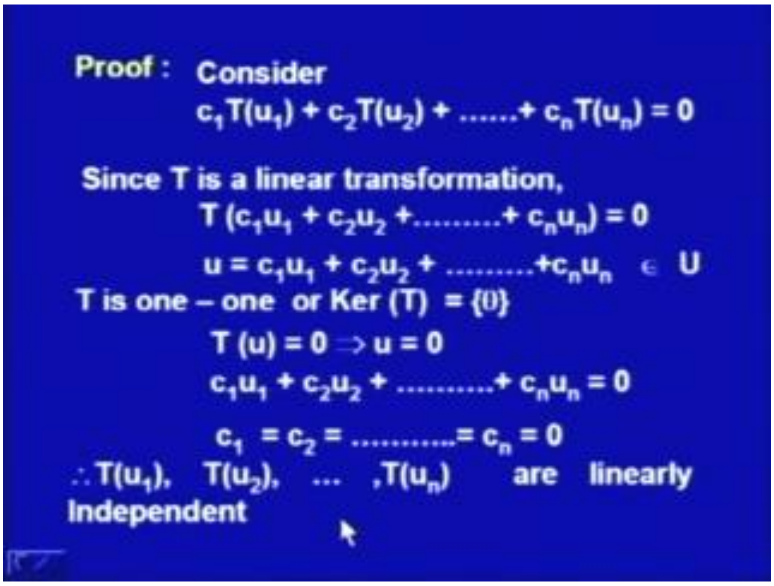

So, here is the proof, we can start with a linear combination of that is T u 1, T u 2, T u n. And since, T is a linear transformation. So, will have T of c 1 u 1 plus c 2 u 2 plus c n u n equal to 0. And u will belong to U, because u is a vector space, the linear combination will also belong to U and T is one-one or kernel T is equal to 0.

So, we are been given that T is one-one transformation and we have earlier prove that kernel T is equal to 0 if T is one-one. That means, T of u is equal to 0, so this vector u has to be 0 because, kernel T is equal to 0. And that means, c 1 u 1 plus c 2 u 2 plus c n u n is equal to 0, but we have u 1, u 2, u n has to be 0; that means, c 1, c 2, c n has to be 0. So that means, we have a linear combination which is 0 and these ((Refer Time: 51:39)) comes out to be 0 and that means T u 1, T u 2, T u n are linearly independent.

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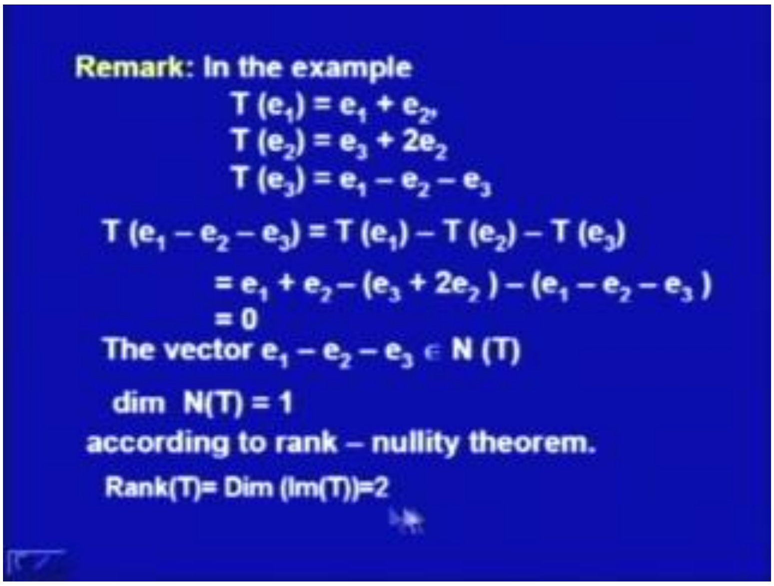

So, in the of this condition will have a linear independent vectors will go to linearly independent set of vectors. Now, what happen in the earlier example, which was T e 1 is equal to e 1 plus e 2, T e 2 was e 3 plus 2 e 2, T e 3 is e 1 minus e 2 minus e 3. In this example, the vector e 1 minus e 2 minus e 3 is actually linear combination of as T e 1 minus T e 2 minus T e 3 and this is actually 0. And that means, the vector e 1 minus e 2 minus e 3 belongs to N of T and dimension of N T is equal to 1 in that case. So, according to rank nullity theorem, the rank of T is dimension of image T which is 2.

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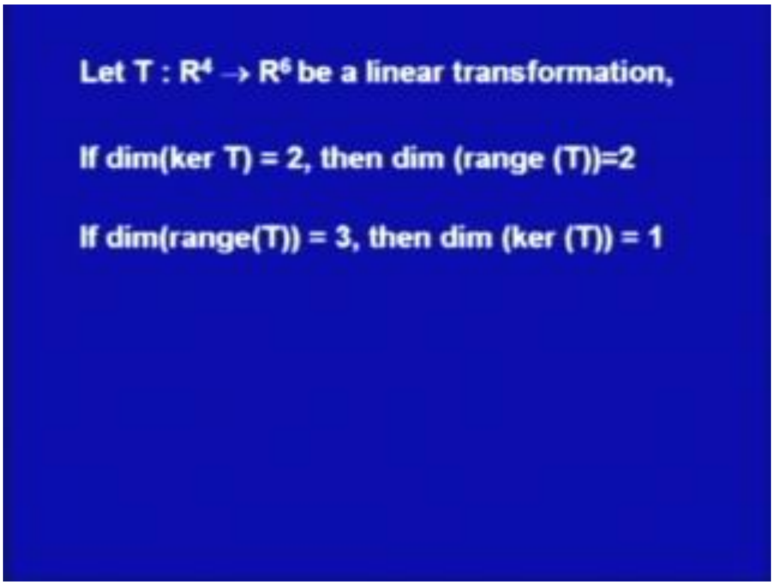

And in this case they are actually not mapping to the three vectors cannot be linearly independent. Now, I have one more simple result, that if I have a linear transformation from R 4 to R 6. The linear transformation and if dimension of kernel T is given to be 2. Then dimension of range of T or image of T can be computed as 4 minus 2, it is 2, you have to make it clear, that the dimension of V is to be considered in rank nullity theorem not the dimension of W, so there is result here is 2.

Similarly, if dimension of range T is 3 then dimension of kernel T is 1, so 3 plus 1 is equal to 4. So, this rank and nullity theorem can be use to obtained the dimension of kernel T if other two things are given can be obtained for dimension of range T, if dimension of V is given and dimension of kernel T being given to us. Viewers, with this we have come to the end of this lecture to summarize, what we have done today.

I have started with the definition of linear transformation I given some example, I have illustrated with the help of examples, what do you mean by linear transformation? What do you mean by additive property? What do you mean by homogeneous property. And then we have discuss several results related with linear transformations I have introduce the concept of kernel and nullity. And we have finally, established theorem relating range, dimension of range and dimension of kernel with the dimension of the vector space V, we have we will continue with this will discuss more concept related to this in my next lecture.

Thank you.