Fluid Mechanics and its Applications
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Indian Institute of Technology, Delhi
Lecture – 16
Similitude and Scale Factor Approach

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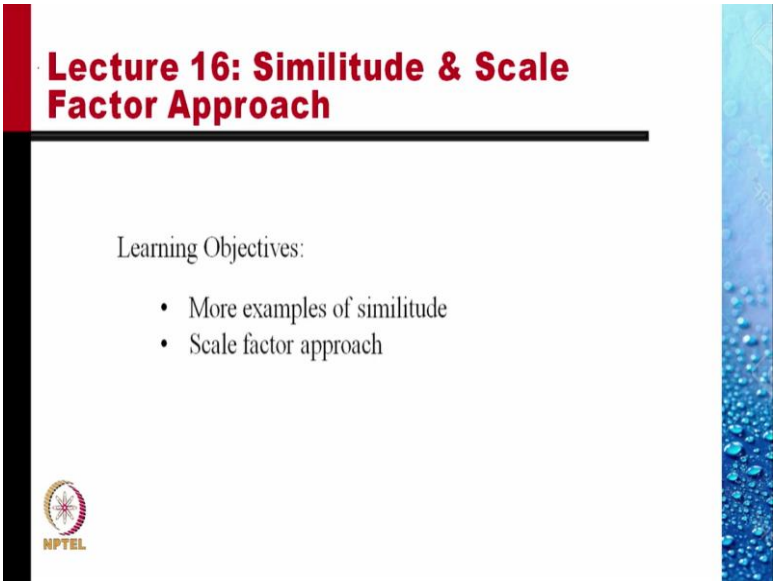

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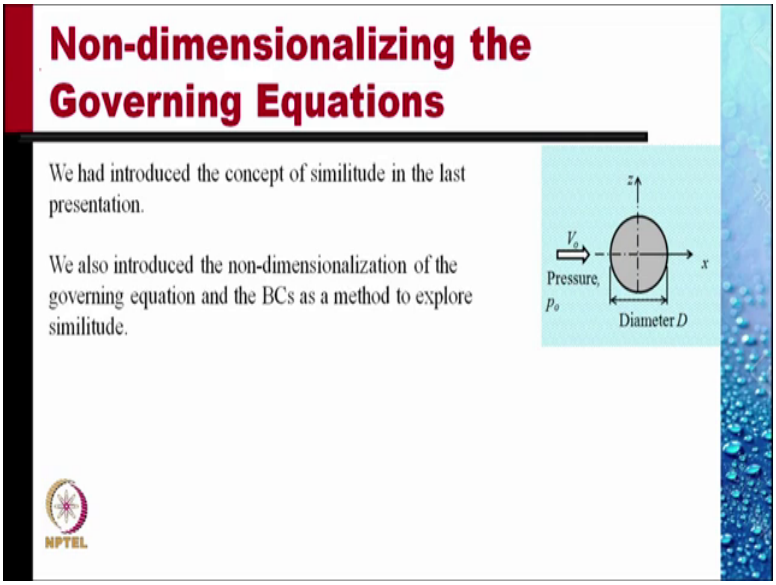

Welcome, back.

In the last lesson we had introduced the concept of similitude. We also introduced the non-dimensionalization of the governing equations and the boundary conditions as a method to explore similitude.

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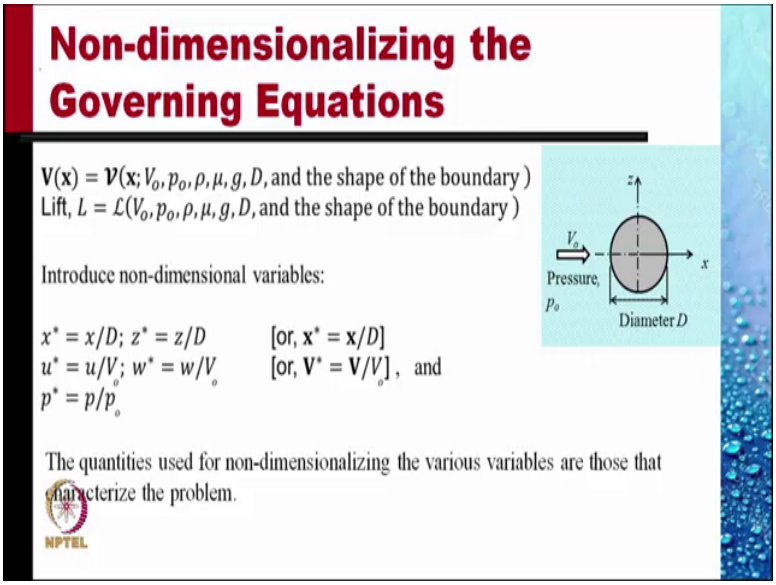

We started doing a problem in which we considered the flow past a sphere. Any dependent variable like velocity V_x is a function of the independent variable x, and the independent parameters like velocity V_o far away, p_o, the pressure far away, the material properties, density, viscosity, the acceleration due to gravity, and the geometry of the body which includes its size represented by the diameter here.

Similarly, any dependent parameter like lift, would depend only on the independent parameters, and the shape of the boundary. We non-dimensionalized by introducing the non-dimensional variables $x^* = x/D$, $z^* = z/D$, the velocity is u^* and w^* by dividing them by the free-stream velocity V_o, and the non-dimensional pressure p^*as p divided by the pressure far upstream. The quantities used for non-dimensionalizing the various variables are those that are characteristic of the problem.

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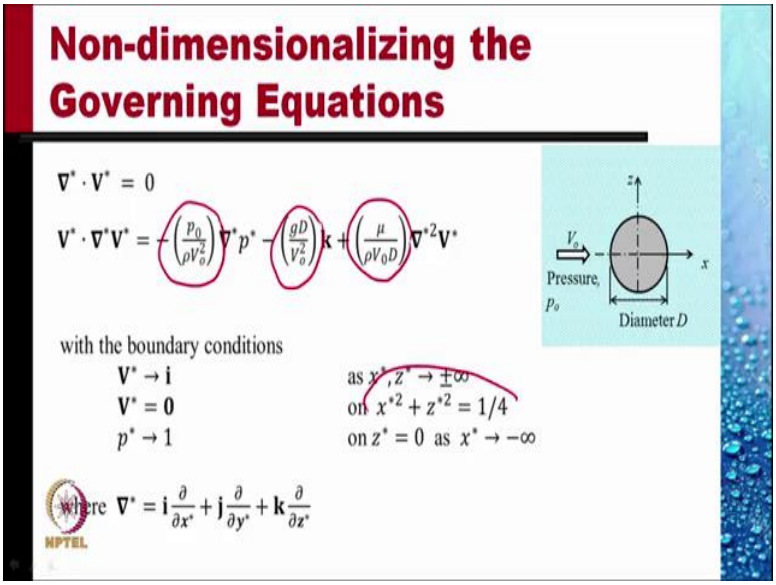

The non-dimensional equations that we got after some manipulation are these. In these equations, the independent parameters are now collected together, and these are the only three locations the independent parameters occur. The boundary conditions become this. There are no independent parameters in the boundary condition except for the shape of the body.

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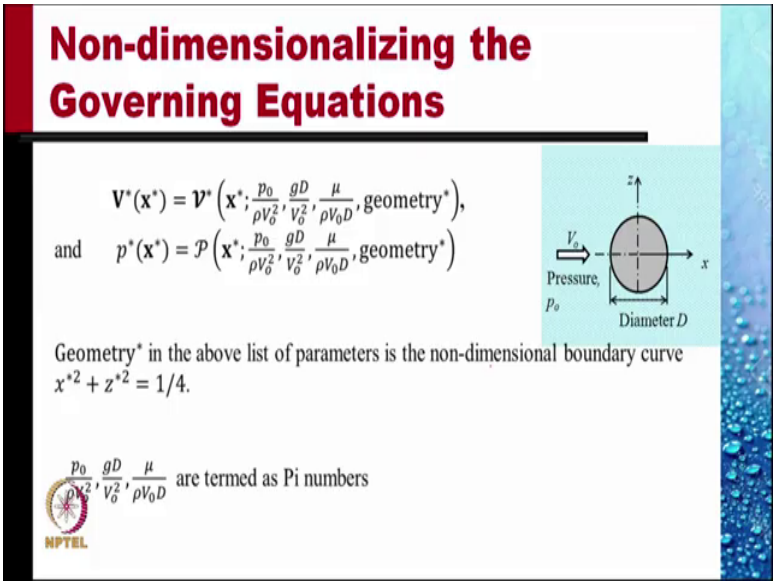

Now, this represents a great saving from initially a dependent variable being function of independent variable x^* and the independent parameters, six of them. Now we had only three groups of independent parameters. These three independent groups of parameters $\frac{p_0}{\rho V_o^2}$, $\frac{gD}{V_o^2}$ and $\frac{\mu}{\rho V_0 D}$ are termed as Pi numbers. Note that these are all dimensionless.

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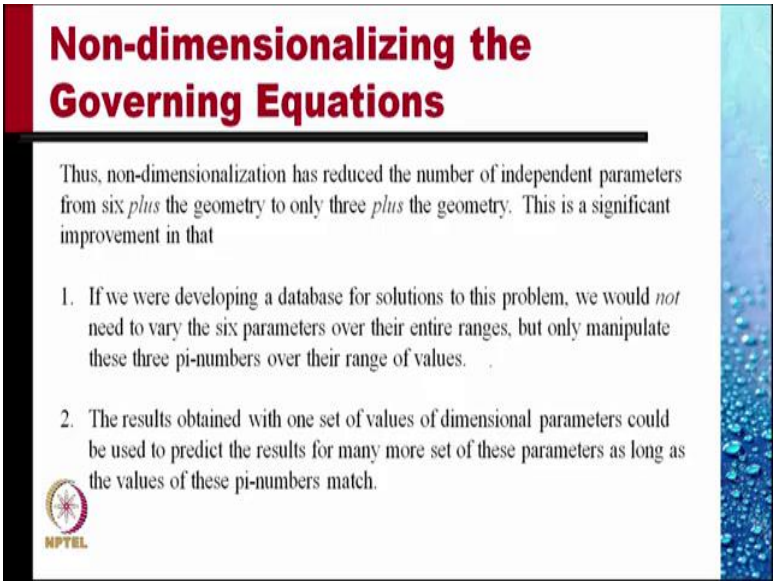

Thus, non-dimensionalization has reduced the number of independent parameters from six plus the geometry, to only three plus the geometry. This is a significant improvement in that, if we were developing a database for solutions to this problem, we would not need to vary the six parameters over their entire ranges, but only manipulate these three Pi-numbers over their ranges of values.

The result obtained with one set of values of dimensional parameters could be used to predict the results for many more sets of these parameters as long as the values of these pi-numbers are equal.

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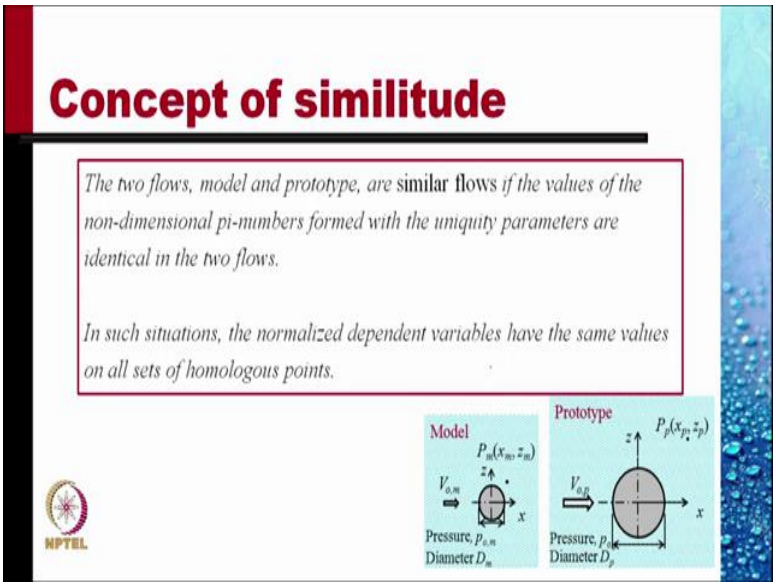

We introduce the concept of similarity in the following words: The two flows, the model and the prototype flows, are similar, if the values of the non-dimensional pi-numbers formed with the uniquity parameters are identical in two flows.

In such situations, the normalized dependent variables have the same values on all sets of homologous points. Homologous points are those points which have the same values or non-dimensional locational variables in the model and the prototype.

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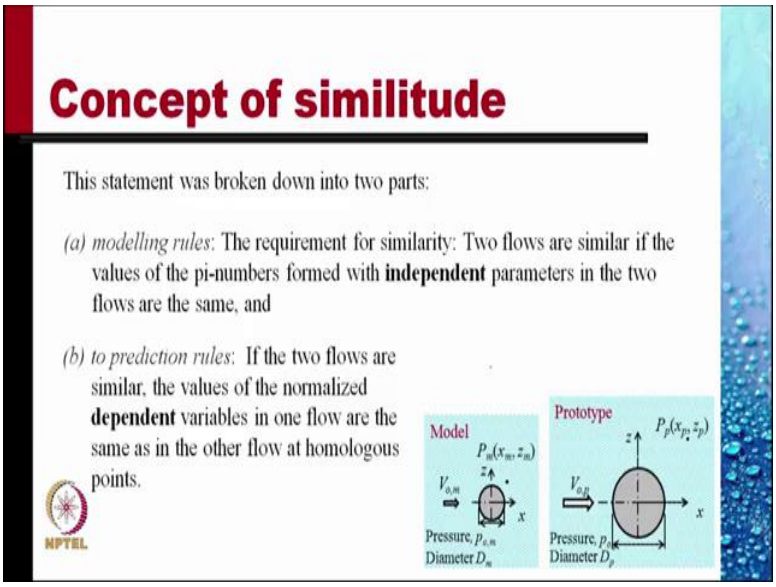

This statement was broken down into two parts: the modelling rules, the requirement of similarity. Two flows are similar if the values of the pi-number formed with independent parameters in the two flows are the same. And the prediction rules: if the two flows are similar, the values of the normalized dependent variables and parameters in one flow are the same as in the other flow at homologous points.

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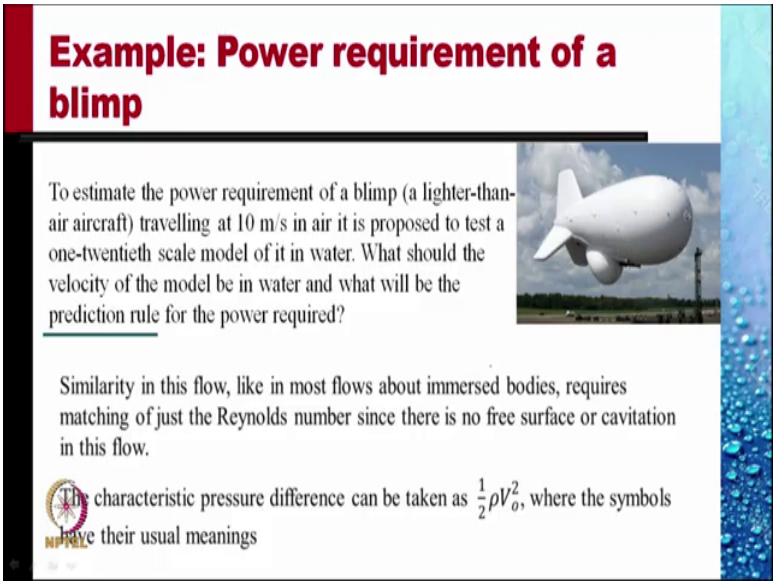

Let us do one example to illustrate this method. We are required to estimate the power requirement of a blimp which is a lighter than air aircraft traveling at 10 meters per second in air. It is proposed to test a 120^{th} scale model of it in water. What should be the velocity of the model in water? And what will be the prediction rule for the power required, if we measure the power from the blimp, model blimp?

Similarity in this flow, like in most flows about immersed bodies, requires matching of just the Reynolds number, since there is no free surface or cavitation in the flow. This was explained in the previous lecture. The characteristic pressure difference in this case can be taken as $\frac{1}{2}\rho V_o^2$, where the symbols have their usual meanings. This is possible because there is only one pressure. So, the pressure difference is not defined a priori. So we begin with $\frac{1}{2}\rho V_o^2$.

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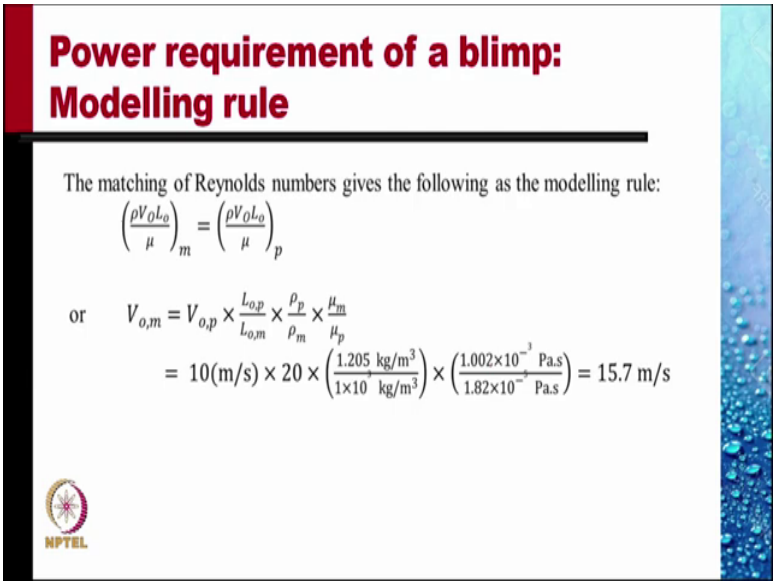

The matching of Reynolds number gives the following as the modelling rule. The value of the Reynolds number in model and in the prototype should be identical. And if we manipulate this, the characteristic value of the velocity in the model should be the characteristic velocity in prototype multiplied by $L_{o,p}$ divided by $L_{o,m}$, that is, the characteristic length in the prototype divided by characteristic length in the model into density of prototype divided by density of model into viscosity of model divided by viscosity of prototype.

This is just recasting the Reynolds number equality in a different form. Here everything on the right-hand side is known, $V_{o,p}$ is the velocity at which the prototype moves, and so the velocity at which the model moves is evaluated at 15.7 m/s when you plug in all the values. So, we need to move the model blimp at a speed of 15.7 m/s to have a flow similar in the prototype air which is moving at 10 m/s.

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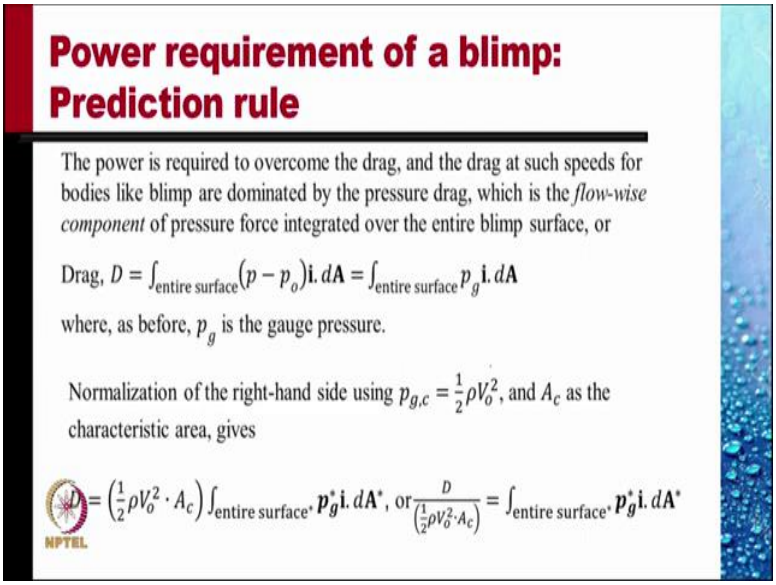

The power is required to overcome the drag, and the drag at such speeds for bodies like blimp are dominated by pressure drag, which is the flow wise component pressure force integrated over the entire blimp surface or that the drag Drag, $D = \int_{\text{entire surface}} (p - p_o)\boldsymbol{i}.\, d\boldsymbol{A} = \int_{\text{entire surface}} p_g \boldsymbol{i}.\, d\boldsymbol{A}$.

Normalization of the right-hand side using the characteristic pressure difference as $\frac{1}{2}\rho V_o^2$ and A_c as a characteristic area gives drag $D = \left(\frac{1}{2}\rho V_o^2 \cdot A_c\right) \int_{\text{entire surface}^*} \boldsymbol{p}_g^* \boldsymbol{i}.\, d\boldsymbol{A}^*$.

Now, this integral is non-dimensional. So, if the flows are similar, the model flow and the prototype are similar, then the value of this integral would be identical in the two cases. So, $\frac{D}{\left(\frac{1}{2}\rho V_o^2 \cdot A_c\right)}$ should be the same in model as well as in the prototype.

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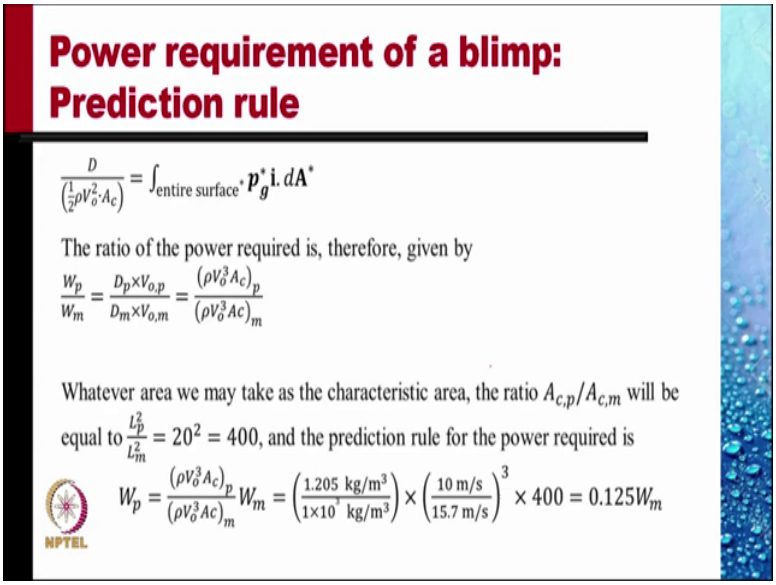

The right-hand side is same, so the left-hand side must be same. So, drag would be proportional to $\rho V_o^2 \cdot A_c$, and the power required, the ratio of the power required would be the ratio of the drag times the ratios of the velocity, since the power is drag times the velocity.

So, the ratio of the powers $\frac{W_p}{W_m} = \frac{D_p \times V_{o,p}}{D_m \times V_{o,m}} = \frac{(\rho V_o^3 A_c)_p}{(\rho V_o^3 Ac)_m}$. Whatever area we may take as a characteristic area the ratio $A_{c,p}/A_{c,m}$ will be equal to $\frac{L_p^2}{L_m^2} = 20^2$, the length scale factor is 20.

So, this area ratio is 400, and the prediction rule of the power required becomes $W_p = \frac{(\rho V_o^3 A_c)_p}{(\rho V_o^3 Ac)_m} W_m = \left(\frac{1.205\ \text{kg/m}^3}{1\times10^3\ \text{kg/m}^3}\right) \times \left(\frac{10\ \text{m/s}}{15.7\ \text{m/s}}\right)^3 \times 400 = 0.125 W_m$. Thus, the prototype would require only one-eighth of the power that the model uses. The model is smaller, then why is it using more power? Because it is being towed in water rather than air. Water is more dense, so the pressure forces are larger in water.

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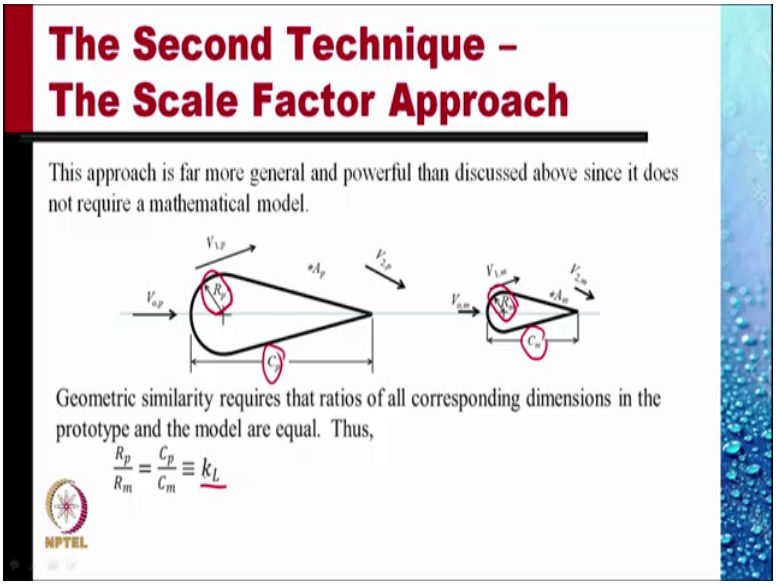

Now, we introduce another technique. This technique is known as the scale factor approach, and this is far more general and powerful, and it does not require a mathematical model to begin. Consider a model and a prototype which are geometrically similar. If the bodies are geometrically similar, this means that the ratios are all corresponding dimensions in the prototype and the model are equal.

Thus, the ratio of the radius of the nose in the prototype to the radius of the nose in the model is the same as the ratio of the chord of the prototype to the chord of the model. And let that ratio be represented by k_L. This is read as scale factor for length k_L. This is the ratio of the length of the prototype divided by the length of the model.

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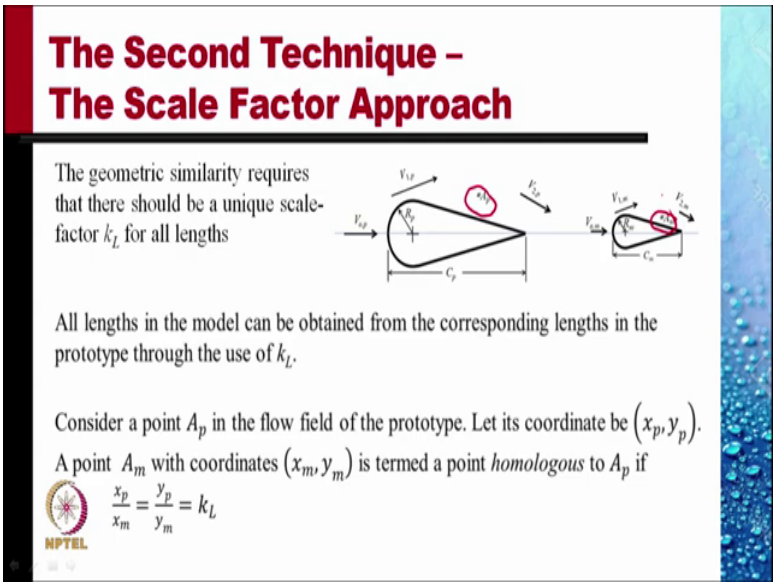

So, in other words, the geometric similarity requires that there should be a unique scale factor k_L for all lengths, whatever be the lengths. All lengths in the model can then be obtained from the corresponding lengths in the prototype through the use of k_L.

Next, we introduce the concept of homologous points. Consider a point A_p in the prototype. A point A_m in the model is termed as a homologous point, if the ratio of the x coordinates and the y coordinates of these two points are exactly equal to k_L, x_p by x_m is equal to y_p by y_m is equal to k_L. In other words, if we enlarge every dimension of the model, and the flow picture of the model by factor k_L, then it will superpose the prototype exactly and the point A_m would then lie exactly over the point A_p on the prototype. Every length, every coordinate is scaled by the same factor k_L.

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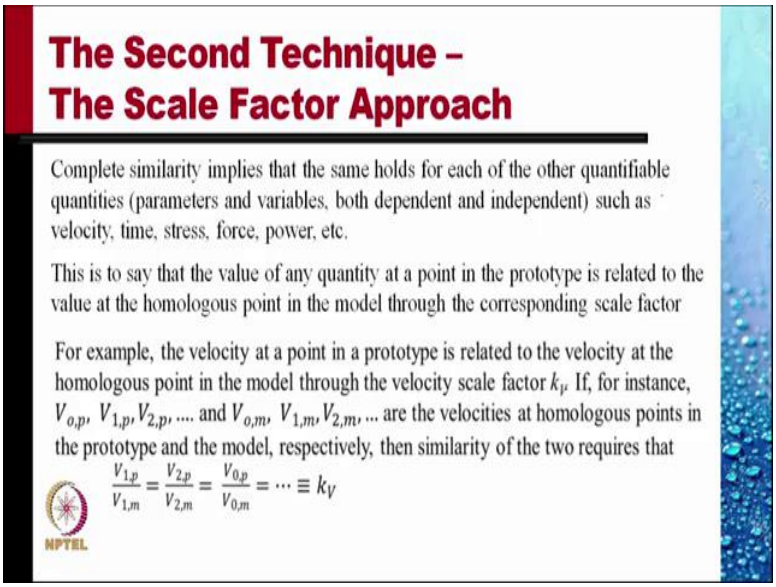

Complete similarity implies that the same holds for each of the other quantifiable quantities, parameters and variables, both dependent and independent, such as velocity, time, stress, power, etcetera. This is to say that the value of any quantity at a point in the prototype is related to the value at the homologous point in the model through the corresponding scale factor.

There is a unique scale factor for velocity, a unique scale factor for time and time intervals, a unique scale factor for all stresses, shear stresses, normal stresses, a unique scale factor of all forces, a unique scale factor for power. For example, the velocity at a point in the prototype is related to the velocity at the homologous point in the model through the velocity scale factor k_V.

If for instance, $V_{o,p}$, $V_{1,p}$, $V_{2,p}$ and $V_{o,m}$, $V_{1,m}$, $V_{2,m}$ are the velocities at homologous points in the prototype and the model respectively, then similarity of the two requires that $V_{1,p}$ over $V_{1,m}$ is $V_{2,p}$ over $V_{2,m}$ so on, all equal to k_V the velocity scale factor.

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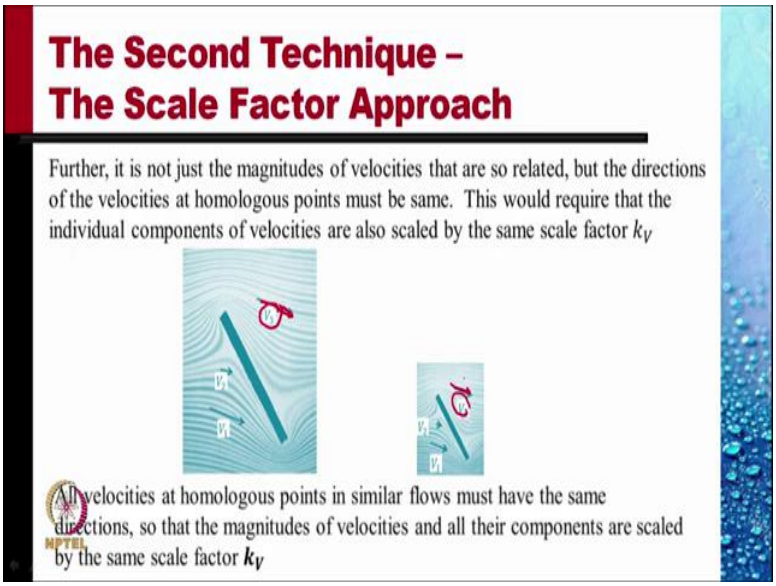

Further, it is not just the magnitude of the velocities that are so related, but the direction of the velocities at the homologous point must be the same. This would require that the individual component of velocities are also scaled by the same scale factor k V. In this picture not only is the velocity V 3 in the prototype same as the velocity V 3 in the model at the homologous point, but also the direction of the velocity is exactly the same.

This means, the components of velocities in the two pictures are related by the same scale factor k_V. This is true for all homologous points. So, all velocities at homologous points in similar flows must have the same direction, so that the magnitude of velocities, and all their components, are scaled by the same scale factor k_V.

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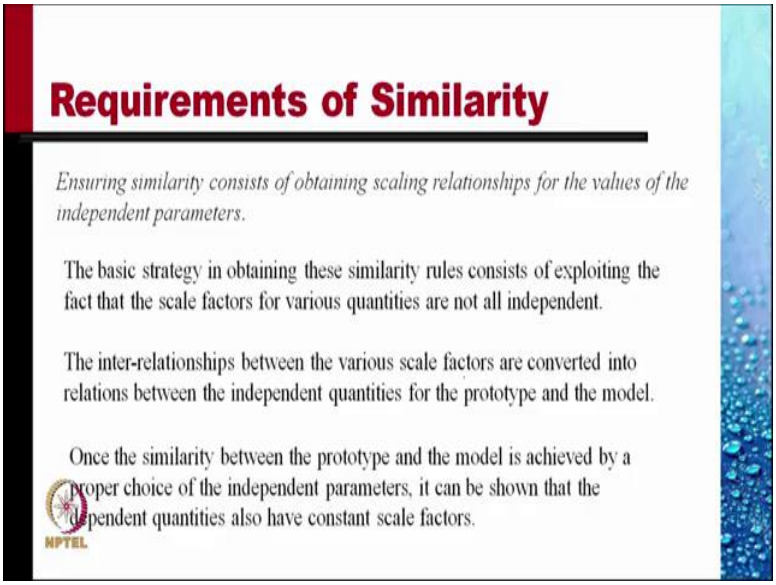

Now, what do we require for similarity? Ensuring similarity consists of obtaining scaling relationship for the values of independent parameters. Given a prototype experiment, we can construct a model experiment which will have a similar flow, but only if we select the values of the independent parameters, which were named uniquity parameters, properly such that they ensure that all quantifiable quantities have the same scale factors at all homologous points.

The basic strategy in obtaining these similarity rules consists of exploiting the fact the scale factors for the various quantities are not all independent. The inter-relationships between the various scale factors are converted into relations between the independent quantities for the prototype and the model.

Once the similarity between the prototype and the model is achieved by the proper choice of the independent parameters, it can be shown that the dependent quantities also have constant scale factors, and those scale factors can be predicted from the scale factors that have been used for the independent parameters.

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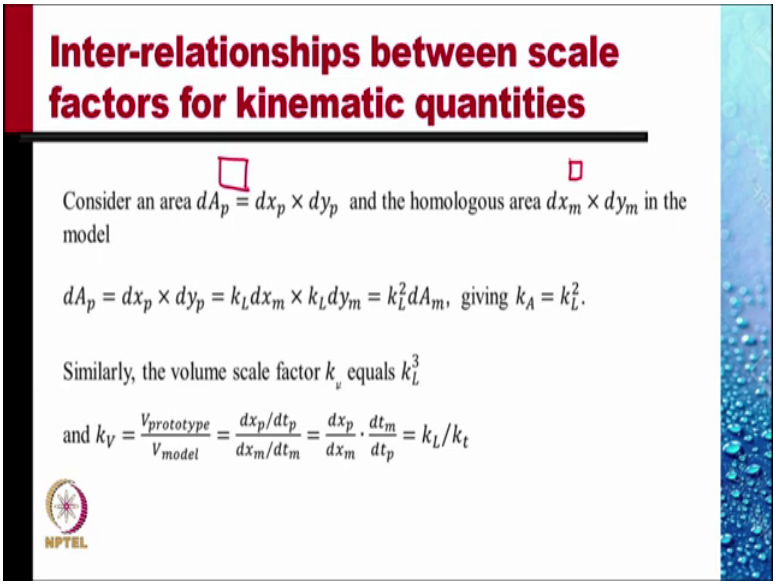

To begin this consideration, let us consider a small area in a prototype with one-dimension dx_p another dimension dy_p. So, $dA_p = dx_p \times dy_p$. The homologous area in the model has the length dx_m in the x direction and dy_m in the y direction. This is the homologous area in the model.

Clearly $dA_p = dx_p \times dy_p$, can be written as k_L times dx_m, since k_L is the scale factor for all lengths, into k_L dy_m the same k_L, which gives you $k_L^2 dx_m dy_m$, which is dA_m, the model area. Thus, k_A which is dA_p over dA_m is nothing but k_L^2.

Similarly, the volume scale factor k_V equals k_L^3. Area has two dimensions of length, so the area scale factor has two k_L's, k_L^2. Volume is three-dimensional length so k_V equals k_L^3. And the velocity scale factor is velocity in the prototype divided by velocity in the model, and the velocity in the model would be dx_m/dt_m.

And this simplifies to $\frac{dx_p}{dx_m} \cdot \frac{dt_m}{dt_p}$, the first one is k_L, and the second ratio is the inverse of k_t, because k_t is dt_p/dt_m . So, the velocity scale factor is length scale factor k_L/k_t. Again, the dimensions of velocities are L into T^{-1}, so scale factor of velocity is k_L/k_t.

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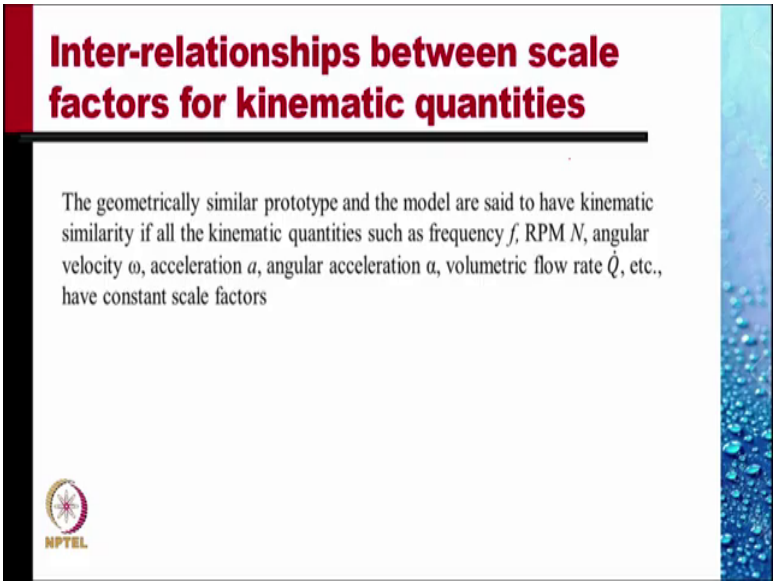

Geometrically similar prototype and models are said to have kinematic similarity if all the kinematic quantities such as frequency f, RPM N, angular velocity ω, acceleration a, angular acceleration α, volumetric flow rate $\dot{Q}$, etc., have constant scale factors. One scale factor for frequency, one scale factor for RPM, one scale factor of angular velocity, one scale factor for volumetric flow rate, etcetera.

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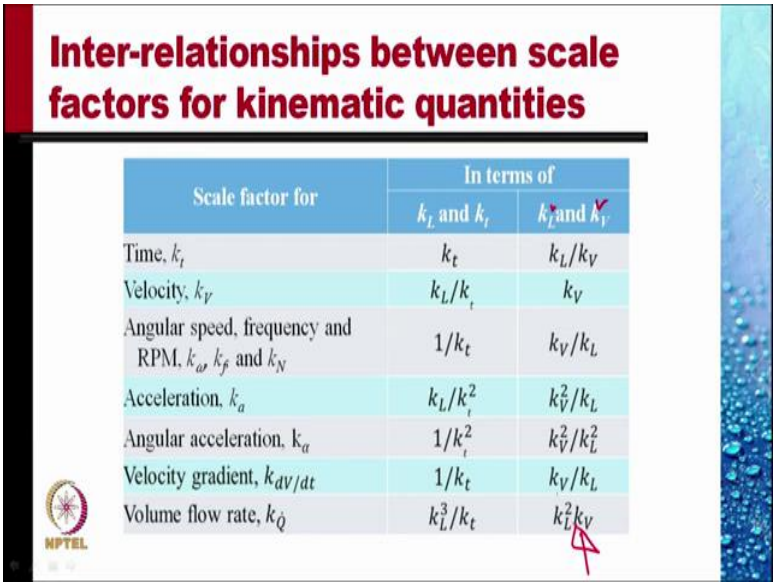

So, that we can relate the scale factors of kinematic quantities with the scale factors k_L and k_t, or k_L and V. Usually in fluid mechanics we use length and velocity as the primary independent parameters. So, it is convenient to express the scale factors for various quantities

in terms of k_L and k_V. And these are the values obtained using the method described in the earlier slides.

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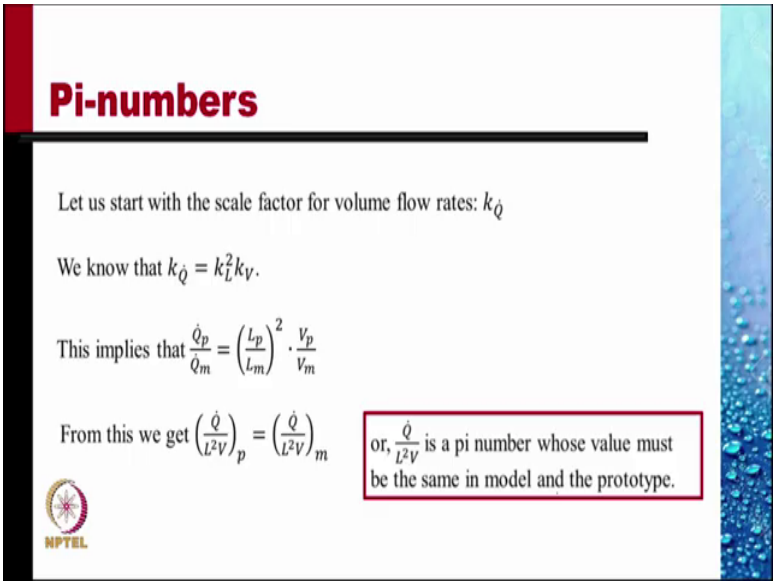

Let us start with the scale factor for volume flow rate as an example, $k_{\dot{Q}}$. We know that $k_{\dot{Q}}$ is like $k_L^2 k_V$. Why? The volume flow rate through a pipe, for example, is equal to the velocity through the pipe multiplied by the cross-sectional area of the pipe. So, the scale factor for volume flow rate would be the scale factor for area of the pipe times the scale factor for velocity, $k_L^2 k_V$.

This implies that $\frac{\dot{Q}_p}{\dot{Q}_m}$ which is the value of $k_{\dot{Q}}$, the scale factor for volume flow rate is equal to $\left(\frac{L_p}{L_m}\right)^2 \cdot \frac{V_p}{V_m}$. We separate on the left and the right-hand side, the parameters and values corresponding prototype, and on the other side the values associated with the model. And we get $\left(\frac{\dot{Q}}{L^2V}\right)_p = \left(\frac{\dot{Q}}{L^2V}\right)_m$.

Here these quantities are assumed to be characteristic quantities, L for the prototype and L for the model measures the same corresponding lengths. If in an aircraft wing, we are talking of chord in one case, it has to be chord in the other case. If we are talking of the thickness in one case, then it has to be thickness on the other case, whatever we choose as the characteristic value of the length.

Similarly, V here is understood to mean that it is a characteristic velocity. We should be choosing the same type of velocity in the two flows. So, if in one flow it is the velocity far

upstream, in the other flow also it should be the velocity far upstream. What does it mean? That the parameter $\frac{\dot{Q}}{L^2V}$ is a Pi number whose values must be the same in the model and the prototype. It is understood that the quantities are the characteristic values. This is a requirement for similarity, if $\dot{Q}$, L and V are independent parameters. But if one of them is a dependent parameter it becomes a prediction rule. We will explain that a little later.

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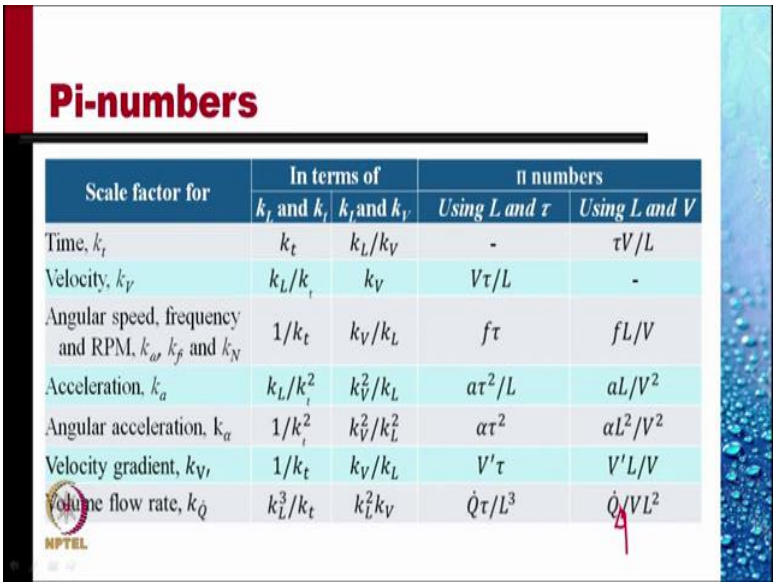

So, we convert the relations between the various scale factors that we obtained earlier into pi numbers. These are the pi numbers using L and V, the length and velocity. For angular speed, frequency, RPM, the pi numbers fL/V. fL/Vin the prototype and in the model must have the same value. $\alpha L^2/V^2$ is a pi number corresponding to angular acceleration. These must have the same values in the model and the prototype, every one of these.