Stability of Structures

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WEEK-01

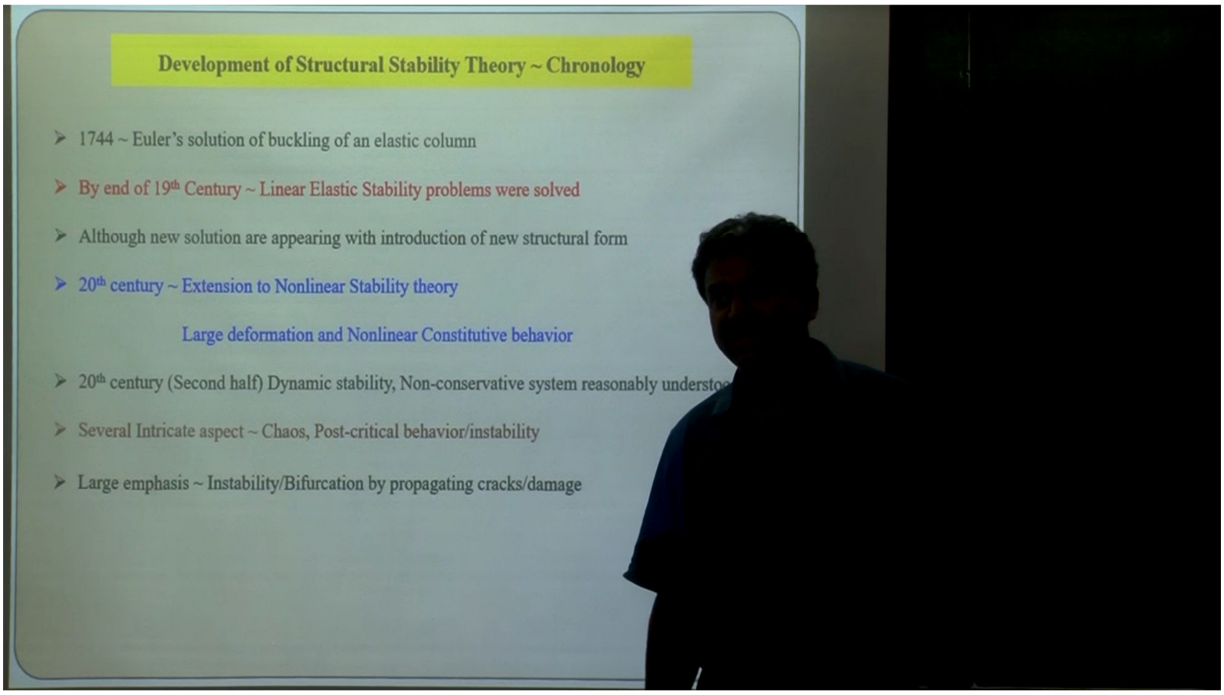
Lecture-01 Introduction to Stability of Structures

Welcome to the first lecture of stability of structure. So first, I will start with the little PowerPoint presentation where I am going to tell you the history of structural stability. Why structural stability is important and how the theory has developed and at the end. I will refer to you the some of book, it will be useful. And then of course, I will start with the deriving equations and the theory, but before that I am going to use the power point actually. So, I am going to review stability of structure its past, present and future. The theory the chronology of development and then the retrospection what how the theory how the subject developed and who are the people who scientist who contributed in the development ok. So, the chronology is going to evolve. So, this is based on a review. By Bazant, Professor Bazant who is one of the authorities in this theory of stability, structural stability and he has a book which is can be treated as Bible, okay.

So, he published a paper or in international journal of solids and structure on structural stability, okay. He is from Northwestern University civil engineering department. So, what overview of structural stability we are going to present here would include in-elastic stability, elastic stability first and then in-elastic stability as well, although in that course we are going to restrict our discussion to elastic stability only. and that itself is a vast subject. Then static and dynamic instability, the importance of non-linear behavior, initially we will treat it as linear right and formulation of stability using thermodynamics, why? The use of thermodynamics may be useful or appropriate for formulaic stability problem and then time behavior stability behavior of time dependent system like Visco is elastic system and Visco-plastic of three portrait dependent plasticity problem and stability under fatigue and a under fracture and damage which is a combination of the geometric instability and then the material instability ok. You see there have been a several failures that has triggered the structural instability ok. Of course, 1940 is the Tacoma bridge collapse was the very famous or notorious whatever that basically that happened on Tacoma narrow bridges in Washington US and it collapsed by aerodynamic instability. Even before that, that is most famous example and then all of we are aware about that and then we when we studied in our school level, we have seen that the picture and then videos are available everywhere to see the how the collapse is happening. So, this is one of the little more complicated which we will cover at the end of our course. But that is definitely a very compelling example of studying why to study stability.

Before that 1907 there was a collapse of Quebec Bridge in Canada over St. Lawrence River. Then 1978 collapse of a space drone in Hartford Civic Center in Connecticut US. And then collapse of 1978 reticulated dome in college theater Brookville New York. There has been a collapse in Melbourne, Australia, which is one of the greatest, biggest, greatest industrial disaster. The collapse of a steel box garden 1970s, all of them involved, of course, not the Tacoma Bridge collapse, but other involved lots of casualties and then death, right? And 1965 collapse of Ferry Bridge cooling tower. So, these are just few examples. There are numerous other examples. So, all these failures, actually, so we have some learning from failure right, that basically the reason why I think this underscores the need for studying stability ok. Why we need to have a very sound knowledge of structural stability, because we want to prevent this failure and we want to design our structure to be safe ok.

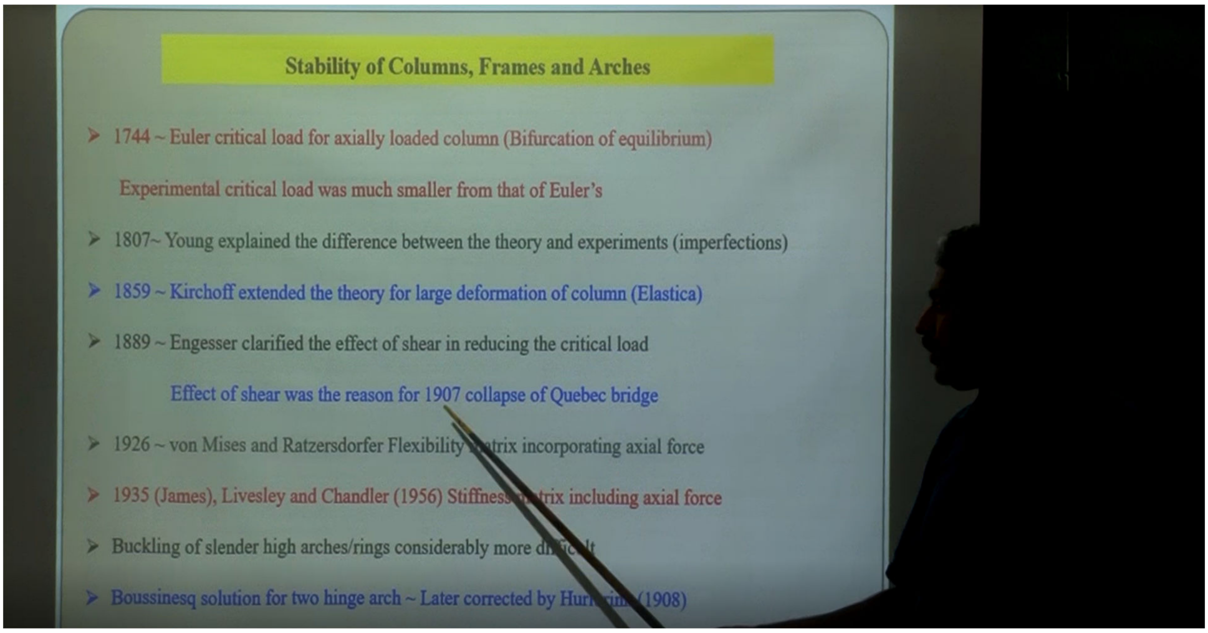
So, if you go the development of the, as far as the development of the theory is concerned, you see it is very brittle old. It is 1744, where Euler's basically given provided his very famous solution of buckling up an elastic column wall of, as we have studied this buckling up column in our undergraduate course, right, that was way back 1744, okay, long almost like more than 200 year or more than even more. By the end of 19th century along with other development we have pretty much good a reasonable understanding on linear elastic stability that was all this problem was solve of course never delays because there is introduction of new type of structure or development of new material. So, we need to have of course the same solution elastic stability solution for these applicable to these specifically these structures right. By 20th century extended the stability theory for non-linear stability theory. So, our linearized stability theory has been extended to non-linear stability theory but let me tell you that today also it is mostly even all research we make use of the linearized stability theory.



Non-linear stability theory is not very simple and not very easy to do and it has been shown that we can capture very useful information even with linearized stability theory. and they include large deformation and non-linear constitutive behavior. So, it is basically geometric instability which is responsible we make use of the geometric non-linearity to develop stability theory. But along with that, there can be interaction with non-linear constitutive behavior. That means non-linear stress-strain relationship of material is also important. So, that basically get along the geometric instability, okay. By the second half of 20th century, we have a good pretty good understanding of dynamic stability, non conservative system. And then there are several intricate aspects what people started studying post critical instability chaos. And then today, the larger emphasis or even the research is concentrating on instability bifurcation.

Under material instability that means in the form of propagating crack front or the damage material which has damage, damage plasticity for example concrete, some composites and others ok. The stability of column, frames and arches, you see these are the chronology of development. Euler's critical load was given 17th century and then once this theory was given there has been lots of doubt among the mind of people that they did not try to believe that, that this whatever he has proposed is good. So, because there is lots of huge disparity between the experimentally observed critical load and whatever was given by Euler's critical load, okay. So, that disparity was explained by Young in 187 by using imperfection, the concept of imperfection, that is the initial curvature, okay, of the column was responsible for this. Today also it is acknowledged that imperfections play a very important role in stability, ok. We are going to study all this, ok. Then of course later Elastica was introduced where large deformation of column has been taken care by Kirchoff in 1859, you see. And then subsequently there are other important effect like shear deformation, effect of shear on buckling by 1889 by N. Gieser. And unfortunately, this was one of the reason ok, lack of understanding of effect of shear on critical load was one of the reasons for the failure of 1907 collapse of cubic bridge ok. And we know these things the effect of shear deformation would have prevented such collapse. Then for the frame and arc, for frame we have flexibility matrix method and then stiffness matrix method and stiffness matrix method that was formulated. The problem with the flexibility matrix is that it gives inconsistent result and sometimes it is they suffers from numerical instability, okay. So, but then both of them were replaced by the modern finite element technique which was developed around 1950s.

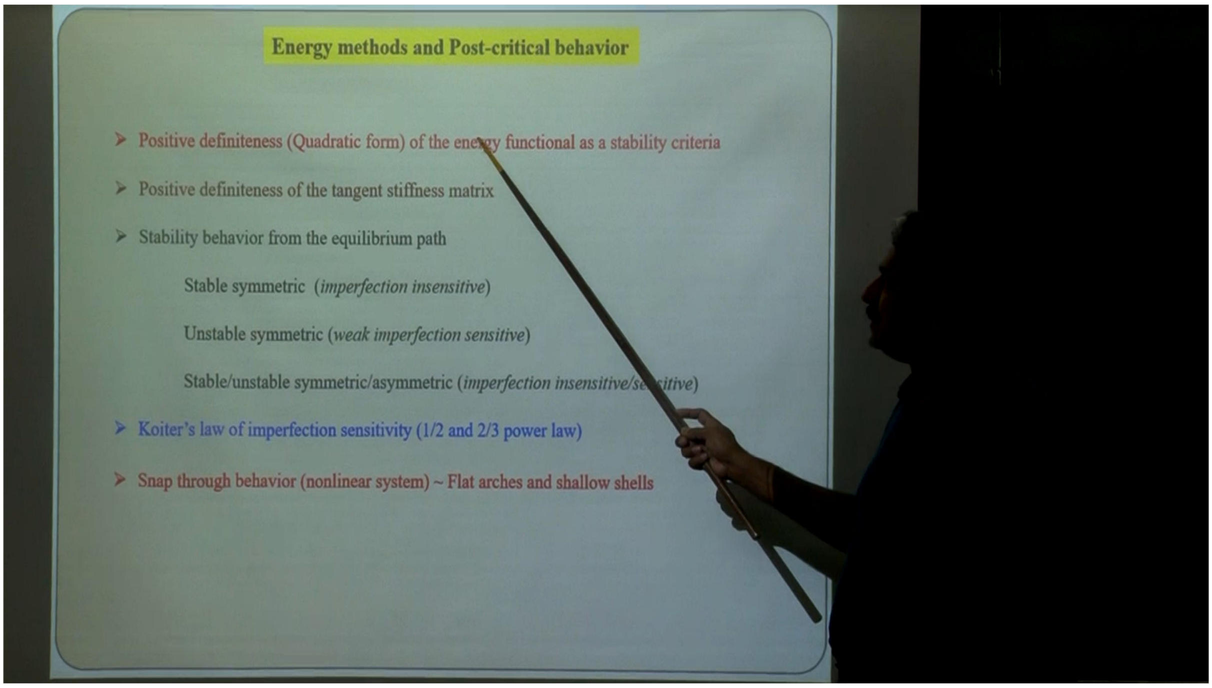
Just see, this approach, flexibility matrix approach 1926 and then the stimulus matrix by James, Livesley and Chandler, 1935. Shortly after that around 1950s also finite element was gaining prominence, okay. and then they will replace this approaches ok. And then of course that theory has been extended by other one-dimensional structures like buckling of cylinder, high arches, rings ok. That was considerably more difficult because you will see arches and others, they will have pre buckling non-linearity and then it has been seen that many people who tried to formulate it using linearized theory. They end up getting wrong result and inconsistent result that is because all these things significantly affected by pre-buckling non-linearity. But the first solution that has been proposed by Boussinesq, Boussinesq solution for twin hinge arch and then later it was corrected by Hurlbrink ok. And by 1970s that for arbitrary statically indeterminate arches were understood ok. Then for the dynamic instability, for all the static instability, the notion of dynamic instability was given by 1893 by Lyapunov, Russian scientist. He generalized the definition of stability for dynamic system, so you will see that. And then there have been other system like parametrically resonance column, a column when it buckles when it is axially compressed subjected to axially compressed load and then it beyond a critical value of the load it will buckle.



But you will see if the axial load is changing over time that means it is a fluctuating it is oscillating ok, with some frequency then there is something called parametric resonance things. Ok, it happens where the lateral vibration have a double frequency than that of axial frequency of the column right. So, load should for the resonance to occur that should be 2 times of that frequency. So, that is solved by using Matthew differential equation by Rayleigh 1894 and then all these basically things when we do try to analyze the static stability, we all are concerned about positive definiteness of the potential energy ok. Then that has been generalized to the topology of the energy surface including kinetic energy for dynamic and then we synthesis something called Liapunov functions, okay. That was also given by Lyapunov.

So, there are alternate choice of liapunov functions that include the potential energy as well as kinetic energy and these things were developed, okay. There has been theory for the and there are lots of misunderstandings, initial understanding where what we consider traditionally the forces have some direction but there can be forces which basically changes its direction under the deformation of the system. For example, a column subjected to follower force. So, force means follower force is one which follows the tangent of the of the teeth. So, initially this problem was not solved until bolotin and in 1963 and others, they started to solve it. So, initial static approaches all failed because unless and until you consider inertia, this is one of the non-conservative systems. So, all elastic theory cannot be applicable. So, by incorporation of the inertia and then subsequently formulating the equation, they solve this, the string stability problem of column under follower force. Now, the very famous 1940 Tacoma bridge collapse, it was investigated by von Karman, the great aero-elastician, and great fluid dynamics. He submitted report to the Federal Work Agency on Tacoma bridge corruption. Subsequently, there have been lots of surge in aerodynamic instability of flexible bridges and namely cable straight and cable suspension bridges and then people by now it is well understood that what really happens for case of aerodynamic flutter of bridges. And then mostly for stability analysis we try to use energy functional because that is a very the most standard approach and then most convenient because it is very easy to work with the energy being a scalar function right. So, we try to ensure the quadratic form of the energy potential functions positive definiteness okay and then we try to see the tangent stiffness matrix in case of nonlinear behavior that way the tangent stiffness is positive definite. That basically ensures that the Hessian of the potential energy functions has is greater than 0 right. Now with this based on this energy approach stability behavior can be classified in different system we will study each of that there can be stable symmetric wiper case and unstable symmetric wiper case whenever there is a loss of stability what happen a system attains an alternate equilibrium configuration.

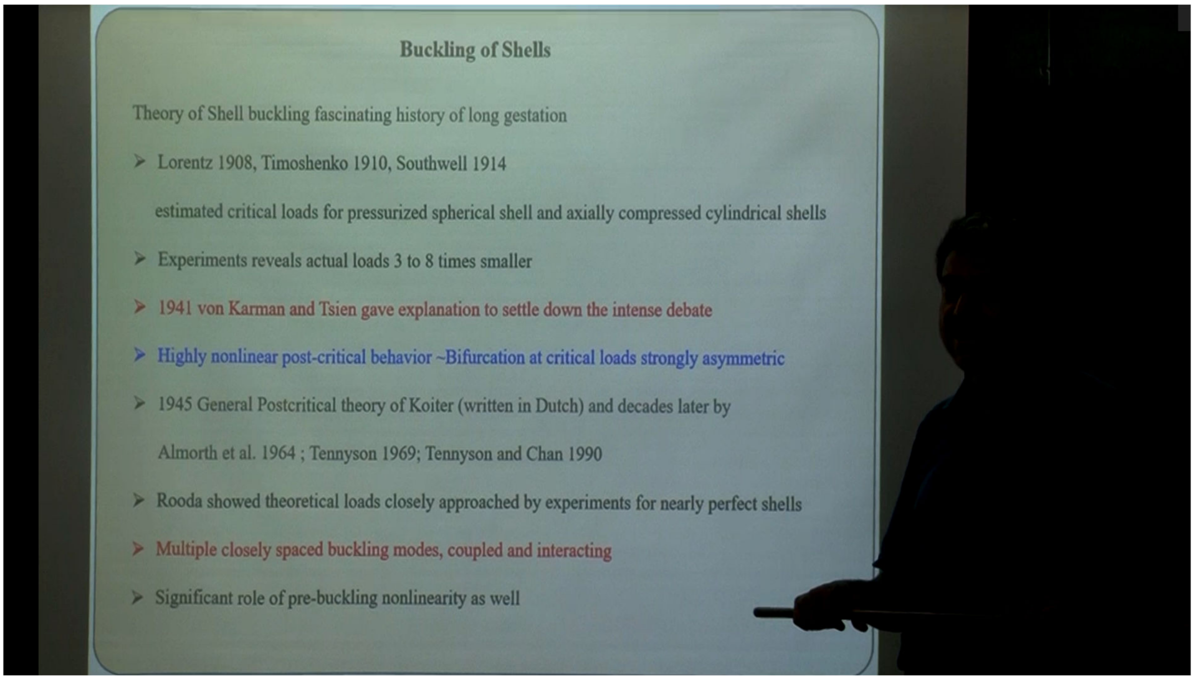
So, the stable equilibrium bifurcates into stable or unstable equilibrium ok. So, that is what the bifurcation is associated to the buckling right. So, you will see stable bifurcation, symmetric bifurcation, stable asymmetric bifurcation or stable symmetric asymmetric bifurcation. So, these different classes of behavior we will study and along with that depending on the system they can be imperfection sensitive or imperfection insensitive meaning in presence of geometric imperfections this system behavior can be drastically different they can lower down the critical load significantly and that is called imperfection sensitivity and there are universal laws for this imperfection sensitivity that that were developed by Koiter's so and those are called weak and strong sensitivity half and two third power law were derived. Now there are other system in which are non-linear system from the very beginning, which is the bifurcated system as not in the immediate vicinity. When a column basically buckles it just achieve a little deflected configuration right. But there is system like Von Misses truss, or we will say arch which on buckling, they will immediately jump into a different configuration further apart.



So, there is a huge gap between the alternate configuration and the previous configuration. So, those are called snap through behavior. So, that we are going to show, but those systems are nonlinear systems and that cannot be meaningfully linearized, that we are going to demonstrate.

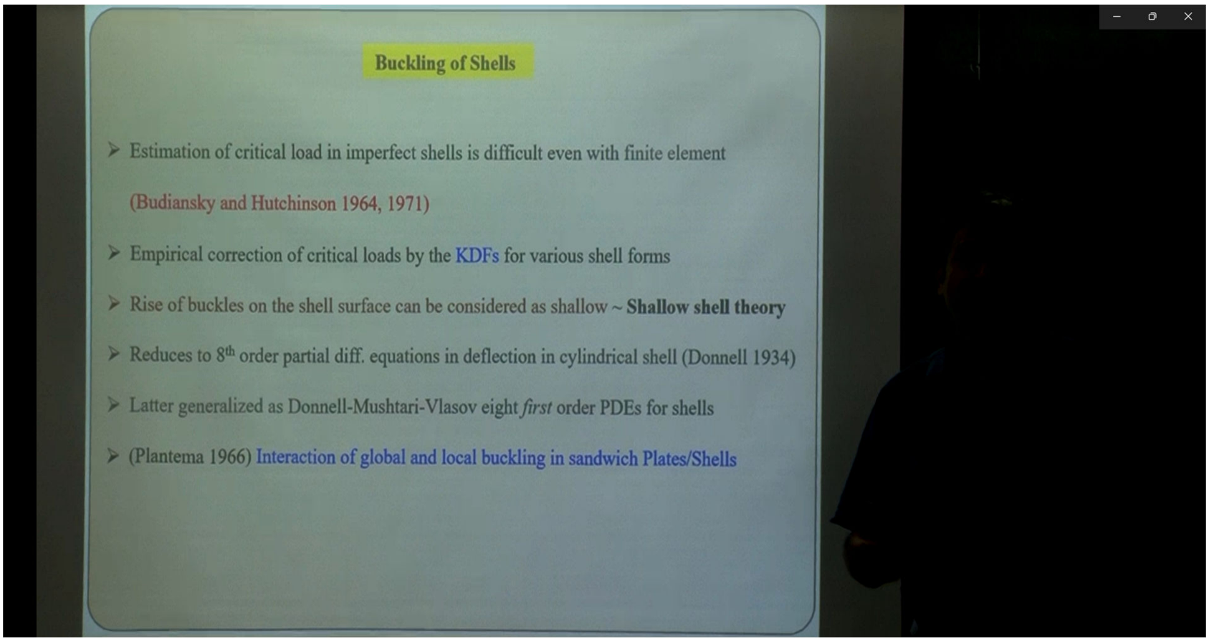
Along with that there are other sections in which those are thin wall section where the along with this venant torsion there are warping torsion meaning those cross section deform under torsion. So, you have to consider the associated force variable to be bi-moment ok. Those of which you are familiar with the advanced structure analysis of a thin wall section you might know what is bi-moment. So, then there is formulation for lateral and lateral torsional buckling and there are solutions ok. All these things ok, that we are all developed. Then what we are going to do. Now comes the more intricate thing which is buckling of shell. So, buckling of shell has a very fascinating history ok and it is a long long gestation period ok. So initially this Lawrence, Timoshenko and Southwell they start started estimating critical load for pressurized spherical shell and axially compressed cylindrical shell. These are the two most common example axially compressed cylindrical shell and spherical shell under external pressure ok. So, they find out some solution, of course, based on linearized stability theory. By the time nonlinear theory was not very well developed. And then what they saw that, oh my God, the experimental critical load that they were getting is very, very low comparing whatever was theoretical prediction from the linearized stability theory, 3 to 8 times smaller. Can you understand the difference? So, then that created lots of confusion among the mind of researcher. And so, shell buckling always remain a very important area of research, ok. And it is a very interesting problem, ok, even today also. And then you will see that shell buckling community is there. So, the people contributions are always documented there and they had a great collection of the literature. So, then it was first the von Karman, once again the great fluid dynamics, it was the von Kármán in 1941 and then tsien, they gave the meaningful explanation to settle out this huge intense dispute or intense debate that why this thing is happening and they attributed it to the high non-linearity that is involved in the system, unlike all the system. like beam, column or even frame, all this system what happen is that they are the effect of influence of non-linearity is not that significant on the critical load ok. And but here what happen is that from the very beginning non-linear plays a very important role.

Even linearized analysis whatever you are going to get is a bunch of buckling mode. Because, of course, it you will have multiple buckling mode right and they are all coupled by geometric non-linearity. So, they will interact and as a result of both these modal interactions the it causes a reduction in the critical load and all these buckling modes are basically highly, they so high asymmetric bifurcation. So, they are highly imperfection sensitive ok. And then comes the general post critical theory by koiter it was initially written in Dutch in my lecture later you will see I made a mistake I referred it to be German but it was written in Dutch later it was translated ok. And then of course, subsequently Armour, Tennyson and Chan they are the great great minds actually which throw light in this in this debate and they started in providing a meaningful explanation of this huge disparity between the experimental and theoretical load in the in the shell. Roda was the first to show that critical load can closely approach by experiment for nearly perfect shell. So, what happened earlier days, experiments were difficult to conduct because the way you fabricate a shell that has lots of bearing on the critical load you obtain because all these primitives fabrication technology. used to have lots of imperfection in the shell geometry. I mean lots of imperfection means whatever little that was significant. But today with 3D printing this and that you can have a very like shell geometry which is virtually perfect. So, if you do so then you will see that asymptotically with a reduced magnitude of imperfection that can asymptotically approaches to the right theoretical value. So that was a great observation okay. So not only the multiple closely spaced buckling mode they are see the buckling mode you will see that when we linearize there are critical load which are very close to each other and because of this closeness or because of their multiplicity even single there can be two modes which are same buckling critical load. They interact and not only that there are huge role of pre-buckling non-linearity and the imperfections as well.



So, that is a still a very fascinating subject, okay. Even with finite element today, see today's stability analysis all we rely on finite element because we have this commercial software and that becomes a billion-dollar industry and we all try to use because it is simple. Gone are those days when people used to derive lots of complex differential equations and we used to try. So, you have much less effort today, but even with finite element Budiansky and Hutchinson, they demonstrated that it is extremely difficult to capture the theoretical load because we have very inappropriate representation of the exact imperfections. Traditionally, imperfections were assumed to be self-defined to the buckling mode geometry, but it is not, sometimes it is random. And that is why probability theory plays a very important role in defining this geometry and that is what people are doing today. So, in order to avoid these things, empirical corrections were given on the theoretical critical by multiplying with a reduction factor, which is called knock down factor. In fact, this formulas, knockdown factor formalism is used, this formulation is used by NASA in US for all their space program to use a KDF which is geometry dependent and dependent on the loading to have a KDF for different geometry and loading condition of shell, ok. This is used. Then later there have been shallow shell theory from the Johnson's that risk and there are these buckles which are formed the rise of buckles on the shell surface can be considered as shallow and using that the Donnell defined an 8th order partial differential equation for the cylindrical shell which is very famous Donnell's equation for the buckling of cylindrical shell.

And then later that has been generalized by other like Mushtari and Vlasov, so that is, they give 8 first order PDE for any general shell geometry. Of course, shell geometry itself, to define it you need to have a lot of differential elements or differential geometry you have to, you have to understand that. But ultimately it gives a Donnel-Mushtari-vlasov equations for that and then of course there depending on the what kind of material like sandwich or composite plates and so on, there are interaction between the global and local mode. Even then global and local mode interactions are also there in column and frames. For example, if you have a reticulated column or the last column okay, there can be interaction of the global mode and local mode.

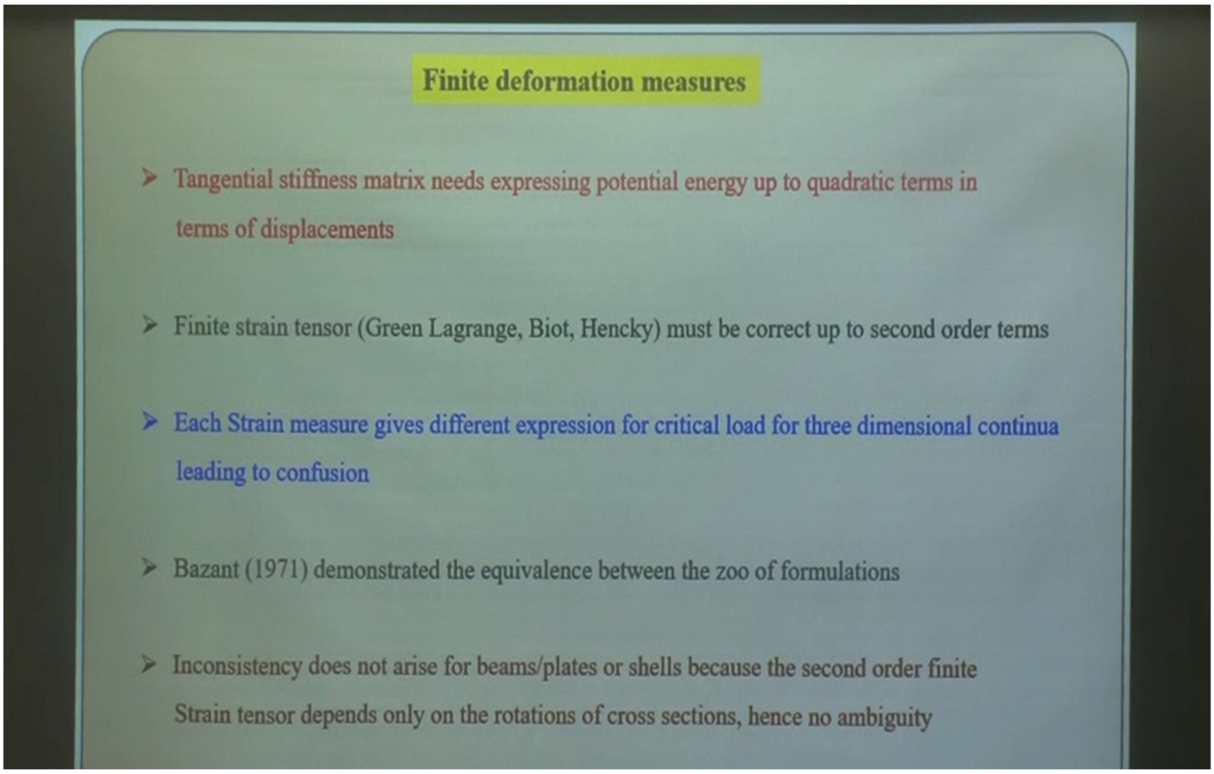


I have presented those things in my lectures. Then there is inelastic stability, elastoplastic stability. So, the first it was attempted by engesser once again the one scientist who investigated the effect of shear deformation on buckling load. So, then he said that inelastic column you can just replace the formula pi square EI by L square the critical load by E should be replaced by the tangent modulus for the inelastic analysis. But then later he reversed himself, 1895. He said no, let us use for the geometry dependent weighted average for the loading and unloading modulus which was referred as reduced modulus. But then people saw that there was huge disagreement between the theoretical whatever was given by reduced modulus and with the actual experiment on the aluminum alloy. Yeah, of course, aluminum alloy was mostly used by aluminum alloy in aerospace industry because of the lightness, okay. Then, in 1947, Shaley, he used inelastic simplified model for the column and demonstrated that whatever is the original proposition of engager using the tangent modulus that is the correct proposition. And because the assumption, there is a problem in the assumption. Column does not buckle at constant load but at increasing load because whenever these inelastic things you can have a hardening kind of behavior as well ok. That was a main difference, okay. So, tangent modulus was found to be correct. Then later it was generalized by Hill's and Shanley's theory. So, Shanley's theory is very well acclaimed and later it was generalized Hill for elasto-plastic structure, not only for the column but any structure. So, Hill's theory for the inelastic buckling or elasto-plastic buckling of structures, okay. there is huge influence of the locked in stresses or residual stresses that has been demonstrated by disparate huge disparity between the theoretical load and then the experimental load in hot rolled sections. So that was attributed to the to the residual stresses because there was a huge difference between the tangent and reduced modulus load okay.

Here people have that has also important bearing in formulating the code provisions because in code when you deal with this role sections you need to take into account. So, there are some modifications has been done okay. Not only that, see it is not that always uniform healing will happen. There are instances in which plastic strain localizes. For example, whenever you do a tensile testing, you see that there is Lloyd's line that falls, right. So, shear band, so those are some kind of localization of plastic strain, necking kind of behavior. So, sometime plastic strain localization instability is also important, okay. other than that, So, what we have seen that here you see all elastic structure is fine, but inelastic structure include basically dissipation. in the form of like sometimes inelastic dissipation of energy because this was hysteresis, they so loading unloading path are different so that is what they lead to dissipation. So that can be best accommodated in using irreversible thermodynamics okay and then you can use a different alternate energy functional like what we have learned in thermodynamics Gibbs or Helmholtz free energy functionals. And then thermodynamic criteria have also been developed. See in stability analysis using energy functional you have a hessian of the energy functional like for equilibrium you have the minimization of potential energy. and then second order of potential energy or for multi degree of freedom system the hessian of the potential energy functional determinant of that must be greater than 0 to be the stability criteria. So, whatever was done using energy functional that can also be equivalently transformed into thermodynamics context.

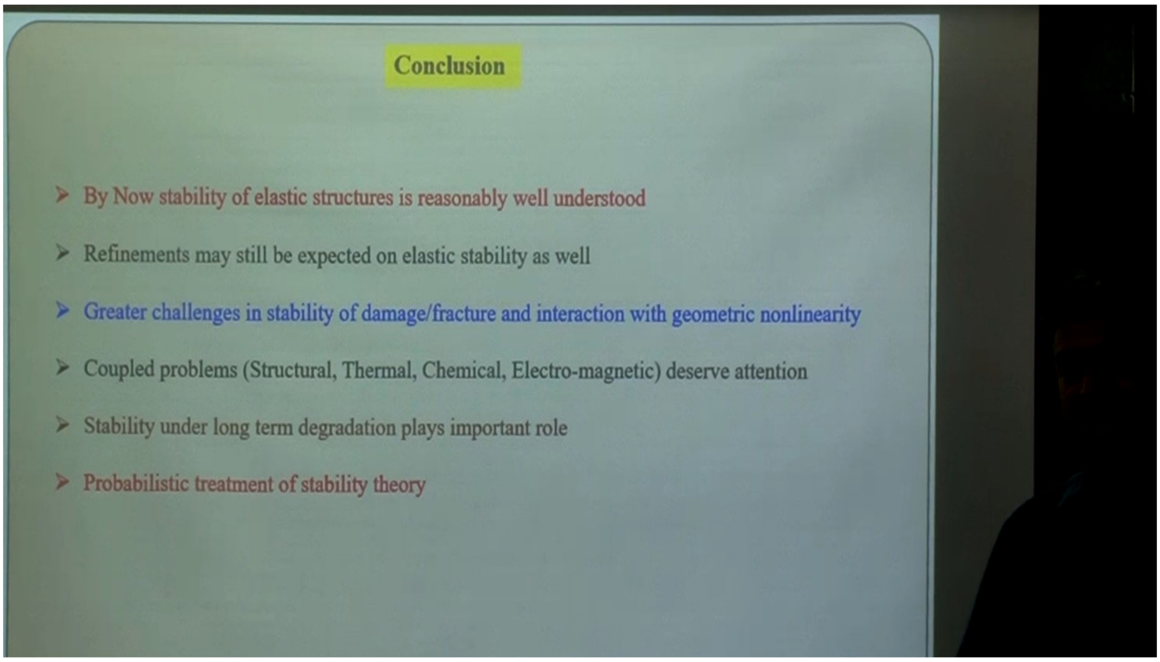
So, positive definiteness of the tangent stiffness matrix is shown to be equivalent to the negative of the entropy increment. You see, we cannot measure entropy, but change in entropy can be measured, right. We can clearly find out that the entropy increments for a system if it is negative that is equivalent to the positive definition of tangent. So, you see the thermodynamics for inelastic system or the system which has lots of dissipation thermodynamic framework can be used very efficiently to tackle the inelastic stability aspect. Also, there is another important thing which must be mentioned that you see sometimes whenever there is equilibrium path sometimes bifurcate into alternate equilibrium path, okay. So, multiple branches occur. Now, which branch should be taken by the structures? That is a problem in numerical simulation, in finite element. So, instead of using the energy formulation, if you use this in thermodynamic formulation. then you can find out very easily which parts to be followed with the maximum. of the second order increment of entropy. So, that is another thing that has been given. Now there are localization instability due to damage. So, these are a material, so material instability because what happen is that damage, sometimes damage localization happens due to in strain softening like concrete after reeling it, it basically there is softening branch, it does not strain hard. So, for this you see descending peak, there are loss of positive definiteness due to softening and then you will see for this elastodynamic boundary envelope probably changes from hyperbolic to elliptical, these are the important things that has been. And there is something called single acoustic tensor that can be formulated using the constitutive matrix and using this. orientation of the whenever there is strain localization, there is a jump discontinuity in the strain field, okay. Displacement will be continuous but the strain will be discontinuous, okay. So those things can be found out by searching for the singularity of the acoustic strains. So that can also be one kind of instability, of course in terms of material instability. Then instability due to propagating fracture, there is single fracture, multiple fracture, so these are all instability induced by crack, propagating crack and others, right.

And then time dependent behavior by viscoelastic and viscoplastic structure. So, viscoelastic means you see there is equivalence principle. So, you just replace the modulus of elasticity with the viscoelastic operator okay and they solve the equation of course, it will be time dependent. But here the important thing is that there is a long-time buckling problem because with time the buckling load is going to change here. So, the way it is formulated by replacing the elastic modulus with the respective viscoelastic operator. which can be expressed either in rate type using differential equation or integral type using an integral equation. So, that has been referred to Freudenthal who was the beginning in structural safety. So, it is load that is not important rather the time take, time must be large enough so that to reach the critical load. So, for a viscoelastic system, the time required to reach that critical load of that means should be long enough. Okay, comparing the design lab of the that way you ensure safety of the structure. Similarly, for the viscoplastic only difference is that here it is once again the similar approach that you need to have ensure that finite critical time for the deflection. So, whatever deflection in viscoplastic material what happened is that the deflection it increases with time like strain get accumulates.



So, that is what we refer as creak and then it is due to viscoplasticity. So, these things are also very important. Of course, this there is this elastic because Visco plasticity the yielding happens and then every time instant there is a change, right. So, there have been model for that and this is very extremely important for the aerospace engine and others where the material source viscoelastic vary the elevated temperature ok. So, you have to ensure that first critical time for the deflection triggered by infinitely small imperfections become finite ok. So, a small imperfection can have a huge implication for the visco plastic because it accumulates right. Then there is another approach see what we all we are talking about in the tangent stiffness matrix. But then tangent stiffness matrix for finding out the geomagnetic it is Hessian and others right. You need to have potential energy that must be quadratic correct until quadratic term in terms of displacement when you express. However, there are alternate definition of finite strain tensor because when we deal with finite deformation there are alternate definition and conjugate definition of finite strain tensor like Green Lagrange, Biot, Hencky and that must be correct up to second order. But all these things used to give different expressions because of this for the three-stability analysis for the three-dimensional continua. And this inconsistency was leading to lots of confusion. See this is not important for plate, sailor, beam column. Why? The second order finite stress tensor depend only on the rotation for the, these things, del w, del x, right? So, for that, that is why this distribution was not important for plates and still another. But for the three-dimensional continuum, when you want to study surface buckling or buckling of the 3D continuum, then this you have to make sure that potential energy function quadratic terms in terms, are expressed in terms of this, but then the, all this alternative strain tensor they are producing different result, but equivalence between all these things has been proved by Bazant, okay, was one of the prominent figure, okay. because there are huge geo-formulations, there have been formulation presented to remove these discrepancies, okay. So those of you who are familiar with finite deformation theory will try to understand these things, we will not cover this thing.

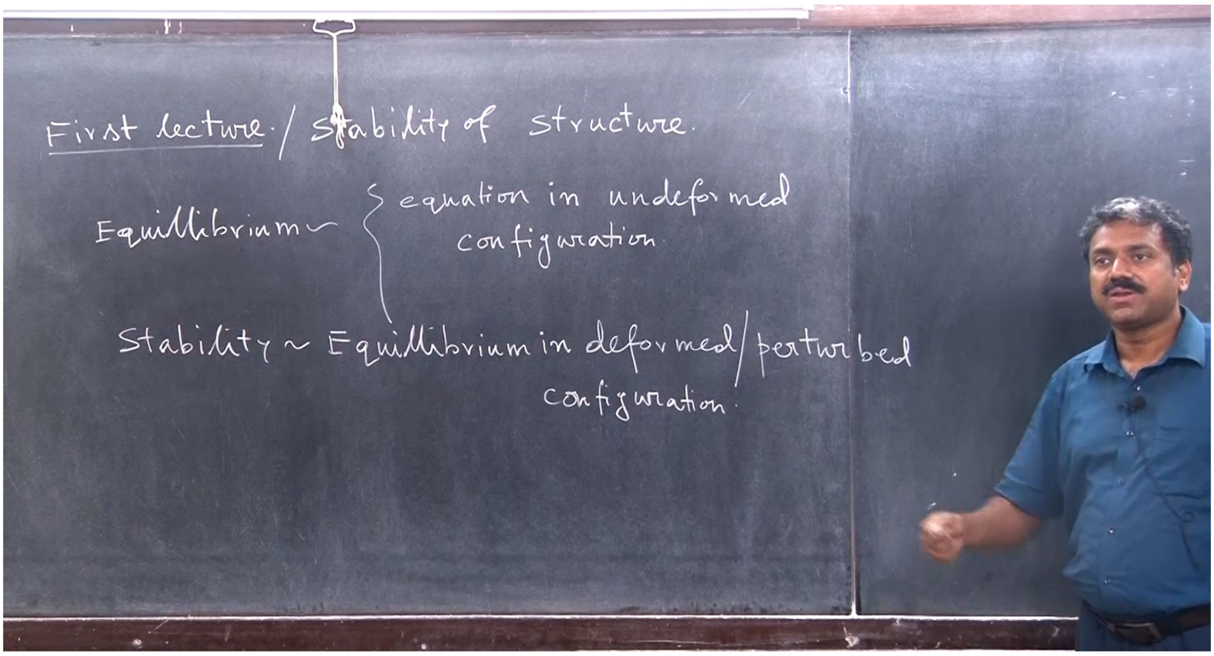
So the conclusion is that by now elastic structures, stability of elastic structures is well understood and refinement is still expected for the elastic stability as well, especially with the advent of new structural configuration and material behavior, but greater challenge is today is that stability In presence of material instability in the form of fracture, in the form of damage, softening damage and strain localization those things ok. And coupled problem where the not only the mechanics, but mechanics is coupled with thermal phenomena, mechanics is coupled with chemical kinetics, mechanics is coupled with the electromagnetic, okay. So, for this stability coupled problem, these are still remained unsolved and then I think more thoughts are required for this and people are doing that. And also, for the long-term degradation, for example, until chemo mechanical degradation, okay, what is happening, that is. And along with that, of course, a very important is the probabilistic treatment of stability theory. Why? Because no matter what you do, imperfections will always be very important, especially for structures which are imperfection sensitive. Even there exist a large class of structure, which are imperfection sensitive. And then imperfection when you want to describe physically you need to use probabilistic means, okay. So that is another important thing. So, with these things, reasonably we reviewed that whatever is the, what is the need of structural stability, I mean, and what great minds has been done, have been doing and over the past two and more than two and half century starting from 1744 by Euler and todays with the latest development, okay. Now with these things we will start you know, but before that I would just refer to several books for this which are important for you.



So, there are many books, good books for the general theory of concept of course Timoshenko and Gere are the one which is a classic in the theory of elastic stability. Then there is a book by Alexander Chazes principle of elastic stability. I refer it is not that the years are given may not be correct because there has been a newer edition but that does not really matter. This is from McGraw Hill, this is print as well and then there is this book Bazant and Cedolin in which I follow, of course there are in southern places you have to be cautious but this is I mean I treat to be Bible of stability, this is very good, very thick and that cover almost all aspect of stability. may be little difficult to follow with the unless and until you have a background in a reasonable background in structural dynamics, reasonable background in finite deformation, elasticity, plasticity, little bit of background fracture, little bit of background in damage mechanics All this background if you do not have, you cannot follow all of it, but definitely whatever we are going to cover, you can definitely follow. And then there is a good book by Simitses, An Introduction to Elastic Stability by Prentice Hall and then Brush and Almorth, this is a very good book, of course, I think they lack new edition, but this is one of the very good books what I sometimes consult Buckling of Bar, Plates and Cell. Now, all these things mainly with the theory of stability. Of course, they refer time to time, but briefly to the design thing, but if you want to have a comprehensive treatment on the design guidelines based on stability theory, then you must consult Galambos, the guide to stability design criteria for metal structure. This is a very good book and only for dynamic stability, not that all these books don't cover dynamic stability, but if you want to go to explore more details, then Volotin, these two books about, non-conservative theory of elastic stability and dynamic stability of elastic systems. One is Pergamon press. VOLOTIN one of the prominent names in dynamic instability. Then for the finite element method of course, today's stability analysis all depends on finite element. Everybody want to do simulation rather than dealing with the complex differential equation and that is to their solution is even more complicated and one of those days. So, for even for you have to be cautious with using finite element theory because sometimes it can be very very misleading unless used with proper judgment and proper clarifications, okay. So finite element method of course, you have this very simple book which we follow for finite element Cook, Robert de Cook, Malkus and Plesha concept of application of finite element method. Along with that, there is another book by Sames which is also good and Rubinstein, this is a simple book, not only on structural stability, but he also covers static dynamics and stability aspect Rubinstein. For advanced things which you do not require for this course but nevertheless I thought I will put it Thompson and Hunt. They are also contributed very huge contribution in development of the stability theory.

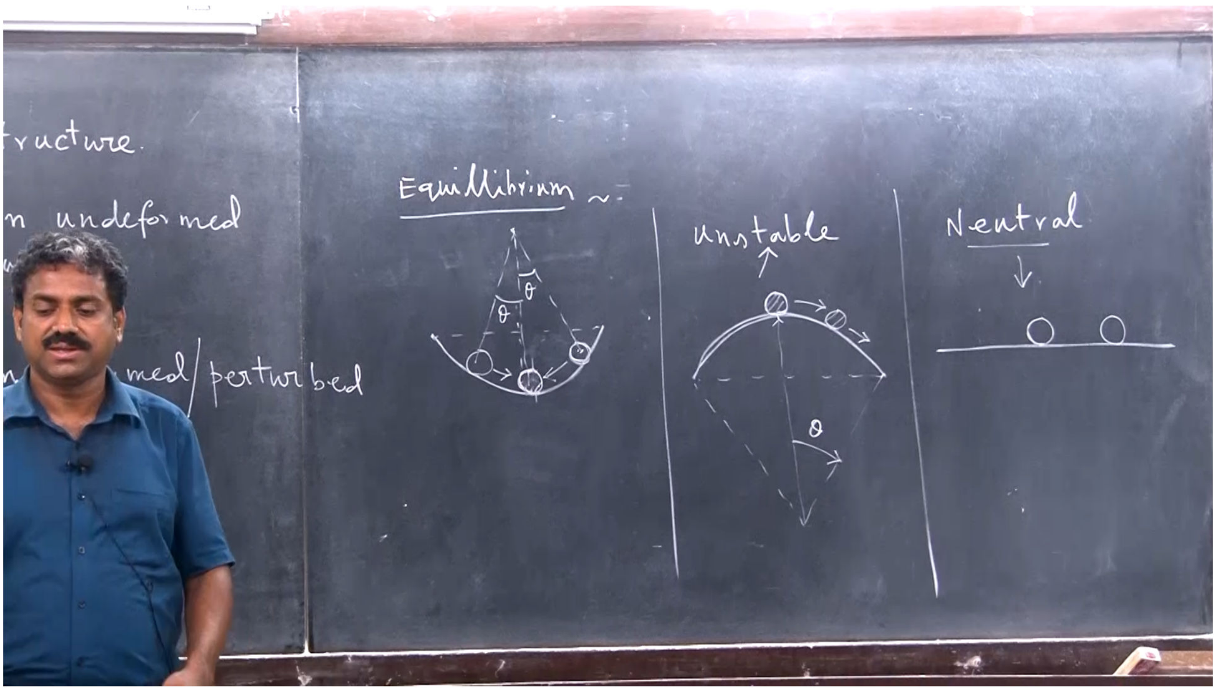
Thomshon and Hunt and their paper you will read a general theory of elastic stability and then life, welds and all these things are for the advanced treatment.

So, with these things I will stop here. So, we will start with this, the stability class and I will use blackboard for this. So, this is the first lecture along with this. So, first lecture on stability of structure. So, the first lesson to learn actually for stability of structure is that, what we are concerned about in all other courses, it is the equilibrium, right. In civil engineering system structures whatever we were concerned about the equilibrium, right. And in equilibrium, what we do? We write down the equation, equilibrium equation in undeformed configuration, right? That is what we write. But in stability, there is little difference, okay? In the first lesson in a stability class is that we write the equilibrium in the deformed configuration. We have to analyze equilibrium give them in deformed configuration or perturbed configuration. I will try to explain why, perturbed configuration, why that is so?. Let us go back to our school level physics, okay.



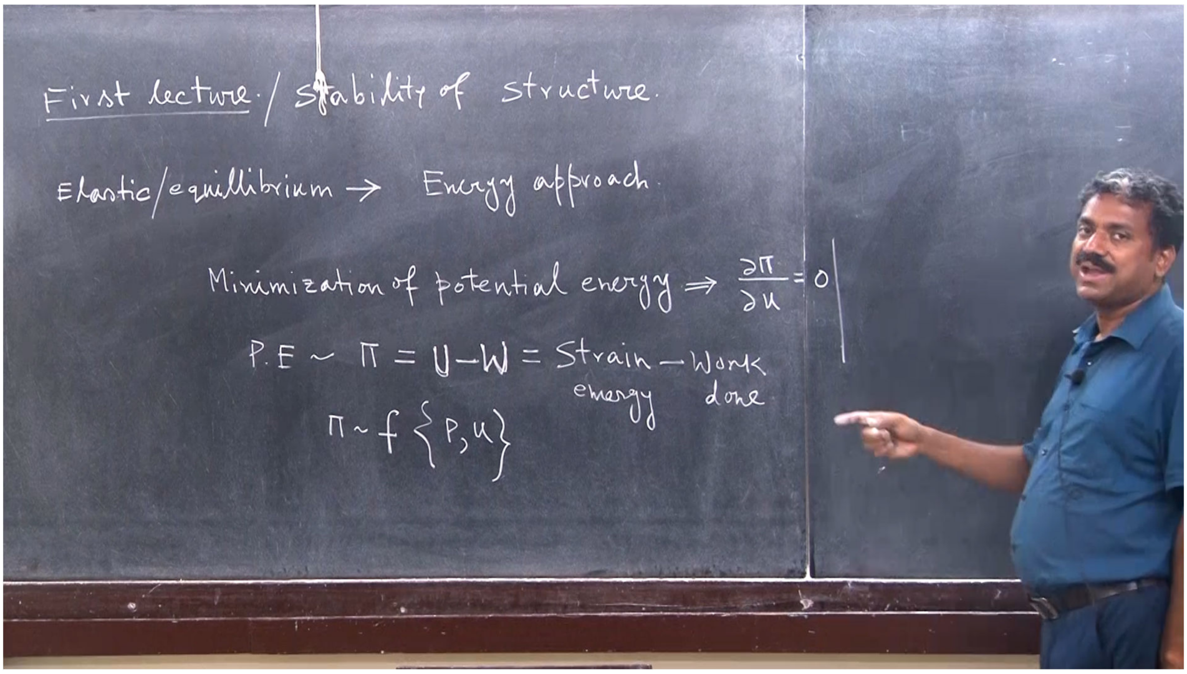
When we have learned the concept of equilibrium and then we are given, so whether a system is in equilibrium or not, that was the zeroth order information or first order information, right. And then once we know the system is equilibrium, we have to inquire that whether that equilibrium is stable or unstable or neutral. So that is a higher order information, right. So, equilibrium have 3 different things whether the equilibrium is stable or equilibrium is unstable or there was neutral.

So, what we have learned in our school level so from there I will see we have made use of this simple example, consider a bowl kind of things right and then we are putting a small ball here right, here. Now this small steel ball may be any ball okay, and this ball is in equilibrium because that weight is acting and so downward and it has an upward reaction right. So, R is basically this weight W and the reaction R, right? So, $R = W$, right?. So it is in equilibrium, right?. Now what we want to know we want to study the nature of equilibrium. So, what really happened to this ball? So how will you see that whether the equilibrium is stable or not?. So, for that, what we did? We want to perturb it, so we took it here, we have given little perturbation and wanted to see whether it come back to its original configuration or not, right?. You see that or we perturb it here and then we see whether it come back to its original configuration or not. In both the cases, whatever perturbation you see, see this perturbation can be expressed in terms of θ, angle θ or angle θ whatever right. In both the cases it is coming back. Why it is coming back? Because a component of its weight mg that is basically giving a restoring force to come back to its original configuration and that is why the equilibrium is stable. So, but in order to see the nature of equilibrium, what we are doing? We are perturbing it. Do you see? We are perturbing it to check what is behavior in a deformed configuration and that is what I am taking. The stability equilibrium we need to assess in deformed or perturbed configuration.



So that is the first difference. So, until now when we were only concerned about the stability or equilibrium itself, we do not care about the perturbed configuration. But here we have to care about the perturbed configuration or deformed configuration and then we have to write down in the equilibrium equation, right. Similarly, if we consider the same bowl but it is put in a concave, okay. Here it is convex, here it is, and then you put the ball over here, right?. And then what is going to happen? You see what is going to happen now?. If you just give a little perturbation, this fellow will come and this fellow will diverge. It will fall. That means it is not coming back to original configuration. Why? Once again, the component of it, weight is basically allowing it to diverge. So, this is unstable configuration. Once again, when we are ascertaining that it is, the equilibrium is unstable, we have to perturb the configuration and this perturbation is given in terms of θ, right?. Now, there can be a third possibility. that there is a plane and on the top of it there is this ball is kept and here the wherever you put the ball it does not really care, it will stay in its in position. So, this is referred as neutral equilibrium right. So, equilibrium is the first order information and then higher order information is the nature of equilibrium, whether it is stable, whether it is unstable or neutral. And the thing we learned that in order to study the nature of equilibrium, we have to see its behavior in the deformed or perturbed configuration. Okay, so that is the first lesson of stability.

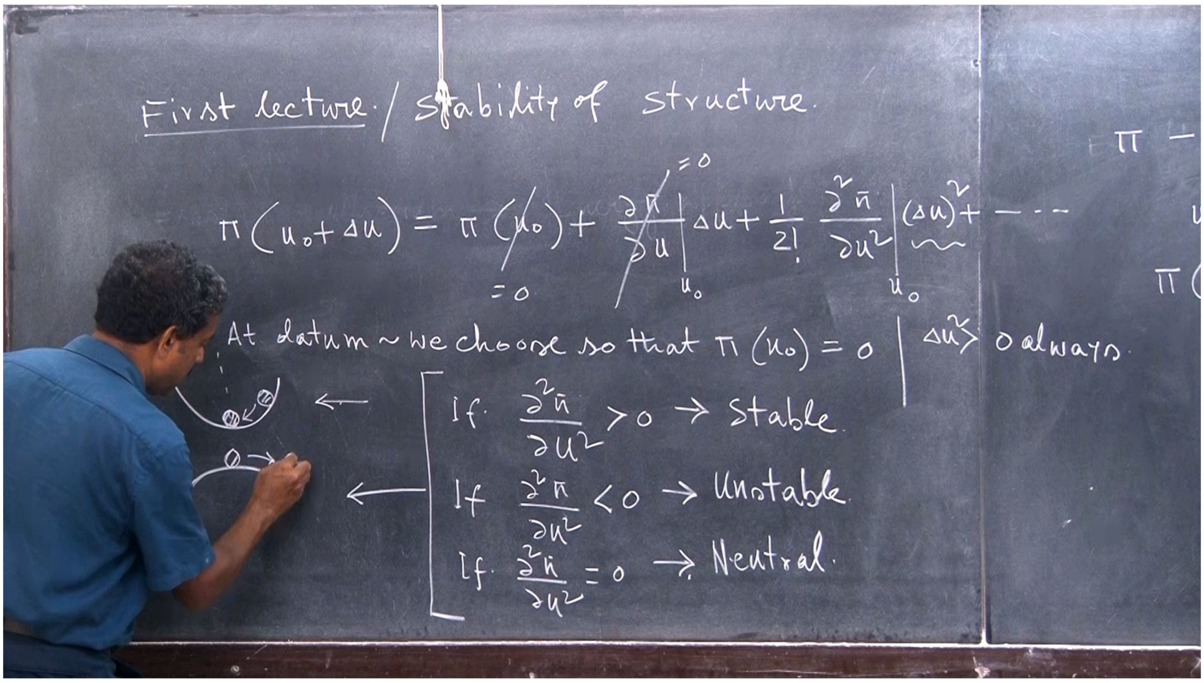
Now, the way we are going to study first, all of you can recall that maybe that will come later. But here what are you going to do that, of course when you study, can we define some energy?. can we define something with some measure of the stability? or we can quantify it mathematically? yes, we can. What we have seen, most convenient way to analyze a system is in the elastic system, and that system is equilibrium. You can recall right, we have our energy approach. In energy approach what we did? Energy approach we have all learned in our strength of material or even in structural analysis right. that all equilibrium is the mini is based on minimization of the potential energy right. That means potential energy we are defining in terms, we are denoting using Π. what is potential energy?. It is the strain energy minus work done. how it is coming. So, this is strain energy, U strain energy minus work done, right?. So, potential energy for an elastic structure is given by strain energy minus work done right. And how this is coming that the strain energy of course is a component of the energy stored in the system. If it is doing some work that means this work this is being the energy is dissipated. So, that is what it must be subtracted. So, that is the basis right. So, once we have potential energy. Functional then, for equilibrium, what we know ? Minimization of the potential energy.



That means, what does it mean? $\frac{\partial \Pi}{\partial u} = 0$. I am assuming that its degree of freedom is given by u. right, u is the degrees of freedom, so a system can have certain degree of freedom, so when we are writing potential energy, Π, Π is a function of what?. It is a function of course the force being applied on the structure and then its displacement and now force is related to displacement anyway, $k \cdot u$ is the elastic structure, so I can express it just solely as a function of the displacement, that is what u is the displacement, so I can write for minimization $\frac{\partial \Pi}{\partial u} = 0$. So, that is a minimization of the potential energy, right. So, if minimization of the potential energy is referring to the equilibrium, then we will see that whether the equilibrium is stable or unstable can be find out by scrutinize or by investigating its higher order derivative. How? So, what we see that it is a function of u, u is the generalized displacement ok. along different degrees of freedom ok. So, Π is a function of u, and we are giving a perturbation, so u we can put it u_0, u_0 is the initial configuration, u_0 some initial or datum, configuration right, so $\Pi(u_0)$ and then when we are giving a perturbation, then it is being $u_0 + \delta u$, δu is the perturbation displacement. So, what is happening in terms of perturbation and displacement?. Now, can we use, so we have learned to use, we are trying to express everything in terms of potential energy. We use potential energy functional; we are giving by perturbation because we know that we have to perturb it. So, can we use Taylor series expansion? Around this datum,

$$\Pi(u_0+\delta u)=\Pi(u_0)+\frac{\partial \Pi}{\partial u}|_{u_0}\delta u+\frac{1}{2!}\frac{\partial^2 \Pi}{\partial u^2}|_{u_0}(\delta u)^2+\cdots$$

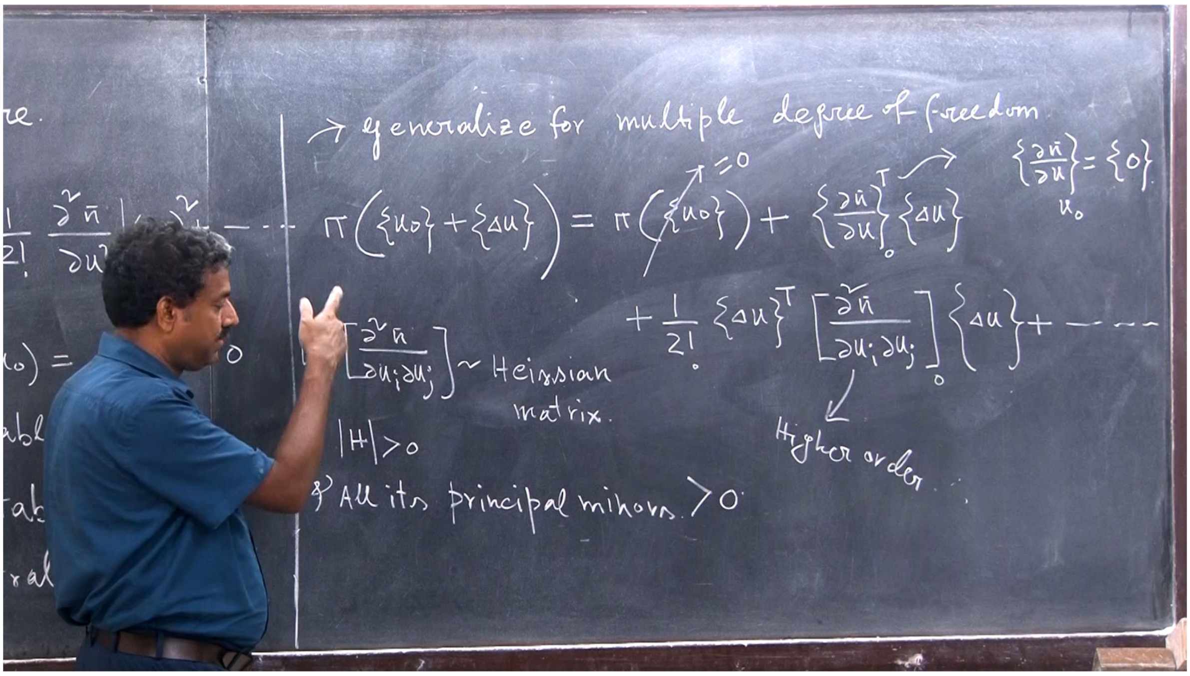
Now, I am assuming that datum we can assume that at datum, we can choose the datum, such that, our potential energy in the datum is 0, if u_0 is 0. How it is 0? For example, you see, if you consider this concave bowl and then at the below, we are putting the bowl and then we are treating that at the lowest point to be the datum, then this is 0, right, so this is 0, if this is 0, then What we have seen for equilibrium? Minimization means this is also 0, right? Minimization of the potential energy, right? Fine.



So, a system is in this configuration and then we are perturbing it to another configuration. Now if I am stable, then I will not go to that configuration, right?. Because when I will go to this configuration, if it lead to an energy, increment in energy, potential energy, that means I will see every system try to attain its minimum potential energy configuration. So, if when I am being perturbed from here to there, if there is an increase in energy, that means that configuration is not conducive for my move, right?. See $(\delta u)^2$, this is always positive, right? always positive So, if we can clearly see, if $\frac{\partial^2\Pi}{\partial u^2} > 0$, that means the system is stable, why it is stable?. Because by perturbing, you are taking it to a configuration in which its potential energy is increasing and all the system try to attain the minimum potential energy configuration. So, this > 0 is a condition for stable equilibrium, right. So that means this configuration of the system will not go, it will try to remain in its u_0 configuration, right?. However, different, that means if $\frac{\partial^2\Pi}{\partial u^2} < 0$, that means when it is going to u_0 to $u_0 + \delta u$, if there is a reduction in the potential energy, that means it will try to attain that. because by doing so, it is minimizing the potential energy, so it will always try to move there, that means whatever is the equilibrium at u_0 that is unstable, clear?. And now if $\frac{\partial^2\Pi}{\partial u^2} = 0$, that means by perturbing from u_0 to $u_0 + \delta u$, there is a no change. So, that it does not really care whether you put u_0, it is neutral equilibrium, right. I hope that all of you understand that why this is associated. So, you can clearly see that the first case here, you see that is here, where the ball is here in the lowest position, right?. You are taking it to this position. It will clearly come here. Why? Because when you are taking from here to here, there is a positive increase in potential energy. So, $\frac{\partial^2\Pi}{\partial u^2} > 0$. This when it is here, from here, you are perturbing it to here, it is a reduction in potential energy. So, $\frac{\partial^2\Pi}{\partial u^2} < 0$ and then for the other. I will just try to do it using, so this is for a system which is single degree of freedom, right?. This system, I am considering that u_0 is the only displacement quantity, that means a single degree of freedom. If I generalize it for multi-degree of freedom system, what will happen? Generalize it, okay?. You see that here is a potential energy, the only thing is that u_0 will be a vector because it is multiple u_{01}, u_{02}, u_{03} and then your perturbation will also have δu. Of course, when you perturb it, you have to keep in mind that you have to enforce the boundary, you have to maintain the boundary conditions, right?. There must be admissibility, right? once again

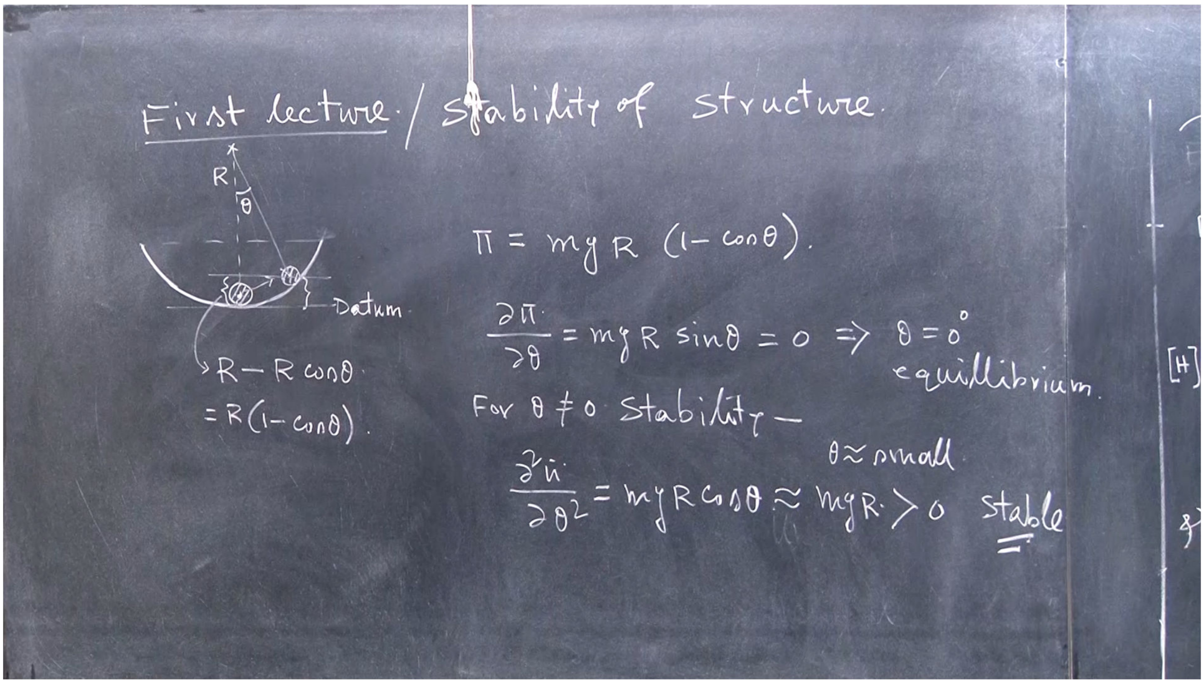
$$\Pi(u_0 + \delta u) = \Pi(u_0) + \frac{\partial\Pi}{\partial u}|_{u_0} \cdot \delta u + \frac{1}{2!}(\delta u)^T \frac{\partial^2\Pi}{\partial u_i\, \partial u_j}|_{u_0}\delta u + \cdots$$

Higher order term we do not consider, not that, it cannot, it should not be considered, please note that if this is 0, that means if it is non-decisive, inconclusive in terms of $\frac{\partial^2\Pi}{\partial u^2}$, then you have to look for higher order derivative, please note that, okay. But if it is here, so just we are omitting the higher order term does not mean that those are irrelevant. There is system in which the stability cannot be asserted using second derivative. Those will be vanishing. So, then you will, we will show using solved example that then you have to look for higher order derivative, fourth order, sixth order, something like that.



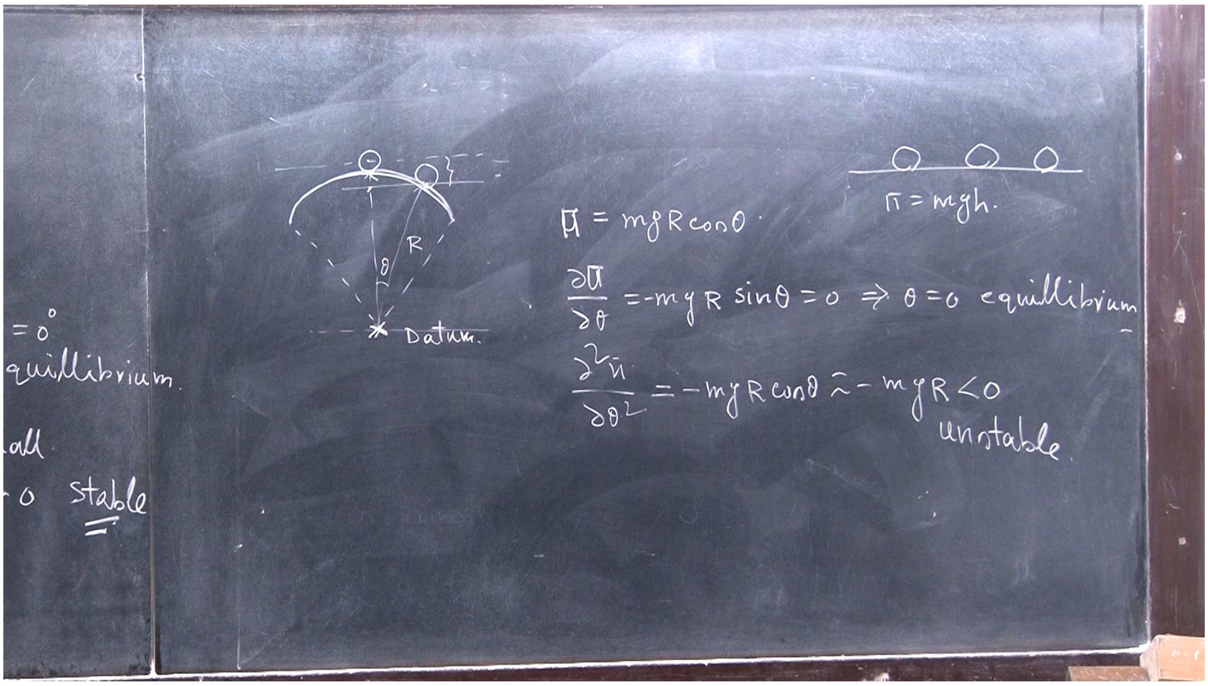
So here you see, minimization of the potential and so if the datum is taken to be 0, minimization of the potential energy, this gives δu, anything, it can be positive, negative, it does not. So, minimization of the potential energy makes your, that $\frac{\partial \Pi}{\partial u}$ at $u = u_0$ is equal to 0, right. This is minimization of the potential energy, okay? That ensures that the system is in equilibrium. So, this is the first order information. That means the system is in equilibrium. Now the higher order information, right?. Higher order information for a multi-degree freedom system. So here what will happen? Of course this is $\frac{\partial^2 \Pi}{\partial u_i \partial u_j}$. This is a matrix. This is called Hessian matrix. This is called Hessian of the potential energy. Those of you who did Hessian matrix, who did optimization course, you will know because this is used in steepest descent method for optimization to find out the minimum point, okay, for the numerical search technique. For the Hessian matrix where, now this is a matrix, so stability condition will be the, this matrix, so I am defining that Hessian H for the potential. So, determinant of Hessian must be > 0 for stability and all its principal minor, what is minor for a matrix, right?. principal minors must be > 0 for the stability. For the stability condition, the determinant of the Hessian matrix must be > 0 and all its principal minor. Minors meaning, the respective to principle minus corresponding to the diagonal term. If you take the diagonal term, then whatever the sub matrices remain, the determinant of that, right?. So, those are principal minus. So, if the Hessian > 0, all of principal minus > 0, then it is stable. If the Hessian matrix < 0 and principle, any one of these principal minuses < 0, then it is unstable. And if all these vanishes, it is stable. So, you understand from the single degree as well as multiple degree. Here the derivative of the potential energy function. determinant of the Hessian, once again this is nothing but the second order derivative of the potential energy function but the only problem because you have a multiple degree of freedom system, you will have a varying combination of the, among the degrees of freedom right, (1, 2),(2,3),(3,1) you want something like that. So, we can solve if you want, I mean you can solve a simple system.

So, this system if you want to solve mathematically maybe here the ball is put, ok. This ball, and then you see that. Here I am assuming this is the radius of curvature r right and then it is perturbed to a different configuration right here and it is going there right this is r, this angle is θ. The first step is the perturb the configuration, perturbing it. So, I am assuming that this I am treating as datum, this point ok, because that means here, when it is here, then it is zero potential energy, right. So, how potential energy is defined here? This position will move here, right? So, what is the potential energy?. $m \cdot g$, the work done, $m \cdot g$, I am assuming mass is m, g is the gravitational acceleration, right?. And then this distance is nothing but this is r and this $r - r\cos\theta$, right?. This distance, right? This distance. So that means $r(1 - \cos\theta)$. So $\Pi = mgr(1 - \cos\theta)$. Now you see for equilibrium $\frac{\partial \Pi}{\partial \theta} = mgr\sin\theta$. And of course, if you want to make it 0, then means what? $\sin\theta = 0$, so $\theta = 0$, that is the equilibrium. You see that? Equilibrium you will get automatically, you see. So, it will be equilibrium when $\theta = 0$, right?. Now, for θ non-zero, for non-zero θ, then for stability, $\frac{\partial^2 \Pi}{\partial \theta^2}$ with a single degree of freedom system because everything potential energy is expressed in terms of single variable, displacement variable θ.



So double derivative is what? $mgr\cos\theta$. So, I am assuming if θ is small, that means for small θ, θ small mgr, right?. So, θ small, very very small quantity and this is always > 0, so that means it is stable, understand that?. So, that way we can prove that, this is stable, right?. Similarly, you can also solve the problem where it is concave surface, right?. here you just write the ball keep that then you are putting it here so from here here you are giving here so I am treating maybe this to be datum here from here so what is happening from here to here when you will get what is potential energy potential energy is what potential energy will be if I treat this to be datum this point then if this is r, so r, this angle is theta, so it is $mgr\cos\theta$, this is the datum this point, point of curve the center of curvature, right, $\frac{\partial\Pi}{\partial\theta} = -mgr\sin\theta$, then $\theta = 0$ is the equilibrium right, equilibrium and double derivative $\frac{\partial^2\Pi}{\partial\theta^2}$ is $-mgr\cos\theta = -mgr$, so always < 0 it is unstable. Do you see that? Unstable. And then if you derive any system in the flat plane here potential energy will be what? mgh. And then difference always it is 0. $\frac{\partial^2\Pi}{\partial\theta^2}$? Okay? So, it does not really. Okay? Clear?.

Now we will solve problem for the multi degree of freedom system in the next class but before that so one more thing that I will just try to emphasize with think what we have learned from our undergraduate course from ok undergraduate course we know how to determine the critical load for a column due to buckling ok.

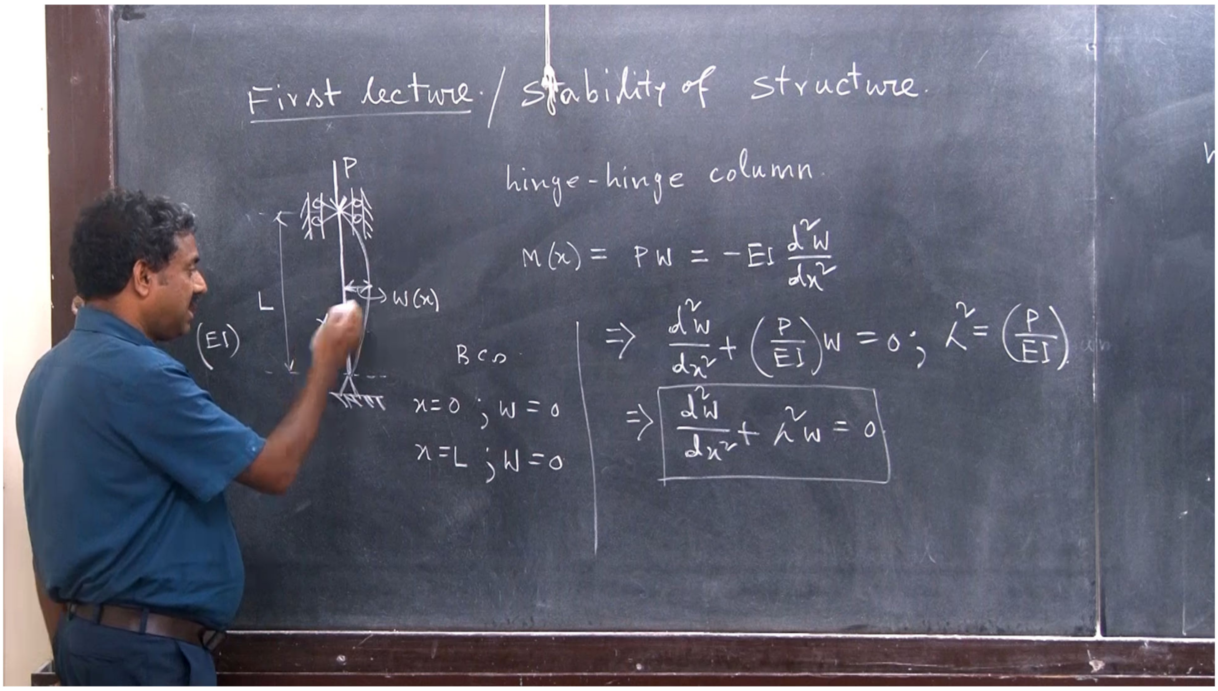


We are going to consider the same with a little more insight. So, for that if we allow it to deform, this is something like this, right. P, load P is given. This is hinge and that is also hinge. I am assuming this is to be length L and I am assuming this is flexural rigidity EI. Now of course we have to allow translation for this. So, then P we this so this is hinge-hinge column right. We will write down, we will try to study its equilibrium its behavior, stability behavior, okay. So, of course, the one way to do this is following the potential energy approach, right. But we have not used that approach in the undergraduate. so, in undergraduate we have written down the we have perturbed it. So, we have perturbed it means we have assumed that well let there be this $w(x)$ and x is assumed in the x right along longitudinal axis. So, when I am a perturbing, once again the first lesson for a stability analysis to perturb the configuration. That's what we are considering ok. Then what we are doing? Because of this perturbation, there is an additional bending moment that is going to come. So, if this load P is there, P, so $P \cdot w(x)$, that is the bending moment, $P \cdot w$, right?. And that is nothing but what? $-EI\frac{d^2w}{dx^2}$, bending, you assume that our deflection is infinitely small, the perturb configuration is infinitely small, we are assuming small deflection, right?. So, then we can write it, so that is what.

$$\frac{d^2w}{dx^2} + \frac{P}{EI}w = 0$$

So, P/EI is that means $\frac{d^2w}{dx^2}$m we are defining $\frac{P}{EI} = \lambda^2$, so

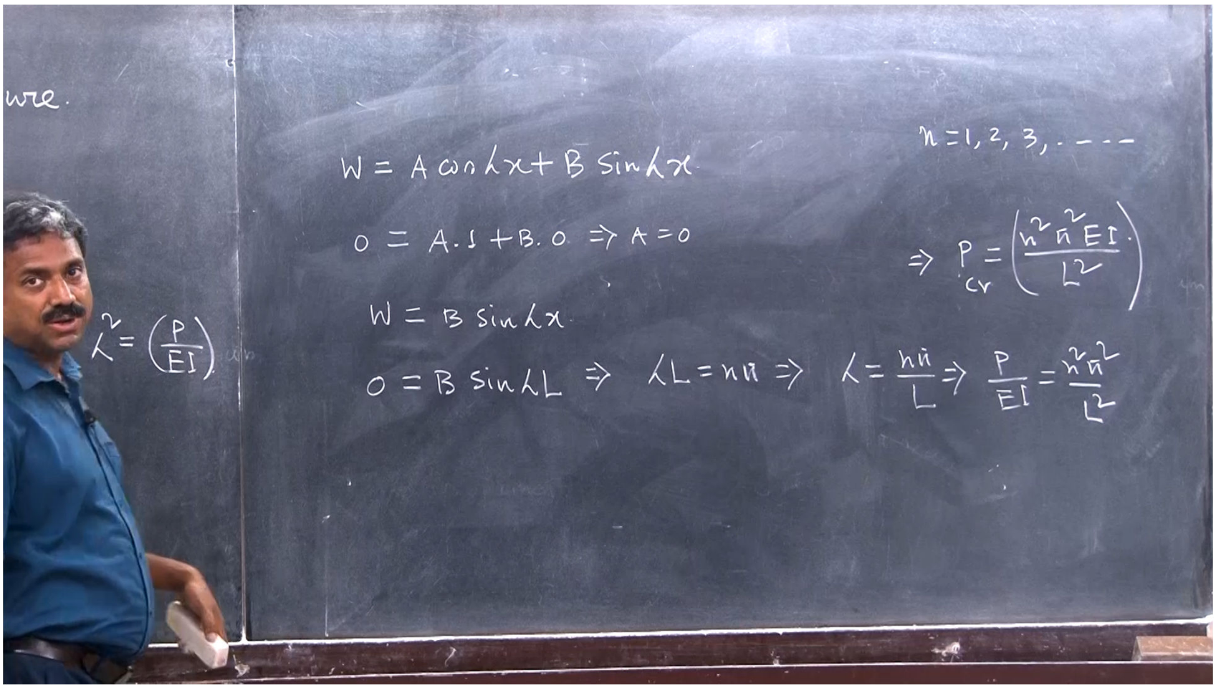
$$\frac{d^2w}{dx^2} + \lambda^2 w = 0$$



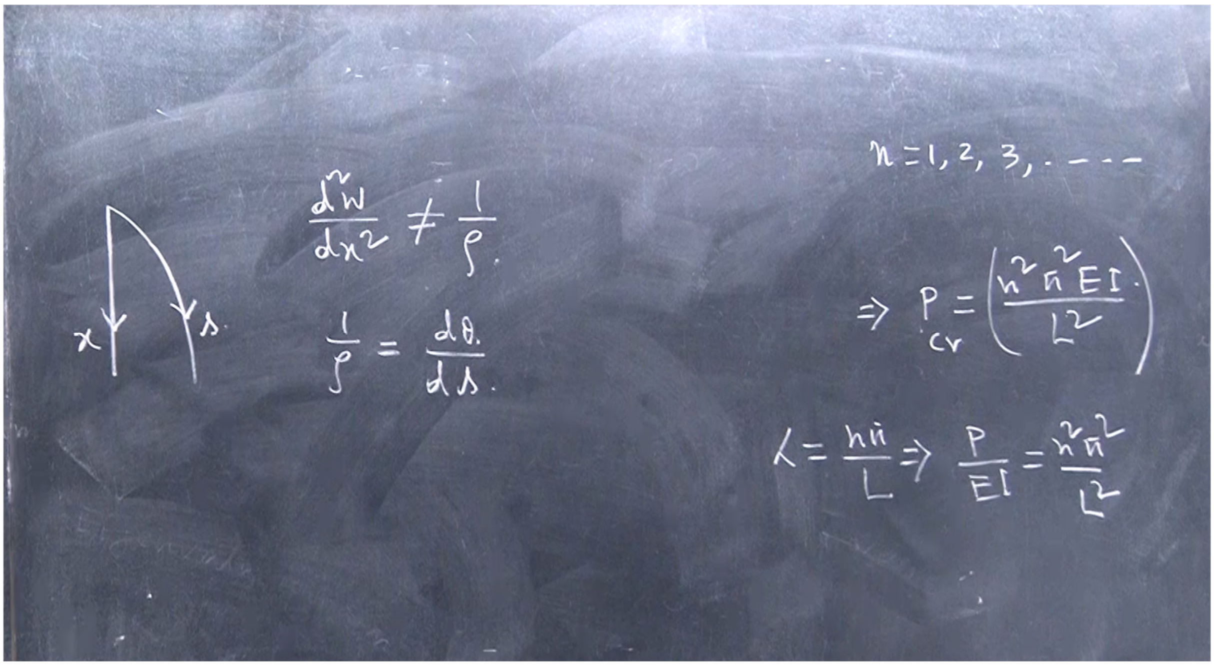
So, it led to a second order differential equation, right?. We know how to solve second order homogeneous differential equation. Of course, we have the boundary conditions. The boundary conditions, what are the boundary conditions? at $x = 0$, $w = 0$, no deflection because it is inch, at $x = L$, L length, deflection is 0, right, two boundary conditions. So, second order equation, two boundary conditions, both are expressed in terms of w, that is permissible. So how to, the solution of this differential equation as you see, $w(x) = A\cos(\lambda x) + B\sin(\lambda x)$, that we know and it is homogeneous equation, so there will be no particular integral. When you enforce the boundary conditions that at $x = 0$, $w = 0$, $x = 0$ means $A \cdot 1 + B \cdot 0$ so essentially $A = 0$ and $w = B\sin(\lambda x)$, $w = 0$ at $x = L$, so $B\sin(\lambda L) = 0$ means $\lambda L = n\pi$. where $n = 1,2,3$, where $\lambda = n\frac{\pi}{L}$, λ^2 means $\frac{P}{EI} = \frac{n^2\pi^2}{L^2}$. or

$$P = \frac{n^2\pi^2 EI}{L^2}$$

So, n is the natural number 1, 2, 3. You see that this is the critical load P_{cr}. What we see? There are several things to note here. Stability analysis, here see buckling, we are finding out the stability in alternate configuration, although right now we are not showing it whether it is stable or unstable. What we want to find out that whether is alternate configuration exist or not. Buckling is one such thing in which there is an alternate configuration exists, that means from the fundamental path at $w = 0$ it going into a bifurcated path, buckling. Because this is bifurcating to an alternate configuration and also another thing is that it is also called symmetry breaking because after buckling it will go only in one direction it cannot go to other direction whether it is right or left it doesn't matter. But once it buckles it will try to diverge in one place, right.



And that led to what is this equation, this is homogeneous equation. From here when I am writing here, so this thing, this is nothing but an eigenvalue problem and these are called eigenvalues. So, these eigenvalue problems are in terms of what? In terms, both are, see, these are the homogeneous boundary condition and homogeneous boundary condition lead to eigenvalue problem, okay. And these are trigonometric eigenvalue problems. Eigen value problem. You will see all these stability analyses will lead to an eigenvalue problem, if it is linearized stability analysis. There is nonlinear stability analysis which will be different, but for linearized stability analysis will lead to an eigenvalue problem and that is how we got it. So, here how the system is linearized? We are assuming the deflection to be infinitely small, okay. However, if you want to study the non-linearity, then you have to consider large deformation that we are going to discuss. Large deformation means $\frac{d^2w}{dx^2}$ is not the curvature, okay. It is not the curvature. Rather, curvature is, will be given as $\frac{d\theta}{ds}$. and where this, this is s and this is x. So, this we are, we will be distinguishing between x and s. Here we are not distinguishing between this x and this x because both are same, we are assuming, right.



That is why this non-linearity is being linearized here, okay. But that we can put in. And we can also define energy potential function here. That maybe subsequent class we will say, okay. But before that we are going to do a multi-degree of freedom system and stability analysis and we will try to first use. the potential energy approach because that itself will give us a huge insight into the problem, okay. And we will see various classes of system and accordingly we will continue. Thank you very much for today's lecture.