NPTEL ONLINE CERTIFICATION COURSES

EARTHQUAKE SEISMOLOGY

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Module 01 : Basic Seismological Theory, Waves on a String, Stress and Strain and seismic waves
Lecture 05: Equations Of Motion, P-and S-wave motion

CONCEPTS COVERED

- **Summary Of Previous Lectures**
- **Equation Of Motion**
- **P-wave and S-wave motion**

Summary of previous lecture

$$\sigma_{ji} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} = \begin{bmatrix} T^{(1)} \\ T^{(2)} \\ T^{(3)} \end{bmatrix} = \begin{bmatrix} T_1^{(1)} & T_2^{(1)} & T_3^{(1)} \\ T_1^{(2)} & T_2^{(2)} & T_3^{(2)} \\ T_1^{(3)} & T_2^{(3)} & T_3^{(3)} \end{bmatrix}$$

And the traction vector is given by (Cauchy Stress Theorem)

$$T_i = \sigma.\hat{\mathbf{n}}$$

The strain tensor is given by

$$e_{kl} = \frac{1}{2}\left\{\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k}\right\}$$

The constitutive relationship for an elastic, isotropic medium is

$$\sigma_{ij} = C_{ijkl}e_{kl} = ((\lambda\delta_{ij}\delta_{kl}) + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}))e_{kl}$$

The relationship can also be expressed as :

$$\sigma_{ij} = \lambda\delta_{ij}\theta + 2\mu e_{ij}$$

Volume shear

$$\theta = \nabla . u = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = e_{11} + e_{22} + e_{33}$$

Cubic dilatation or divergence of the displacement field.

Basic Calculus

Scalar field

Every point in space is assigned a scalar value. Values vary with position and denoted by

$$\phi(x) \quad or \quad \phi(x_1, x_2, x_3)$$

Vector field

Every point in space is assigned a vector. Values and directions varies with position and denoted by

$$\begin{aligned}\mathbf{u}(x) &= \mathbf{u}(x_1, x_2, x_3) \\ &= u_1(x_1, x_2, x_3)\hat{e_1} + u_2(x_1, x_2, x_3)\hat{e_2} + u_3(x_1, x_2, x_3)\hat{e_3}\end{aligned}$$

Spatial variations of scalars, vector, or tensor fields are described using the vector differential operator "del" ∇,

$$\nabla = \left(\hat{e}_1 \frac{\partial}{\partial x_1}, \hat{e}_2 \frac{\partial}{\partial x_2}, \hat{e}_3 \frac{\partial}{\partial x_3} \right)$$

Gradient

Gradient is a vector field formed from spatial derivatives of a scalar field

If Φ(x) is a scalar function of position, the gradient is defined by

$$\text{grad } \phi(x) = \nabla\phi(x) = \left(\frac{\partial \phi(x)}{\partial x_1} \hat{e}_1 + \frac{\partial \phi(x)}{\partial x_2} \hat{e}_2 + \frac{\partial \phi(x)}{\partial x_3} \hat{e}_3 \right)$$

partial derivatives

Divergence

It describes the spatial variation of a vector field u(x), given by the scalar product of del operator with u(x).

$$\operatorname{div} u = \nabla . u = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}$$

Divergence measures the net flow of fluid (or material) from a given point.

If the divergence at a point is positive implies that point is a source

and if divergence at a point is negative indicates that it is a sink

Curl

The cross product of the ∇ operator with a vector field yields a another vector field.

$$\nabla \times \mathbf{u} = \hat{e}_1\left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}\right) + \hat{e}_2\left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}\right) + \hat{e}_3\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right)$$

$$\nabla \times \mathbf{u} = \begin{pmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ u_1 & u_2 & u_3 \end{pmatrix}$$

Curl measures the degree to which the fluid or material is rotating about a given point.

$$\nabla.(\nabla \times \mathbf{u}) = 0$$

Divergence of a curl is zero; represents no volume change give rise to shear waves

$$\nabla \times (\nabla \phi) = 0$$

Curl of a gradient is zero i.e no curl or rotation & give rise to compressional waves.

Laplacian

Divergence of the gradient of a scalar field. Represented by ∇^2

$$\nabla^2\phi = \nabla.\nabla\phi = \frac{\partial^2\phi}{\partial x_1^2} + \frac{\partial^2\phi}{\partial x_2^2} + \frac{\partial^2\phi}{\partial x_3^2}$$

The laplacian of a scalar field an another scalar field.

An important identity related to laplacian:

$$\nabla^2\mathbf{u} = \nabla(\nabla.\mathbf{u}) - \nabla\times(\nabla\times\mathbf{u})$$

Equation of Motion

Equation of motion satisfies "Newton's Second law, F=ma, in terms of surface and body forces .

According to this equation acceleration results from the body forces and $\sigma_{ij,\,j}$, the divergence of stress tensor.

$$\frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{32}}{\partial x_3} + f_2 = \sum_{j=1}^{3} \frac{\partial \sigma_{j2}}{\partial x_j} + f_2 = \rho \frac{\partial^2 u_2}{\partial t^2}$$

? X_2 component of the forces

Note: Equation of motion relates stress tensor to ground motion (or displacement).

In summation convention it can be written as

$$\frac{\partial \sigma_{ij}(x,t)}{\partial x_j} + f_i(x,t) = \rho \frac{\partial^2 u_i(x,t)}{\partial t^2}$$

Because the stress-tensor is symmetric, so

$$\frac{\partial \sigma_{ji}(x,t)}{\partial x_j} + f_i(x,t) = \rho \frac{\partial^2 u_i(x,t)}{\partial t^2}$$

- It is interesting to note that the divergence of the stress tensor give rise to a force, which is a vector, just as the divergence of a vector yields a scalar.
- If the body is at equilibrium, then acceleration must be zero such that

$$\frac{\partial \sigma_{ji}(x,t)}{\partial x_j} = -f_i(x,t)$$ ⟵ Eq^n of Equilibrium

If no body force is applied

$$\frac{\partial \sigma_{ji}(x,t)}{\partial x_j} = \rho \frac{\partial^2 u_i(x,t)}{\partial t^2} \longrightarrow$$

Homogeneous eq^{n} of motion

This is called the homogeneous equation of motion, where “Homogeneous” refers to the lack of forces. This equation describes seismic wave propagation, except at a source, such as an earthquake or an explosion, where body force generates seismic waves.

P-waves and S-waves

Equation of Motion (E.O.M)

- **E.O.M can be written and solved entirely in terms of displacements, because stress is related to strain, which is formed from derivatives of displacement.**
- **The equation of motion relates spatial derivatives of stress tensor to a time derivatives of displacement vector (or ground motion). The resulting solution gives the displacement vector and hence the strain and stress tensors as function of both space and time**

$$\frac{\partial \sigma_{xx}(x,t)}{\partial x} + \frac{\partial \sigma_{xy}(x,t)}{\partial y} + \frac{\partial \sigma_{xz}(x,t)}{\partial z} = \rho \frac{\partial^2 u_x(x,t)}{\partial t^2}$$

Since we are considering homogeneous medium so the above equation do not possess any source term.

A homogeneous equation has no forcing function or source term

$$\sigma_{ij} = \lambda\theta\delta_{ij} + 2\mu e_{ij}$$

$$\sigma_{xx} = \lambda\theta + 2\mu e_{xx} = \lambda\theta + 2\mu\frac{\partial u_x}{\partial x}$$

$$\sigma_{xy} = 2\mu e_{xy} = \mu\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right)$$

$$\sigma_{xz} = 2\mu e_{xz} = \mu\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right)$$

We then take the derivatives of the stress components given in previous slide, we will get

$$\frac{\partial \sigma_{xx}}{\partial x} = \lambda \frac{\partial \theta}{\partial x} + 2\mu \frac{\partial^2 u_x}{\partial x^2}$$

$$\frac{\partial \sigma_{xy}}{\partial y} = \mu \left(\frac{\partial^2 u_x}{\partial y^2} + \frac{\partial^2 u_y}{\partial y \partial x} \right)$$

$$\frac{\partial \sigma_{xz}}{\partial z} = \mu \left(\frac{\partial^2 u_x}{\partial z^2} + \frac{\partial^2 u_z}{\partial z \partial x} \right)$$

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} = \lambda \left[\frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_y}{\partial x \partial y} + \frac{\partial^2 u_z}{\partial x \partial z} \right]$$

$$+\mu \frac{\partial^2 u_x}{\partial x^2} + \mu \frac{\partial^2 u_x}{\partial x^2} + \mu \frac{\partial^2 u_x}{\partial y^2} + \mu \frac{\partial^2 u_y}{\partial y \partial x} + \mu \frac{\partial^2 u_x}{\partial z^2} + \mu \frac{\partial^2 u_z}{\partial z \partial x}$$

$$= (\lambda + \mu) \frac{\partial^2 u_x}{\partial x^2} + \left(\frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial y^2} + \frac{\partial^2 u_x}{\partial z^2} \right) \mu + (\lambda + \mu) \left[\frac{\partial^2 u_y}{\partial x \partial y} + \frac{\partial^2 u_z}{\partial x \partial z} \right]$$

$$= (\lambda + \mu) \left[\frac{\partial}{\partial x} \left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} \right) \right] + \mu \left(\frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial y^2} + \frac{\partial^2 u_x}{\partial z^2} \right)$$

For x-component of equation of motion

$$(\lambda + \mu)\left(\frac{\partial \theta}{\partial x}\right) + \mu \nabla^2 u_x = \rho \frac{\partial^2 u_x}{\partial t^2}$$

For y-component of equation of motion

$$(\lambda + \mu)\left(\frac{\partial \theta}{\partial y}\right) + \mu \nabla^2 u_y = \rho \frac{\partial^2 u_y}{\partial t^2}$$

For z-component

$$(\lambda + \mu)\left(\frac{\partial \theta}{\partial z}\right) + \mu \nabla^2 u_z = \rho \frac{\partial^2 u_z}{\partial t^2}$$

These three equation can be combined to get

$$(\lambda + \mu)\nabla(\nabla . u(x,t)) + \mu\nabla^2 u(x,t) = \rho \frac{\partial^2 u(x,t)}{\partial^2 t}$$

As $\nabla^2 u = (\nabla^2 u_x, \nabla^2 u_y, \nabla^2 u_z)$

It has three component, so it's a vector quantity

Where is P and S-waves ?

We need to separate this into two different wave equation for P and S-waves

We know that :

$$\nabla^2 = \nabla(\nabla . u) - \nabla \times (\nabla \times u)$$

The previous equation will become

$$(\lambda + 2\mu)\nabla\underbrace{(\nabla.\, u(x,t))}_{M} - \mu\underbrace{\nabla \times (\nabla \times u(x,t))}_{N} = \rho\frac{\partial^2 u(x,t)}{\partial t^2}$$

← **Elastodynamic equation**

To solve the above equation, we will use Helmholtz equation (or Helmholtz Decomposition) which decomposes 'u' into its scalar (Ø) and vector potential (γ)

$$u(x,t) = \nabla\phi(x,t) + \nabla \times \gamma(x,t)$$

Here, Ø is scalar potential (P-waves) and γ is the vector potential (S-waves)

We will use the following vector identities

$$\nabla \times (\nabla\phi) = 0 \quad | \quad \nabla.\, (\nabla \times \gamma) = 0$$

Represents no curl or rotation and gives rise to compressional wave

Represents no volume change and corresponds to shear waves

$$
\begin{aligned}
M &= \nabla[\nabla.\, u(x,t)] \\
&= \nabla[\nabla.\,(\nabla\phi(x,t) + \nabla \times \gamma(x,t))] \\
&= \nabla\left(\nabla^2\phi\right) + \nabla.\,(\nabla \times \gamma(x,t)) \\
&= \nabla\left(\nabla^2\phi(x,t)\right)
\end{aligned}
$$

Likewise,

$$
\begin{aligned}
N &= \nabla \times [\nabla \times u(x,t)] \\
&= \nabla \times [\nabla \times (\nabla\phi(x,t) + \nabla \times \gamma(x,t))] \\
&= \nabla \times \nabla \times (\nabla \times \gamma) = -\nabla^2(\nabla \times \gamma)
\end{aligned}
$$

So,

$$(\lambda + 2\mu)\nabla\left(\nabla^2\phi(x,t)\right) - \mu\nabla^2(\nabla \times \gamma) = \rho\frac{\partial^2}{\partial t^2}(\nabla\phi + \nabla \times \gamma)$$

$$\nabla\left[(\lambda + 2\mu)\nabla^2\phi(x,t) - \rho\frac{\partial^2\phi(x,t)}{\partial t^2}\right] = -\nabla \times \left[\mu\nabla^2\gamma(x,t) - \rho\frac{\partial^2\gamma(x,t)}{\partial t^2}\right]$$

One of the solution of this equation can be found if both terms in the brackets are zero

$$\nabla^2\phi(x,t) = \frac{1}{\alpha^2}\frac{\partial^2\phi(x,t)}{\partial t^2} \longrightarrow \textbf{A}$$

With the velocity $\alpha = \sqrt{\frac{\lambda + 2\mu}{\rho}}$

P-waves

$$\nabla^2 \gamma(x,t) = \frac{1}{\beta^2} \frac{\partial^2 \gamma(x,t)}{\partial t^2} \longrightarrow \textbf{B}$$

$$\beta = \sqrt{\frac{\mu}{\rho}}$$

S-waves

Equation (A) $$\nabla^2\phi(x,t) = \frac{1}{\alpha^2}\frac{\partial^2\phi(x,t)}{\partial t^2}$$ Wave equation for the P-wave

The scalar potential satisfying the above equation is

$$\phi(z,t) = A\exp\left(i(\omega t - kz)\right)$$

so, the resulting displacement is the gradient

$$u(z,t) = \nabla\phi(z,t) = (0,0,-ik)A\exp\left(i(\omega t - kz)\right)$$

which has a non-zero component only along the propagation direction z. The corresponding dilatation is non-zero.

$$\nabla.\,u(z,t) = -k^2 A\exp\left(i(\omega t - kz)\right)$$

So, a volume change occurs

As the wave propagates, the displacement in the direction of propagation cause material to be alternatively compressed and expanded . Thus the P-wave generated by scalar potential is called a “Compressional Wave”.

Likewise of S-wave, described by vector potential

$$\gamma(z,t) = (A_x, A_y, A_z)\exp\left(i(\omega t - kz)\right)$$
$$u(z,t) = \nabla \times \gamma(z,t) = (ikA_y, -ikA_x, 0)\exp\left(i(\omega t - kz)\right)$$

whose component along the propagation direction is zero i.e., displacements are perpendicular to the direction of propagation

So,

- P-wave causes change in volume
- Shear wave cause no volume change

Revision of Module 1

- We used force balance to derive the 1-D wave equation.
- We did an overview of parameters that describe harmonic waves. The wave number, k, may be new to you.
- We derived reflection and transmission coefficients.
- We looked at KE and PE averaged over a wavelength.
- An alternative method of solving a differential equation is through normal modes.
- Stress describes the force/area and is a 3x3 tensor
- The traction vector is the surface force per unit area on a plane with given normal.
- As per the engineering convention that tensional (stretching) stress is positive. That is why stress in the earth is negative (compressional) values.

Revision of Module 1

- Stress tensor has following properties
  - Stress is a symmetric tensor
  - can be rotated it into other coordinate system.
  - Coordinate system without the shear stresses gives rise to principal stresses.

    These are the eigenvalues of the stress tensor.
  - The trace of the stress tensor is independent of the coordinate system used.
- The deviatoric stress tensor is the stress tensor minus the pressure.
- The pressure is the mean of the trace of the stress tensor.
- Units of stress are N/m^2 = Pascals
- 33 km increase in depth increases pressure by roughly 1 Gpa.

Revision of Module 1

1. C_{ijkl} completely describe the behaviour of an elastic material.

1. λ does not have any physical meaning, but μ is called "rigidity or shear modulus".

1. A material with large μ is quite rigid and responds to a given stress with less strain & vice-versa.

1. A material in which μ is zero can not support shear stresses, & corresponds to a perfect fluid.

Revision of Module 1

$$\sigma_{ji} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} = \begin{bmatrix} T^{(1)} \\ T^{(2)} \\ T^{(3)} \end{bmatrix} = \begin{bmatrix} T_1^{(1)} & T_2^{(1)} & T_3^{(1)} \\ T_1^{(2)} & T_2^{(2)} & T_3^{(2)} \\ T_1^{(3)} & T_2^{(3)} & T_3^{(3)} \end{bmatrix}$$

And the traction vector is given by (Cauchy Stress Theorem)

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Revision of Module 1

The constitutive relationship for an elastic, isotropic medium is

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Volume shear

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Cubic dilatation or divergence of the displacement field.

Revision of Module 1

Equation of motion

$$\frac{\partial \sigma_{ij}(x,t)}{\partial x_j} + f_i(x,t) = \rho \frac{\partial^2 u_i(x,t)}{\partial t^2}$$

Elastodynamic equation

$$(\lambda + 2\mu)\nabla(\nabla . u(x,t)) - \mu\nabla \times (\nabla \times u(x,t)) = \rho \frac{\partial^2 u(x,t)}{\partial t^2}$$

Revision of Module 1

P-waves

$$\nabla^2\phi(x,t) = \frac{1}{\alpha^2}\frac{\partial^2\phi(x,t)}{\partial t^2}$$

With the velocity $$\alpha = \sqrt{\frac{\lambda + 2\mu}{\rho}}$$

S-waves

$$\nabla^2\gamma(x,t) = \frac{1}{\beta^2}\frac{\partial^2\gamma(x,t)}{\partial t^2}$$

With the velocity $$\beta = \sqrt{\frac{\mu}{\rho}}$$

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THANK YOU !