Introduction to interfacial waves
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Lecture - 08
Vibrations of clamped membranes (Continued...)

We were looking at the vibrations of a circular membrane, which was clamped at its edges.

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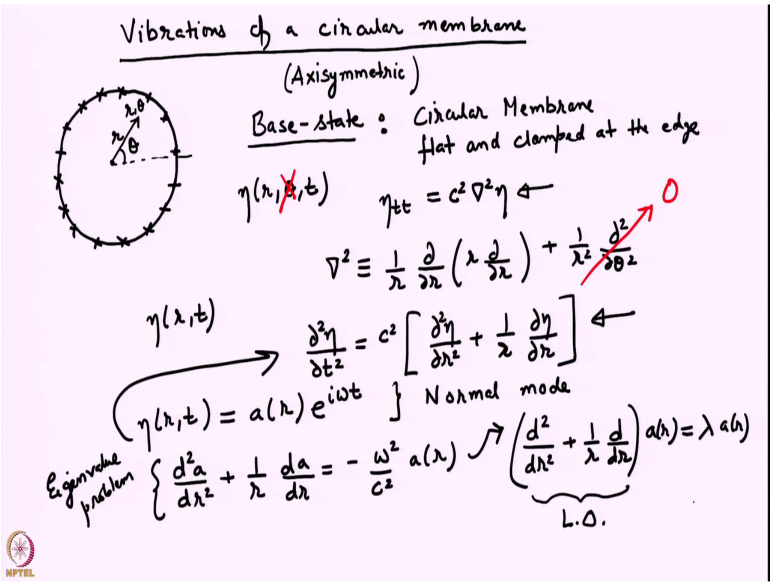

We had seen earlier that the governing equation remains the Laplace equation.

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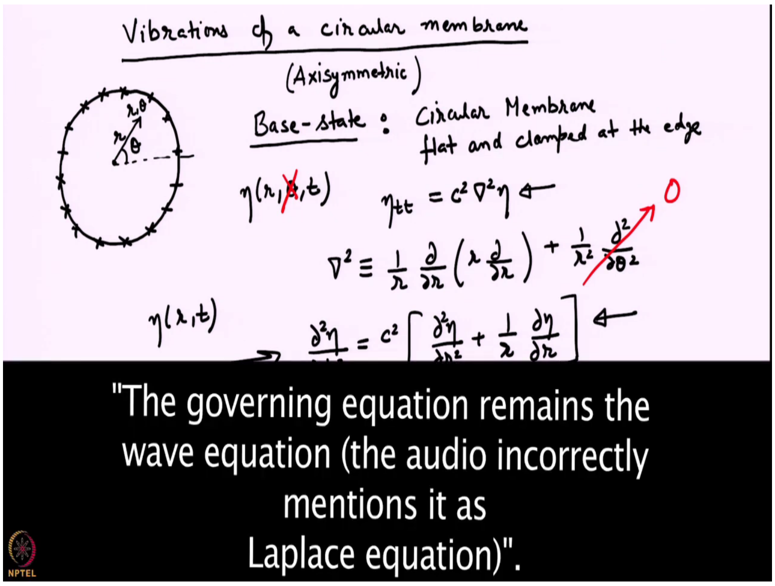

We had converted into it into a one dimensional equation by assuming that the motion of the membrane is axisymmetric. So, all derivatives with respect to theta are 0 and so eta, the vertical displacement of the membrane was just a function of the radial coordinate r and time.

Now, under this approximation, we were led to wave equation in polar coordinates; here theta not being present because of the axisymmetric approximation and we were trying to solve this equation using the method of normal modes. So, once again we substituted eta of r comma t is equal to some Eigen mode into e to the power i omega t.

After substituting, it let us do an Eigen value problem, where the linear operator was of this form. Notice that this in the in this particular problem, the linear operator is slightly more complicated than the previous one. In the previous problem where we looked at vibrations of a rectangular membrane, the linear operator was a constant coefficient operator. Here the linear operator is not a constant coefficient operator, its coefficients are functions of the independent variable r.

So, we will have to worry a little bit more about how to solve this equation. Now, once we substituted this, it let us to an Eigen value problem, where the Eigen value is of the form, where the Eigen value lambda is of the form minus omega square by c square. Now, you can see like before that in this Eigen value problem, only for certain values of lambda will this problem admit solutions and those values of lambda will led lead us to the dispersion relation.

For correspondingly for every such value of lambda, there will be an Eigen mode; but first we have to find out how to solve this equation, let us do that.

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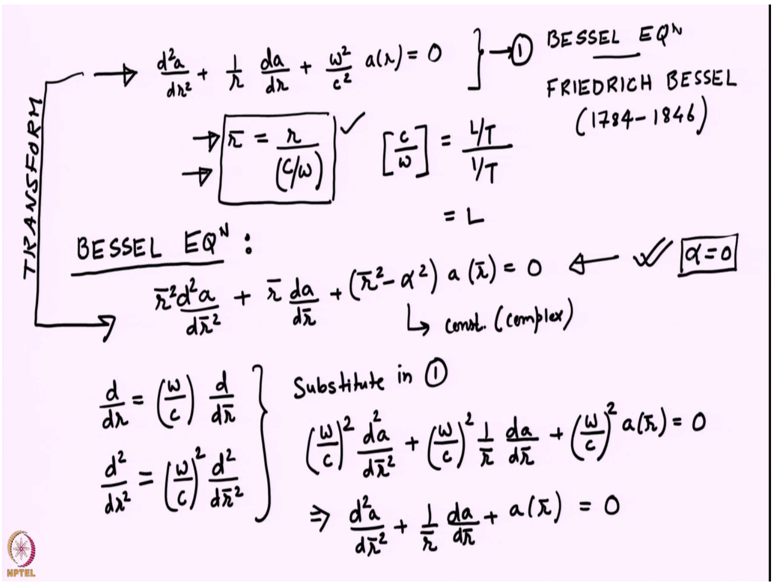

Our equation was of the form d square a; so this is the equation that we need to solve. Now, as I told you before this is a linear, but not a constant coefficient ordinary differential equation. Now, this equation is a well known equation or is a special case of a well known equation, that equation is called the Bessel's equation. This was first found by a mathematician named Friedrich Bessel and is named after him.

Now, let us look at this equation. So, the first step is to the solutions of this equation are known, the arguments of the solutions just like the in the previous case the solutions of the equation were circular functions and the arguments of sin and cos are non dimensional; similarly the arguments of the solutions to this equation will also be non-dimensional.

So, it is useful and advantageous to non-dimensionalize this equation, non-dimensionalize the independent variable r. So, let us non-dimensionalize the equation. So, we will define r r bar which is, so you can immediately see that r bar is defined as r divided by c by omega. So, let us look at the dimensions of c by omega. So, the dimensions of c, c is a speed and omega is a frequency; so c by omega has the dimensions of length.

So, I define a non dimensional r which is r bar, which is the dimensional r divided by something with the dimensions of length. Now, if I if I plug this in into this equation, I will get the Bessel's equation; before I do that, I would like to write down the standard form of the Bessel's equation, so I am going to do that here.

This is the standard form and we will reduce the equation that we have written here to a special case of that standard form, let us write the standard form first.

So, I am writing the standard form of the equation and this r bar is the same as the r bar that we have written here, that is a non-dimensional r. So, there is a square here. So, this is the standard form of the Bessel equation, where alpha is a constant, in general it can be a complex constant; for most applications alpha is turns out to be an integer or a half integer, in this particular case we will see that alpha will be 0.

So, now let us let us transform our equation. So, we will transform our equation into the standard Bessel's equation. For that let us use this transformation that we have defined here already and let us express all the derivatives with respect to r in terms of a derivative with respect to r bar.

So, we can see that d by d r is omega by c d by d r bar, that follows just by taking the derivative on both sides of this equation; because this is a second order equation, I also want the expression for d square by d r square in terms of d r bar square and this is just the square of, the coefficient is just the square of the first derivative, you can check this easily from this relation.

Now, we substitute this. So, if I call this equation 1. So, substitute in 1 and if you substitute, you can readily see that all the terms will have omega by c square. So, I am replacing all the derivatives and the coefficients which are dependent on r in terms of r bar. And so, I can eliminate the omega by c whole square, because that is in general not 0. And if I take this equation and multiply both sides by r square, it immediately transforms to a special case of this.

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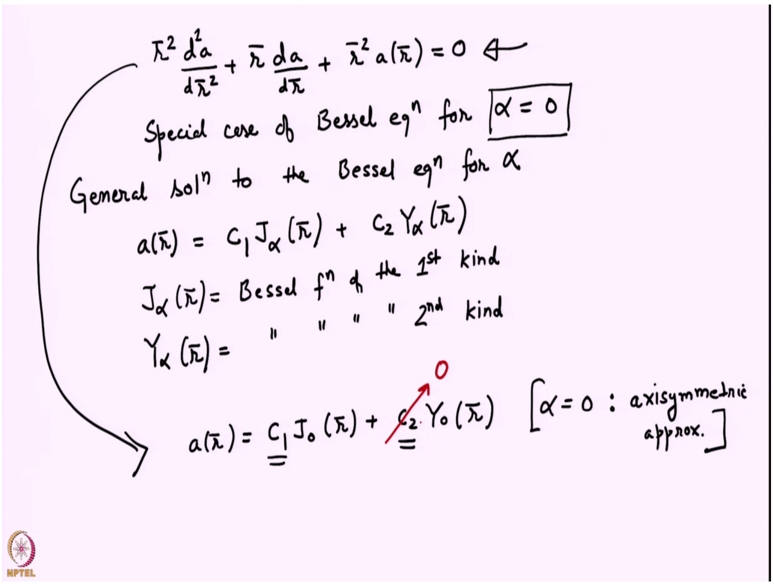

So, I am just multiplying both sides by r bar square. So, now, if you compare this equation with the equation that we had written earlier, the standard form of the Bessel equation; you will see that what we have is a special case of this equation for alpha is equal to 0, this 0 is an outcome of the axisymmetric approximation that we have made.

So, this is a special case for alpha equal to 0. Now, the general solution to the Bessel equation that we had written earlier, for arbitrary alpha is given by, it is a linear second order equation, so there must be two constants of integration and two linearly independent solutions. So, the general solution is given by these J of alpha r bar and Y of alpha r bar are known as the Bessel functions.

So, this J of alpha r bar is basically known as the Bessel function of the 1st kind and Y of alpha is known as the Bessel function of the 2nd kind. These are well known functions, they are tabulated; you can plot them in a standard software package like MATLAB or Mathematica and you can find how their behavior is, they are in general also oscillatory functions.

Now, because our equation is for the special case of this general thing when alpha equal to 0; so it is clear that the solution to this equation is given by C 1, we want alpha to be 0. So, I write J 0 of r bar plus C 2 Y 0 of r bar; alpha equal to 0 is an outcome of the axisymmetric approximation.

You will find and I encourage you to do this that, if you take the two dimensional wave equation for a circular membrane in the theta direction; in that case you will have derivatives with respect to theta also in the Laplacian and you will find that the solutions in the theta direction is a linear combination of cos m theta and sin m theta, where m is a positive integer.

Now, if you put m equal to zero in that expression, you will recover the axisymmetric expression. So, in the general case you would have J m of r bar and Y m of r bar; when m is 0, you go back to the axisymmetric approximation and thus you recover this result, which is a linear combination of J 0 and Y 0. Let us plot, I am just going to plot by hand how these functions looks, so that you have a physical feel for these functions. So, let me just plot these functions.

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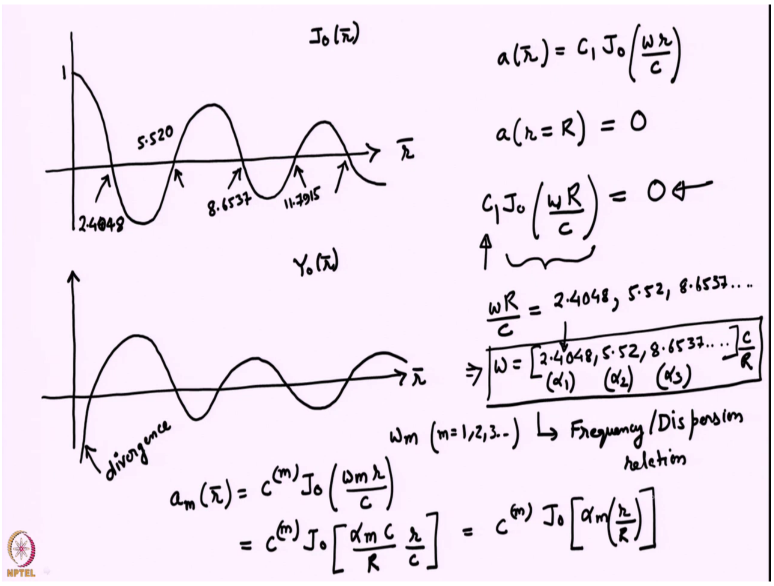

So, this functions are oscillatory, they are not exactly periodic and their amplitude decays for as r bar becomes larger and larger. So, they start, J 0 starts from 1, does not rise up to 1; it is a slow decay as r bar becomes larger and larger. So, this is 1, this is the plot of J 0 as a function of r bar; this axis is r bar and the places where J 0 of r bar becomes 0 is also tabulated. So, these places where it intersects the x axis, this will be of importance for determining the frequency relation as we will see shortly.

So, these points where J 0 of r bar becomes 0 are known and they are tabulated. So, you can read them off a table. So, the first point is about approximately 2.4048; this point is 5.520, this one is about 8.6537, this point is about 11.7915. Similarly, you can look up in a table the further 0's; they are in general an infinite number of them, they are all countable, they form a countably infinite set, this is J 0 of r bar.

How does Y 0 of r bar look? Y 0 of r bar is also an oscillatory function; however it has a divergence at r bar equal to 0, so it looks like this. It is a divergence, it is a logarithmic divergence; so it diverges logarithmically at r bar equal to 0 and it goes to minus infinity. And because r bar equal to 0, so this, so this is Y 0 of r bar, this is r bar.

Now, you can see that if we have a function which diverges at r bar equal to 0 in our solution, that is going to cause my Eigen mode to become unbounded. Because my Eigen mode represents the amplitude of oscillation, this would imply that the amplitude of oscillation at r bar equal to 0, the center of the membrane would also become unbounded, that is physically not meaningful.

And so, in order to keep things finite everywhere, we would like to set the coefficient of any function to 0, which diverges anywhere within our domain. This implies that when we determine these constants, it is it implies that we have to set C 2 to 0 and that is because Y 0 of r bar diverges at r equal to 0. Note that J 0 of r bar does not diverge within the domain and so, we are allowed to retain it.

That eliminates two things that takes care of two things; it eliminates a problematic term which would otherwise diverge, it also determines one constant of integration or in other words C 2 is equal to 0. So, this term is set to equal to 0 and my Eigen mode is just proportional to J 0 of r bar. So, now, let us continue from there and let us impose now the boundary conditions, which basically says that the membrane is clamped at the edge.

So, we have found a of r bar is C 1 J 0 of r bar and if I open up r bar and write it in terms of r, then it is J 0 of omega r by c. Now, we also have to satisfy the zero displacement condition at the edge of the membrane at all times. So, this implies that a of a when r is equal to R is 0; this will in a, this will ensure that eta is 0 at all times at the edge of the membrane. If you substitute this here, this implies C 1 J 0 of omega capital R by C is equal to 0.

You can immediately see that this is C 1 is not 0 in general; so the only way to satisfy this is to find out at which points J 0 is 0. As I said before these are the points at which J 0 is 0. And so, this form a countable infinite set of points at which J 0 is 0, they are tabulated.

And so, we can write that when omega R by C is equal to 2.4048 comma 5.52, 8.6537 and so on; then this boundary condition is satisfied. This you can immediately see that for a membrane C is fixed, because C is determined by the tension force per unit length to the density, the aerial density of the membrane; the radius of the membrane is fixed, so we have no control over C and R.

And so, this essentially ensures that omega is a certain multiple of C by R. So, you can see immediately that omega is a set of numbers dot dot dot into C by R. Once again we get a discrete set of frequencies at which this membrane is allowed to oscillate, there are a countably infinite number of them like before.

So, this gives us our frequency or dispersion relation. So, in general i will write the omega as omega m, where omega where m will go from 1, 2, 3 up to infinity; omega 1 would be 2.4048 times C by R, omega 2 would be 5.52 times C by R and so on and so forth.

Now, for each such omega m, there is also an Eigen mode; I will call the Eigen mode as a m of r bar and this in general is C 1 of m. I can drop the 1 now, because there is no 2; C 2 was set to 0, so this is C of m J 0 and this is omega m and omega m r by c. And what is omega m? So, if I call these values, so if I call these values as. So, this is alpha 1, alpha 2, alpha 3 and so on; so omega m is basically alpha m C by R, ok.

So, then this is the constant of integration into alpha m C by capital R into small r by C; the C and the C get cancelled and we obtain alpha m small r by capital R. You can see that the argument of J 0 is still non dimensional as it should be; alpha m is just a number, it is the number in this set. So, alpha 1 is 2.4048, alpha 2 is 2.5052, 8.6537 is alpha 3 and so on.

So, these are our Eigen modes A m. Now, we know that the most general solution like before is a linear combination of the Eigen modes multiplied by e to the power i omega m t and summed over all possible values of m. Let us write that down.

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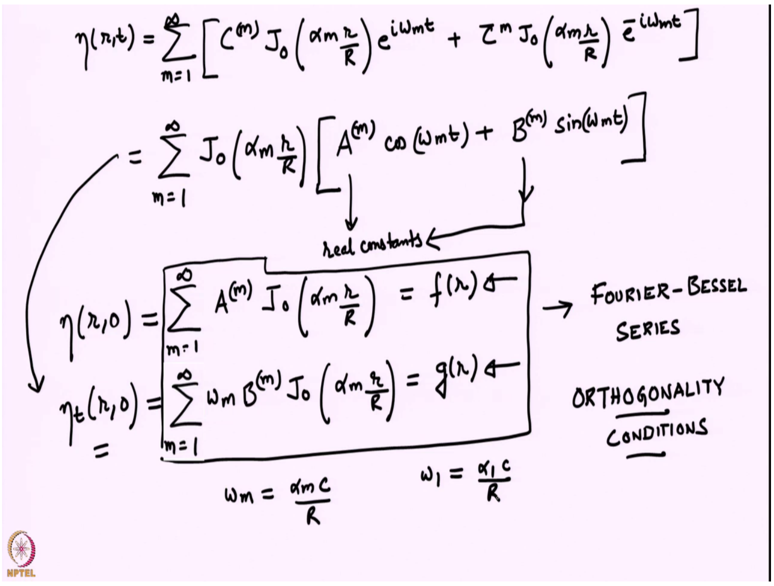

So, the general solution is eta of r comma t is m is equal to 1 to infinity, some constant of integration C m into J 0 alpha m r by R e to the power i omega m t plus C m bar the complex conjugate of C m, the Eigen mode remains the same and then I can collect this because the Eigen function is real.

So, I can pull it outside and then switch to real notation, where the coefficients are also real. So, you can see that I am going to do the same thing C m e to the power i omega m t plus C m bar e to the power minus i omega m t.

If I combine them, I will get C m plus C m bar into cos omega m t and then there will be another term which will be i times C m minus C m bar sin omega m t. C m plus C m bar is real i times C m minus C m bar is again real; I can write it in terms of some other constant, I have chosen those to be A m and B m.

And so, using completely real notation, I write A m cos omega m t plus B m sin omega m t, ok. And these are now real constants; they are not complex constants, because we have shifted completely to real notation. Once again how do we determine these A m's and B m's; there are countable infinite number of them, how do we determine them?

So, we substitute initial conditions. If I substitute t equal to 0 in this, this just becomes; let me write the A m outside and eta of r comma 0 just tells me what is the displacement of the membrane initially at time t equal to 0 at every r. So, this in general would be some function of r, which would be given to us.

So, this forms one infinite series; similarly just specifying the displacement is not enough, we also need to specify the velocity of the membrane at every point at every r at time t equal to 0. If I do that, then I have to take the derivative of that expression.

So, let us use this expression and if I take the derivative, you will see that the sin term becomes cos and the cos term becomes sin and then if we substitute equal to 0; then the sin term which became cos will be the one which will survive and there will be an omega m outside.

So, you will have omega m B m J 0 of alpha m r by R and this function would be some other function g of r; you could set it equal to 0 if you want, but you do not have to, you can even specify an initial impulse to the membrane in the form of a velocity everywhere. So, given two functions f of r and g of r, which represents the displacement and the velocity of the membrane at time t equal to 0; these infinite series say that these functions can be represented as a series of Bessel functions.

This is a generalization of the series that we saw earlier; in those cases we used a series in terms of the trigonometric functions, circular functions, here we are writing down it in terms of Bessel functions, these series are known as Fourier Bessel series. It is a generalization of the idea of a Fourier series.

Once again how do we determine? Given f of r and g of r, note that the only thing that we do not know here are the A m's and the B m's, everything else is known, alpha m comes from a set of numbers which are known.

So, we know what is the alpha 1, alpha 2, alpha 3 and alpha 4; f of r and g of r will be specified as a part of the initial conditions. So, if you want to determine A m and B m from these infinite series, we have to once again use the orthogonality conditions, between the Bessel functions; these are well known available in handbooks, one can use them.

We basically say that Bessel functions are you have to take the inner product and some inner products will go to 0 and that will determine the coefficient as an the coefficient of each of these terms a 1, a 2, a 3 as an integral over f r with multiplied by some kind of Bessel function. Once those integrals can be computed either analytically or numerically and a all the coefficients can be determined.

Again we have the same feature that, in general if you want our membrane to vibrate in a pure normal mode; we will have to make sure that the for example, we could initialize and initialize the membrane by giving it a displacement which is J 0 of alpha 1 into small r by capital R. If we just do this without giving it an initial velocity, then it will vibrate purely in mode 1 and the frequency is something that we have determined.

So, we have seen earlier that the frequency is given by omega m is equal to alpha m C by R. So, in our case the frequency would be alpha 1 C by R. Note that R and C have to be given to you; C is known to you if you know the equation, that is determined by the physical properties of the membrane, the tension that it is under the force per unit length and the density, the aerial density of the membrane.

Alpha 1 comes from that list of numbers that I have written down, so alpha 1 would be 2.4048, the first intersection of J 0 with the horizontal axis. So, this way you can determine what is the frequency with which you can predict, what is the frequency with which the membrane would vibrate if you give it a small displacement.

We have to remember that all this is for sufficiently small displacements. If you give large amplitude displacements to the membrane, it will show features which are not present in this linearized analysis. We will see some of those things later on in this course.

Again if you give it a arbitrary initial condition, if you give it some f of r which is arbitrary; that f of r will have projections along each of the Eigen modes, the various J 0's for alpha 1, alpha 2, alpha 3. And consequently the coefficients a 1, a 2, a 3 and so on will not be 0; the resultant motion may look quite complicated, because the membrane is moving simultaneously in a superposition of many normal modes. The resultant motion will look oscillatory, but not necessarily periodic just like before.

So, we have now looked at various kinds of things. And in the next class, until now we have looked at vibrations of linear systems governed by ODEs and PDEs; in the next class, we are going to move over to a non-linear problem.