Advanced Mathematical Methods for Chemistry
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Module - 04
Lecture - 03
System of 1st Orders, Linear 1st Order ODEs

In this lecture I am going to talk a little bit more about first order differential ODEs, I am going to talk about system of first order ODEs, then I will take talk about a linear system of first order ODEs. Now a system of first order ODEs is something that you have all encountered in chemical kinetics courses, so that is you see it very often and what I will try to do today is to just give you some basics about what the system of ODEs looks like and what are the techniques that you can use to solve it. For the particular case when you have a linear system of ODEs, if your system of ODEs is not linear then of course, you have use to other techniques to solve it. But at least in today's lecture we will focus on solving a linear system of first order ODEs.

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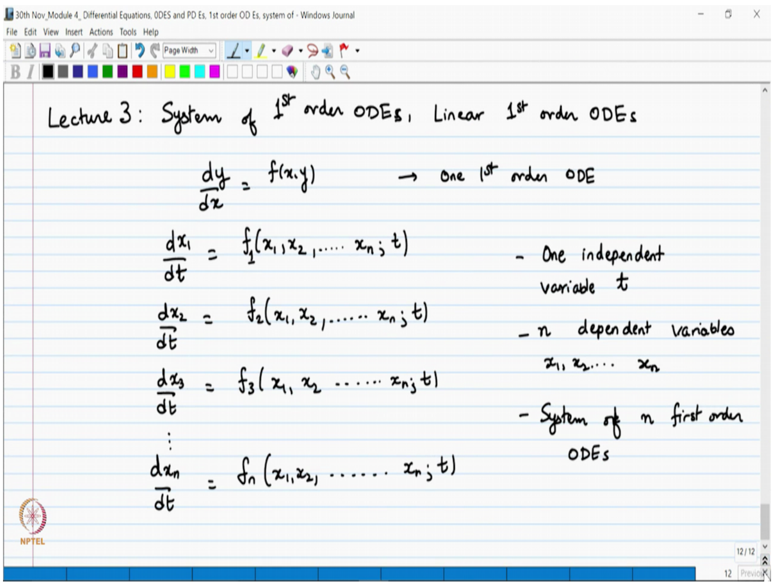

So, first let me tell you what I mean by a system of first order ODEs. So, we saw in the last time that we wrote something like d y by d x is equal to f of x y and we said that this is a first order ODE, this is one first order ODE. On the other hand you can have you can have something like this, so d x 1 by d t is equal to f of x 1 x 2 or f 1 of x 1, x 2 up to x n and t, and you could have d x 2 by d t is equal to f 2 of x 1, x 2 to x n and t d x 3 by d t is equal to f 3 of x 1 x 2 up to x n and t. So, on I will just put a few dots d x n by d t is equal to f n of x 1 x 2 up to x n and t.

So now here I have I have 1 independent variable that is t. So, 1 independent variable t and n dependent variables x 1, x 2 up to x n. So, this what I wrote here is a system of n first order ODEs it is not one first order ODEs, but it is a system of n first order ODEs and. So, this is what we mean by a system of differential equations, now let me take 1 example.

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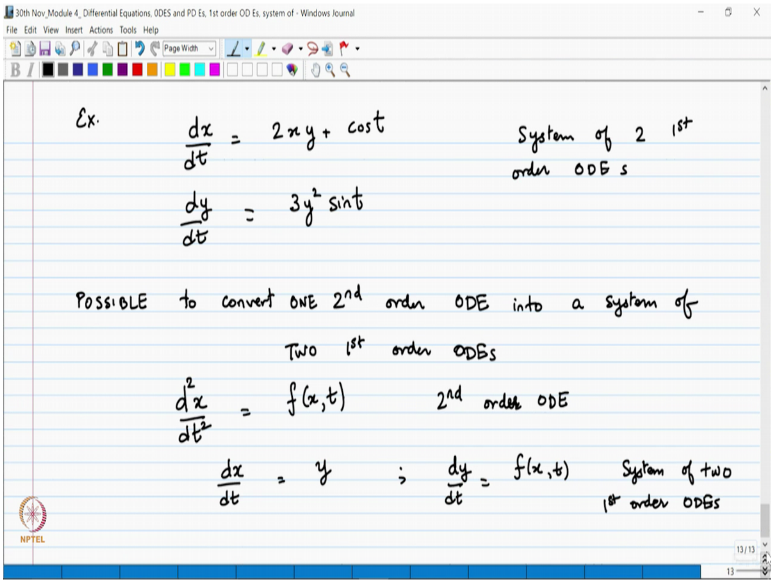

So, suppose you have let us say, let us say you have d x by d t is equal to 2 x y plus cos t and you have d y by d t is equal to 3 y square sin t. So, this is a system of 2 first order, this an example. This is a system of 2 first order ODEs.

Now see it is possible to convert second order ODE. So, one second order ODE into. So, let me emphasize one second order ODE into a system of 2 first order ODEs. So, I will just show you how to do this, suppose I have d square x by d t square is equal to f of x y or f of x t. So, this is a second order ODE. Now I can write this d x by d t is equal to y and d y by d t is equal to f of x t this is a system of 2 first order ODEs.

So a single second order differential equation has been converted to a system of 2 first order differential equations. So, the first differential equation is d x by d t equal to y, the second differential equation is d y by d t equal to f of x t. Now this is a very useful thing to do.

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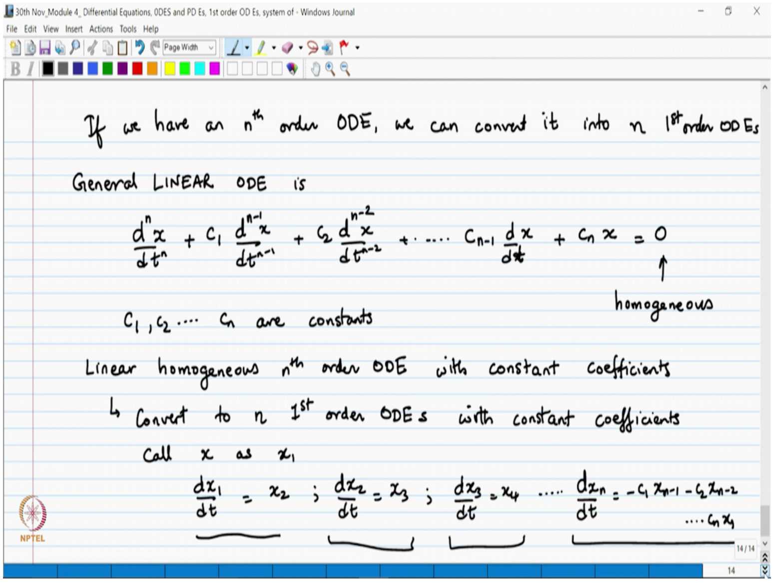

In particular let us consider so, if we have an nth order ODE we can convert it into n first order ODEs, this is just a natural extension of what we did in the previous problem.

So, if you have done, if you had an nth order ODE then you can convert it to n first order ODEs. So, a general linear ODE, a general linear ODE is written as d n I will just take x and t x by d t raise to n plus some constant I will just c 1 d n minus 1 x by d t raise to n minus 1 plus c 2 d n minus 2 x by d t raise to n minus 2 plus so on, up to you have all the way to c you can take you go to let us say c n minus 1 d by d x, d by d t of x plus c n x equal to 0. So, x is the dependent variable x is a function of t. So, you have a constant term, or you have x you have a term that depends on x then you have a term that depends on d x by d t, then you have a term that depends on d square x by d t square and all the way up to d n x by d t raise to n. So, this is a linear ODE and equal to 0, so it is homogeneous. So, since the right hand side was 0 there is no constant. So, each term, each term has a derivative to 1 power and it has nothing else it just has a constant.

Now, such an equation this linear ODE it you can write the solution. So, linear ODE and linear ODE let me emphasize. So, it is c 1, c 2 up to c n are constants. So, what you have is linear homogeneous nth order ODE with constant coefficients, what you can do is you can use you can use the method that we did earlier we can convert it to a to n first order

ODEs with and what will happen is all of them will have constant coefficients, you can easily do this. So, I can write, let me will be call x as x 1 and d x 1 by d t equal to x 2, d x 2 by d t equal to x 3, d x 3 by d t equal to x 4 and you will get d x first derivative second derivative third derivative.

So, when you go to the when you go to the last one then you will get d x n by d t, this will be will be minus c 1 times now will get x n minus 1 and so on; c 2 times x n minus 2 and so on, up to c n times x 1 that was the original differential equation. So, what I did was I replace each of these by 1 of these x 2 x 3 x 4. So, finally, what I got is a each of these is a linear a first order differential equation with constant coefficients and finally, you get this equation. So, now, I can write this.

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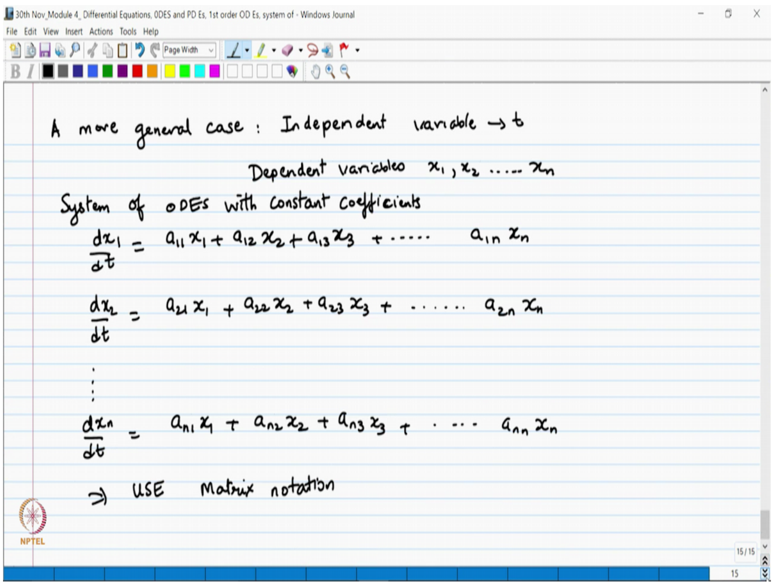

So, now let us consider a more general case. So, independent variables where variable is t and dependent variables x 1, x 2 up to x n. Now system of ODEs, d x 1 by d t and now what I will write is d x 1 by d t, I can, in general it will be a 11 x 1 plus a 12 x 2 plus a 13 x 3 plus all the way up to a 1n x n, these are the constants I am just calling the constants a 11 a 12 a 13 and all the way up to a 1n.

Similarly, I have d x 2 by d t and here here I will have a 21 x 1 plus a 22 x 2 plus a 23 x 3 plus all the way up to a 2n x n and you can go all the way to d x n by d t is equal to a n1 x 1 plus a n2 x 2 plus a n3 x 3 plus all the way up to a nn x n. So, I wrote it in this form and this actually when you have a system of ODEs with constant coefficients. So, with constant coefficients notice that none of the terms contains only the independent variable each term contains one of the dependent variables to 1 power. So, it contains one of the different dependent variables to 1 power. So, this is the system of ODEs with constant coefficients and now I can use matrix methods matrix notation first.

So, how do we use matrix notation? So, what we will do is we will say that let me.

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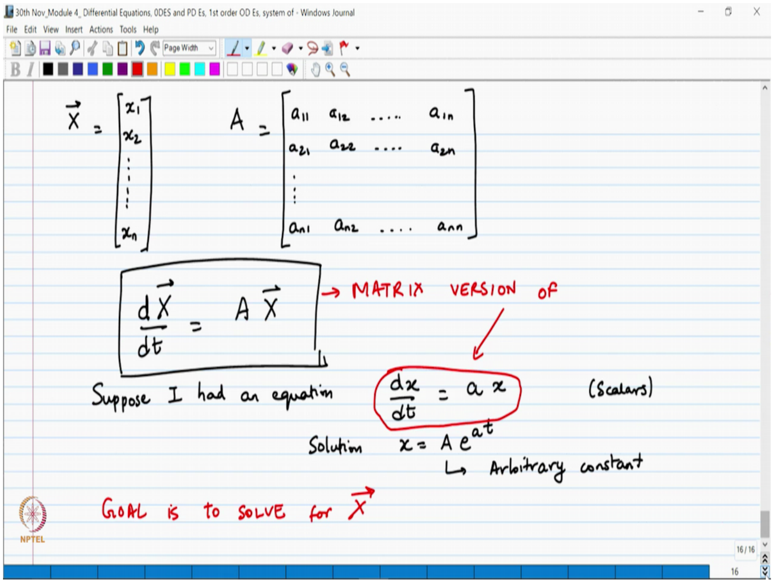

Let me say x vector is equal to what I will call this as a, this will be a column vector x 1 x 2 all the way up to x n and I will write a matrix as a 11 a 12 up to a 1n, a 21 a 22 a 2n, a n1 a n2 a nn.

So with this notation I can write this differential equation in matrix notation as d x vector by d t is equal to A times x vector, A is a matrix x is a vector and I can write in this very simplified notation. So, this system of linear ODEs can be written in matrix notation. Actually there are lot of nice things you can do with this notation it is not weak and we can actually use the matrix to solve for this. Now suppose I had an equation d x by d t is equal to a times x where this is a scalar equations, now a is a constant then you will write solution x is equal to x is equal to A e raise to a t plus B. So, A and B are constants. So, rather you want write that way you will just write x is equal to A e raised to a t, A is an arbitrary constant. So, if you if you just had a scalar equation d x by d t equal to A x the solution would be x equal to A is A constant times e raised to a t.

Now this, what you have here, this is a matrix version of this differential equation. So, this is a matrix version of this differential equation and so you should be able to use very similar methods to solve for it. So, now we can ask the question, what can you do to solve this. So, suppose, remember our goal our goal is to solve for x, in this case you are supposed to solve for the vector x. So, goal is to solve for the vector x. So, what we will do I will write I will emphasize this point. So, goal is to solve for x vector that is all those variables x 1, x 2, x all. So, these n variables we are supposed to solve for them.

So, what is the vector that will satisfy this differential equation? Now here this is where we use some of the matrix methods that we learnt.

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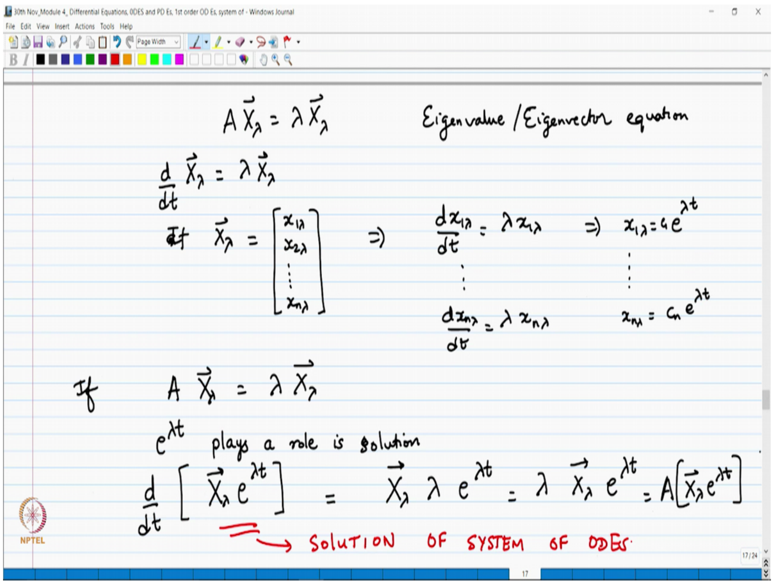

So, now suppose I had a times x lambda equal to lambda times x lambda. So, this is my eigenvalue equation, eigenvalue eigenvector equation. So, these are the, this is an eigenvalue and eigenvector of A. So, x lambda is eigenvector and lambda is the corresponding eigenvalue.

So now suppose I take this, suppose I knew some x lambda which had, which satisfied this then what I can see what I can immediately write is that d by d t of x lambda is equal to lambda times x lambda and what I can see this is that is that if I had if so, d by d t of x lambda equal to lambda times x lambda is actually a system of equations that is if x lambda I was written as x 1 lambda, x 2 lambda up to x n lambda then this differential equation is actually a set of n differential equation d x 1 lambda by d t is equal to lambda x 1 lambda and so on up to d x n lambda by d t equal to lambda times x n lambda. And each of these has a solution x 1 lambda equal to e raise to lambda t times some constant I will just call it c 1 all the way up to x n lambda equal to c n e raise to lambda t. So, all these c's are arbitrary constants.

So, we immediately see that e raise to lambda t is a solution, e raised to lambda c t multiplied by some constant. So, then what I can what you can do is you can, we need x lambda to satisfy A x lambda equal to lambda times x lambda. So, if A x lambda equal to lambda times x lambda then we can clearly see that this e raise to lambda t plays a role in the solution. So, then e raise to lambda t plays a role in solution, so then I can write my solution as I can write x lambda e raise to lambda t. So, if I write x lambda times e raise to lambda t. So, if I write, if I take d by d t of this whole quantity. So, x lambda is clearly a constant. So, this is just x lambda times, now e raise to lambda t if I take the differential of that I will just get lambda e raise to lambda t.

Now if I take the lambda outside I can write this as lambda times x lambda well will I will write it lambda times x lambda e raise to lambda t, now lambda times x lambda is just A times x lambda e raise to lambda t. So, clearly this is a solution, solution of system of ODEs.

So, what we did was that by using this eigenvectors and eigenvalues we can write the solution of ODE. So, x lambda e raise to lambda t is a solution of d x by d t equal to A x, where x lambda is an eigenvector and lambda is the corresponding eigenvalue.

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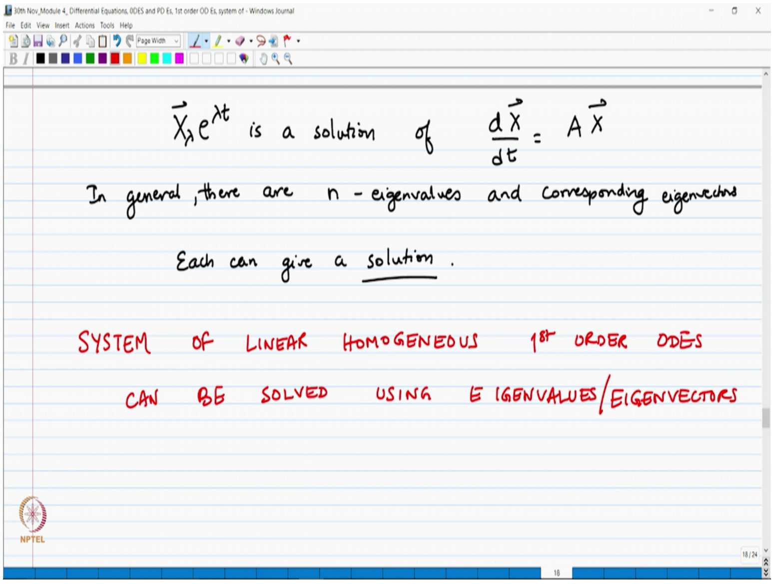

Now in general there are n eigenvalues and corresponding eigenvectors and each of them each can give a solution.

So, what we have seen is that, what we have seen by that is that we have what I whatever what I want to emphasize is that system of linear homogeneous first order ODEs can be solved using eigenvalues and eigenvectors. So, this is a very important message that I want to give from this lecture that if you have a system of linear homogeneous first order ordinary differential equations then you can solve the system of ODEs using the method of matrices. So, there is a very important connection that we are making between matrices and linear ODEs.

So, in the next class, next lecture we will try to see how to write the general solution and particular solution of this system of linear ODEs and we will talk in general about homogeneous ODEs and their solution.

Thank you.