

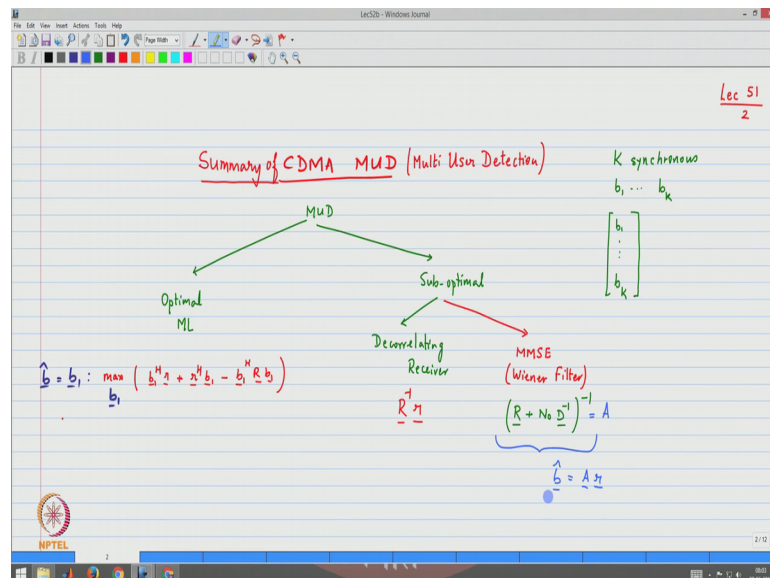
Introduction to Wireless and Cellular Communication
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Lecture – 52

Good morning and welcome to lecture 52 in our series of lectures in the introduction to wireless and cellular communications course. We will quickly review the contents covered in lecture 51 and then focus our attention on the topic for today which will be primarily on the multiple input multiple output framework in which is used widely used in fourth generation and the fifth generation systems.

So, for a quick review of the material covered in the last lecture, the at last lectures last couple of lecturers have been focusing on the topic of multiuser detection, and we have been looking at the three different solutions.

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The first one being the optimal or the maximum likelihood detection of the solution, which is given by the vector b_1 which maximizes the metric given in this expression, we said that this would require us to consider all possible combinations of the vector b_1 which grows exponentially in complexity.

So, therefore, the maximum likelihood receiver is something that would be very complex in terms of implementation, when there are large numbers of users and when the

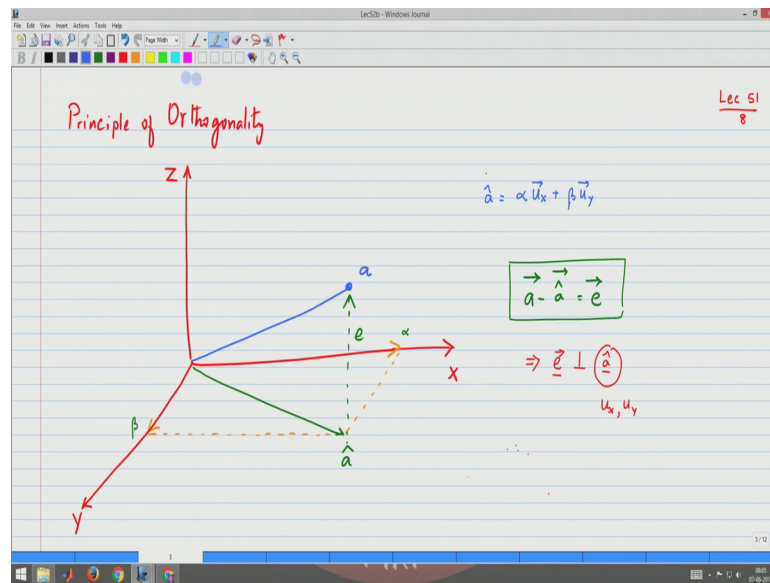
constellation of the signal being transmitted is of a large dimension. On the other hand we have sub optimal solutions, which perform reasonably well in several of the practical scenarios. So, if we have K synchronous users b_1 through b_k each of them have transmitted a symbol, the goal is to find identify the received symbol the transmitted symbol based on the received signal R . decorrelating receiver gives us a very simple and elegant solution, which says we can compute the receiver transmitted symbols by means by computing $R^{-1}r$ and then applying the decision based on this statistic.

The MMSE or the wiener filter which we discussed in the last lecture does something very similar way our goal is to obtain a linear based on a linear transformation of the received vector, the output of the K correlators. If you denote that as r we want to express the received the \hat{b} as A times r , and we have obtained the best expression for which will minimize the error between the decoded symbols and the transmitted symbol. So, that is given by $(R + N^{-1})^{-1}r$, and the whole inverse and this we found was a good way to summarize the three different options the optimal solution the decorrelating receiver, and the MMSE receiver we also showed how the MMSE receiver in the case where the noise is not very significant.

If you can ignore the noise actually corresponds to the decorrelating receiver, and in most of the practical scenarios where the noise may not be negligible, then the MMSE you would be a better option compared to the decorrelating receiver, because the decorrelating receiver by its structure may cause enhancement of the noise. So, that is a summary of the CDMA multiuser detection, the schemes and again I would encourage you to read have up from the different textbooks, and reference books about the different approaches that we can use for multiuser detection.

Now, one of the key points in the derivation of the MMSE solution was the principle of orthogonality, again I will just mention it so that we have a good summary of the recap.

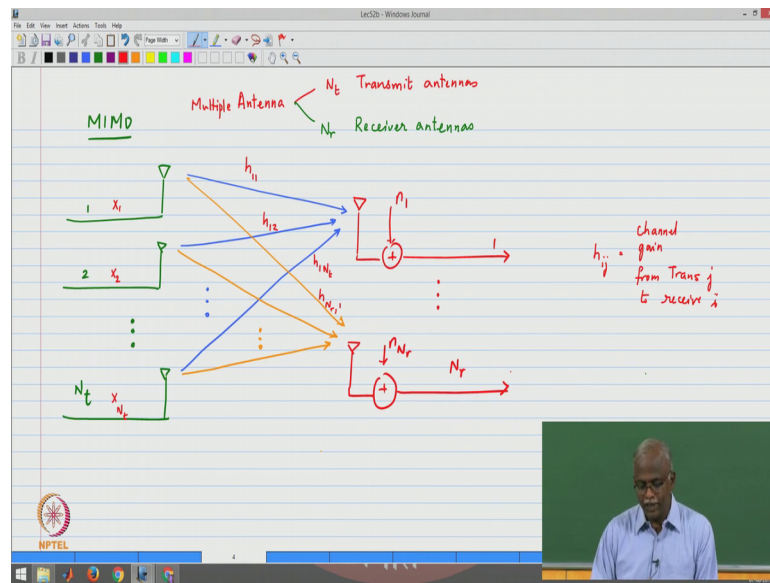
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So, the principle of orthogonality can be explained geometrically in the following fashion. The if a is the vector that we are trying to approximate we are approximating it with the components in the x and y plane. So, the best approximation will be \hat{a} which corresponds to the point perpendicularly below perpendicular projection of the point a into the xy plane. So, the point that the best approximation also happens to be the one that produces the minimum error, and geometrically we can see that the point that produces the minimum error will be perpendicular to the components that constitute \hat{a} and to \hat{a} as well.

So, the principle of orthogonality states that the optimal solution is one in which the error becomes perpendicular to the input. So, that is the principle of orthogonality and it helps us simplify the objective function, and also derive the expression for the MMSE receiver. So, with this we will conclude our discussion on CDMA, and we will move into our next topic which is the study of multiple input antennas at the input, and multiple and antennas at the both at the receiver and at the transmitter.

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So, in the last lecture we introduce the following diagram, where we have a set of \$N_t\$ transmit antennas, each of the transmitter transmit antennas is transmitting a corresponding a symbol \$x_1, x_2\$ to \$x_{N_t}\$, notice at each of these antennas are transmitting different symbols, each of the symbols or transmissions are received by the \$N_r\$ received antennas and we are now required to process this information to identify what were the symbols that were transmitted at on the transmitting transmitter side. So, we would now like to go into this subject little bit more in detail.

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Inter-antenna interference Lec 52
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$$y_i[n] = h_{i1}[n] x_1[n] + h_{i2}[n] x_2[n] + \dots + h_{i, N_t}[n] x_{N_t}[n] + n_i[n]$$

$$\begin{bmatrix} y_1[n] \\ y_2[n] \\ \vdots \\ y_{N_r}[n] \end{bmatrix} = \begin{bmatrix} h_{11}[n] & h_{12}[n] & \dots & h_{1, N_t}[n] \\ h_{21}[n] & h_{22}[n] & \dots & h_{2, N_t}[n] \\ \vdots & \vdots & \ddots & \vdots \\ h_{N_r, 1}[n] & h_{N_r, 2}[n] & \dots & h_{N_r, N_t}[n] \end{bmatrix} \begin{bmatrix} x_1[n] \\ x_2[n] \\ \vdots \\ x_{N_t}[n] \end{bmatrix} + \begin{bmatrix} n_1[n] \\ n_2[n] \\ \vdots \\ n_{N_r}[n] \end{bmatrix}$$

$N_r \times 1$ $N_r \times N_t$ $N_t \times 1$ $N_r \times 1$

* inc. capacity
 * diversity benefit
 * complexity ↑

$y = Hx + n$

Given y
Known H Estimate x

So, again a looking at it in the pictorial form and translating it into equations we can write down the received signal for the first antenna as h_{11} which represents the transfer function between the first transmitter antenna and the first receive antenna, likewise you have the different transmission gains h_{ij} the notation would be h_{ij} is the gain or the channel gain from the j th transmitter to the i th receiver channel gain from transmitter j to receiver i antenna i ok.

So, that is the notation that we are following. So, we are now trying to write down all the elements or the components of the signal that is received by antenna one. So, h_{12} would be from the second transmit antenna to the first received antenna likewise all the different N_t antennas and their corresponding symbols that they have transmitted, and of course, the receive antenna one will have a component of noise which corresponds to the thermal noise in the receiver electronics.

So, if we were to write down for each of the N_r received antennas, then we would get the expression in the form of a matrix, where the dimensions of the matrix are important we have N_r cross 1 received vector, N_t cross 1 transmit vector and a transfer function matrix which is of dimension N_r cross N_t again the dimensions are important. So, please make note of the dimensions, and the dimensions of the noise vector. Since this corresponds to the receive antennas and then there are N_r received antenna this would also be an N_r cross 1 vector that would be. So, as we mentioned in the last lecture, the MIMO problem statement is that we can write down the input output equation for a MIMO system in the following way, y is equal to $h x$ plus n the MIMO problem statement says given y , that is you can observe the outputs of the MIMO system or the received signal vector at the receiver antennas, and assuming that the channel transfer matrix h is completely known between the N_t transmit antennas and the N_r receive antennas, and we are now required to estimate x .

So, this is the problem statement maybe a couple of comments that we would like to make even to motivate the topic. Now as you can see when we want to detect the signal transmitted by antenna 1 x_1 , and we are observing the received signal received by antenna 1 y_1 .

So, signal transmitted by antenna one, signal received by antenna one. So, basically this is what we are interested in we find that there are number of other terms that will

interfere in or without observation. So, in other words MIMO system is one in which we have the inter antenna interference. So, we have noticed that this could be described as inter antenna interference. When I am trying to detect the symbol transmitted by antenna one, I find that the signals from the other antennas are also interfering; however, notice that there are multiple observations in which the contribution of x_1 is present.

In fact, in each of the N_r observations n one x_1 is present. So, given this environment what we find is that what we would have considered as impairment inter antenna interference actually turns out to be to our benefit, because we actually have multiple copies or multiple observations of the transmitted symbol x_1 of n . So in fact, we can make a better and more robust decision regarding x_1 of n , while at the same time being able to make robust decisions on each of the different transmitted signals. So, that is the strength of a MIMO system.

So, we are able to control or manage the interference, such that we are able to make better decisions or in other words we are able to get additional performance because of the diversity benefit of the multiple antennas and. So, we are able to improve the performance and notice we are also at a given time transmitting multiple symbols. So, we are improving or increasing each other that total capacity of the system. So in fact, in the interference environment is not impairment, but actually becomes a benefit to us.

So, in the case of a MIMO system, what we find is that we are able to increase capacity. So, that is a added benefit that we are able to get. We are also able to improve the performance. So, we are able to get the diversity benefit and the combination of these two is what makes MIMO systems very interesting, that we can get both improvement in performance and improvement in capacity.

Now, where did we pay the price? The receiver now becomes a more complex receiver because the, but that is something that we can of course, be willing to pay because we are able to be getting such a significant benefit. So, the receiver complexity goes up. So, at the cost of additional signal processing, we are able to get better capacity and better performance and that we believe is a very good tradeoff in our context and also our ability to work with the systems.

So, given the scenario now let us see if we can go a little bit further and describe the problem formulation. So, in this context first we would like to make our clarify our state

some of the assumptions that we are making. So, here are assumptions that we are making about the quantities that are in our expression.

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Assumptions

$y = Hx + n$

- $H = [h_{ij}]$ $N_r \times N_t$
 h_{ij} Complex, Gaussian, zero mean
 $|h_{ij}|$ Rayleigh Distribution
 IID
 Channel flat fading Rich scattering
- $n_i[n]$ white
 Complex, Gaussian, zero-mean
 $R_n = E[\eta \eta^H] = \sigma_n^2 I_{N_r}$
 $\eta = \begin{bmatrix} n_1[n] \\ \vdots \\ n_{N_r}[n] \end{bmatrix}$
- $x[n]$ Complex Gaussian
 Zero Mean
 σ_x^2

So, we have an expression Y is equal to Hx plus n . So, we will have assumptions about the n elements of H , the elements of n and also elements of x . So, we will call it as the assumptions that we are making. So, the first one are is the assumptions about the elements of H .

Now, as we mention the matrix H is a dimension N_r cross N_t matrix, which has elements h_{ij} they represents a complex gains between transmitter antenna transmit antenna j to received antenna i . So, h_{ij} the elements of the transfer matrix, these are complex values h_{ij} are complex, they are also Gaussian and they are zero mean. So, complex Gaussian zero mean therefore, their envelope would be Rayleigh as we have discussed before, and this is a assumption that each of the channel coefficients are Rayleigh a distribute.

So, magnitude h_{ij} is a has a Rayleigh distribution. So, this is a assumption that we are making because working in a wireless environment and we are so, basically Rayleigh distribution would mean that there is no line of sight component, but in the transmitter and receiver, but the received signal envelope has got a Rayleigh distribution and we also make the assumption that the each of those elements channel gains are independent of each other.

So, basically they are independent complex Gaussians with zero mean. So, that is the assumption about the channel matrix. Now this also corresponds to an important characterization such a description of the transfer function basically says that the channel is flat, in other words the channel gain between the transmit antenna and the received antenna corresponds to a single coefficient, it is not a transfer function. So, it is not if it is a flat fading channel.

So, an important classification is that it is a flat fading channel, which has got really characteristics and because each of these large number of channel gains are all independent of each other such an environment is called a rich scattering environment. So, the assumptions that we are making is that we have N_t transmit antennas N_r received antenna and that it is a rich scattering environment which means that it encompasses the following, that each of the channel gains is an is a complex value, it is a flat fading channel and each of the channels are independent of the other channel. So, that is the summary of the statement that we have. Now the second statement would be about the noise elements. So, we describe the noise components in each of the receive antennas as white spectrally white they are complex, they are complex valued because we are looking at the complex baseband signals. So, it is they are complex, they are Gaussian and zero mean. So, basically it would be a zero mean AWGN type of environment additive white Gaussian noise which is zero means.

So, just as you would characterize the properties of a white noise, basically we can write it down in terms of the correlation matrix or the co variance matrix which as that if I take the expected value of the vector n times n Hermitian, where the vector n corresponds to the noise terms in each of the antenna received antennas. So, this is $n \times 1$ of n all the way to $n \times N_r$ of n , if this is the vector presenting the noise samples then expected value of n times n Hermitian because they are complex white Gaussian noise samples, and they are uncorrelated with each other we can write down this expression as σ_n^2 times I the identity matrix time with the dimensions of that would be N_r . So, that would be the description of the noise components.

Now, in order for us to do the capacity calculations, we will also make a third assumption. The third assumption is about the characteristics of x . Now x is the transmitted signal typically in a digital communication system this would represent a QPSK or a QAM symbol. So, we normally assume that x is the transmitted signal

represents a constellation point in the IQ plane, but for the purposes of capacity calculations we will also make an assumption which is different from the usual one, but again more from the point of view of capacity calculations.

So, we will also make the assumption that x of n , the single that is translate single that is transmitted by each of the antennas x of n is also a Gaussian distributed variable. So, we will assume that x of n is x_i of n is also a complex Gaussian with zero mean and variance σ_x^2 . Again the reason for this would become clear when we talk about the description or the calculation of capacitors. So, we have made three assumptions, all of which are pointing to Gaussian assumptions in this equation the channel gains are complex Gaussian.

The received signal noise samples are complex Gaussian, and again the third assumption for the purpose of capacity calculations will be the assumption on x .

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The image shows a digital whiteboard with handwritten notes in blue and red ink. The title 'Lec 5A' and the number '7' are in the top right corner. The notes are as follows:

- MIMO** $\checkmark \checkmark$

$$\underline{Y} = \underline{H} \underline{X} + \underline{n}$$
- Possibility 1**

$$\underline{x} = \underline{H}^{-1} \underline{y}$$

channel inversion

⊙ \underline{H} is a square, invertible matrix
- Possibility 2**

$$\underline{H}^{-1} \underline{x}$$

Pre-coding

$$\underline{y} = \underline{H} \underline{x} + \underline{n} = \underline{H} \underline{H}^{-1} \underline{x} + \underline{n}$$

$$\underline{y} = \underline{x} + \underline{n}$$
- If \underline{H} is not a square matrix**
- Singular Value Decomposition (SVD)**
 Any rectangular (real or complex) \underline{A}
 of dimension $N \times M$

$$\underline{A} = \underline{U} \underline{\Sigma} \underline{V}^H$$

where \underline{U} = unitary matrix = $[\underline{u}_1, \underline{u}_2, \dots, \underline{u}_N]$ are the N left singular vectors of \underline{A} (eigen vectors of $\underline{R} = \underline{A} \underline{A}^H$)
 \underline{V} = $[\underline{v}_1, \underline{v}_2, \dots, \underline{v}_M]$ are the M right singular vectors of \underline{A} (eigen vectors of $\underline{R} = \underline{A}^H \underline{A}$)

Now I would like to go back and re look at the equation that we have written down. So, the MIMO equation a MIMO system can be represented in terms of \underline{Y} is equal $\underline{H} \underline{x}$ plus \underline{n} , and we are told that we are we have the observations of \underline{y} and \underline{H} is known, and we are trying to find out what is \underline{x} .

Now, given this an environment and for a moment if we can ignore the presence of the noise, one possible way for us to do it if \underline{H} were invertible would be the following. So,

one a possibility is that you transmit x possibility one, that if you received x and you know H , you can do h inverse times y . So, this is what we referred to as channel inversion, this makes some assumptions first assumption is that H is a square matrix square in and it is in invertible matrix, that is its non singular. So, this itself is a fairly restrictive assumption which tells us that the number of transmit antennas and the number of received antennas must be the same, and the transfer matrix must be an invertible matrix or a non singular matrix.

So, again we cannot guarantee this in all scenarios because some of these are random phenomena and we cannot guarantee this type of an assumption. However, if in when you think along these lines we can also think of a possibility 2 number 2 and the possibility 2 says that you do before you transmit the signal x what you can do is you pre code the transmitted signal. So, basically instead of transmitting x you transmit h inverse times x . So, at the transmitter itself we are making a modification of the transmitted signal.

So, instead of transmitting x we are transmitting each inverse times x , again this introduces a way of thinking. So, if H is known at a transmitter and this is what we referred to as CSIT channel state information at the transmitter, then we can do a pre coding option, this is what we referred to as a pre coding. Pre coding is something that is done at the transmitter with the knowledge of the transmission channel. So, we do H inverse times x , and we call that as x tilde. Now if x tilde is transmitted then what I receive will be Y is equal to H times x tilde, and this will be nothing, but H times H inverse times x and a plus noise, plus noise and this would also give me y is equal to X plus the noise.

So, in other words we are able to obtain the received signals as a signal or a decision process where you do not have the effect of the inter antenna interference, again we are able to make a decision independent of the other. Again this is only under a very limited set of conditions and when the when it is a H is a square matrix and it is also invertible of course, we would be interested in a much larger family of solutions where H is not a square matrix. So, if H is not a square matrix, then what are our options?

So, H is a rectangular matrix N_r cross N_t , the channels the coefficients of this matrix are complex Gaussians now under this assumption, what are the options that are available?

And if it turns out a very general solution can be obtained which is what we will now derived. So, the result that we are going to be looking at is something that many of you would have studied in the context of a linear algebra and in a complex variables, this would be the concept of singular value decomposition. Singular value decomposition again I am assuming that most of you would be familiar with it. So, I will just give a quick overview of the key results that we would need for our discussion and again I would refer you to any book on the linear algebra for the results of or the main elements that we would need for the singular value decomposition.

So, the statement of the singular value decomposition states that any rectangular matrix which would fit our description of our matrix h , any rectangular matrix which can be real or complex valued, in our case it will definitely be complex valued and. So, such a matrix A which is of dimension N cross M where N and M are different. Now the statement of the singular value decomposition or the SVD theorem states that the matrix A can be written in the following form, where A can be written as U times Σ times V^H where U is a matrix which is a unitary matrix.

So, this is a unitary matrix again the definitions of unitary matrix, we have already indicated. So, U is a unitary U and V are unitary matrices, which is given by the following vectors u_1, u_2 up to u_N . N is the row dimension. So, these are the what are called the left singular vectors, these are the N left singular vectors of A of the matrix A , it also turns out that we can relate these N vectors as the Eigen vectors, we can also relate them to the eigenvectors of an N cross N matrix of a matrix R which is given by A times A^H Hermitian A^H being N cross M we can verify that R is a N cross N matrix it is a positive semi definite matrix and using the factorization of a positive semi definite matrix we can then obtain the different vectors or u_1 through u_N .

So, the next item that we want to define is the matrix V , V is also a unitary matrix it consists of the following vectors V_1, V_2 all the way through V_M , M is the column dimension these are the m right singular vectors right singular vectors of A and like before we can relate these to be related to the Eigen vectors of a positive semi definite matrix in this case it is given by $R = A^H A$. So, if you look at the two variance of A one is $A^H A$ Hermitian this one is $A A^H$ Hermitian A this would have dimension m cross m and these would be the Eigen vectors corresponding to that.

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Now, sigma matrix can be written as a diagonal matrix with the following values sigma 1 through sigma L, where sigma k is given by root lambda k, where lambda k can be related to the non zero Eigen values of A Hermitian, AA Hermitian. It can also be related to the non zero Eigen values of A Hermitian A. Both of them have the same set of nonzero Eigen values. So, this can also be read it can also be related to these. So, what we have is that the matrix A can be written as u times sigma V Hermitian where the dimensions we will write down this is a N cross N matrix, this is a M cross M matrix, this would be an N cross M matrix.

So, one of the things is the what is the different basic reviews at term L, and L is related to the rank of the matrix a and basically we can make the following statement. So, if you look at our matrix H this has dimension N r cross N t and we can make the statement that the rank of H the rank of the matrix H it has to be less than or equal to it has to be less than or equal to the minimum dimension between N t comma N r. So, this is what we referred to as the dimension L the rank of the matrix of the so, assuming that we will make the following assumption, assume that the matrix H is full rank, because we have assumed a rich scattering environment.

So, if this case is full rank then the value L is equal to the minimum of N r comma N t the rank is. So, in other words let me to say this, the rank this equal to L is less than or equal to in the general case and when the H is full rank this is the case.

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The image shows a digital whiteboard with handwritten mathematical content. At the top, it says "Lec 5A" and "9". The main content is divided into two cases for the singular value decomposition of a matrix A of size $N \times M$.
Case 1: $N \leq M$. The matrix Σ is shown as a diagonal matrix with singular values $\sigma_1, \sigma_2, \dots, \sigma_N$ on the diagonal and zeros elsewhere. The text $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_N$ is written to the right.
Case 2: $N > M$. The matrix Σ is shown as a diagonal matrix with singular values $\sigma_1, \sigma_2, \dots, \sigma_M$ on the diagonal and zeros elsewhere. The text $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_M$ is written to the right.
A small video inset in the bottom right corner shows a man in a blue shirt speaking.

So in our case we can have two options it could be either the case where N is less than or equal to M , where we are looking at the matrix A which is of dimension N cross M ; that means, the number of rows is less than the number of columns in such a case then the rank would be equal to N which is the lower dimension.

So, in such a case the matrix sigma can be given in the following form, it would be a sigma 1, sigma 2 it is a diagonal matrix all the way to sigma n , and since that the number of columns is more than the number of rows, the rest of the entries would be 0. So, this would be the form when N is less than or equal to M , this would be the form that you would get you would have the n non zero Eigen values, which they are represented along the diagonal matrix and then you have the zero elements. Now for the second case where N is greater than M , then now the rank becomes determined by the column dimension M .

So, in such a case the sigma matrix can be visualized in the following form, it is sigma 1 sigma 2 dot dot dot all the way to sigma m that is the column dimension, and since the number of rows is more than the number of columns what we have the for the rest of the entries it is 0. So, in one case where the number of rows is less than the number of columns, we get the upper form and the second case when we have the number of rows more than the number of columns then we get this form.

Now, of course, you can visualize the case where the matrix A is not full right, in which case there would be minor modifications of this structure no keeping in mind that only a

subset of these are non zero, and we have made the following assumptions that the singular values are rank ordered that is sigma 1 greater than or equal to sigma 2 greater than or equal to sigma n. So, if more of these are become 0 because the rank is not full rank, then the dimensions of the non zero portion of sigma will reduce and you will get more zero entries I would maybe leave that as an exercise for you to look at what it, how it is this matrix sigma will look like if the matrix is not full rank.

But in our case we will make the assumption that the matrix is full rank so, the summary of the material that we have presented so far.

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Parallel Decomposition (MIMO) $N_r \times N_t$

Known $\underline{H} = [h_{ij}]$ $i=1, \dots, N_r$
 $j=1, \dots, N_t$
 full rank $\tilde{x} \rightarrow \underline{H}$

$$\underline{H} = \underline{U} \underline{\Sigma} \underline{V}^H$$

$$\underline{y} = \underline{H} \underline{x} + \underline{n} = \underline{U} \underline{\Sigma} \underline{V}^H \underline{x} + \underline{n}$$

Pre-coding @ Tx $\tilde{x} = \underline{V} \underline{x}$

$$\tilde{y} = \underline{H} \tilde{x} + \underline{n}$$

$$\tilde{y} = \underline{U} \underline{\Sigma} \underline{V}^H \underline{V} \underline{x} + \underline{n}$$

Post-processing @ Rx $\underline{y} = \underline{U}^H \tilde{y} = \underline{U}^H \underline{U} \underline{\Sigma} \underline{x} + \underline{U}^H \underline{n}$

$$\Rightarrow \underline{y} = \underline{\Sigma} \underline{x} + \underline{U}^H \underline{n}$$

If you look at the parallel decomposition of a MIMO channel which is N_r cross N_t in terms of dimension, that is the task that we have we assumed that we know the following matrix which is h_{ij} where i is equal to 1 through N_r j is equal to 1 through N_t it that this is known, this is known and its full rank full rank ok.

So, basically we will assume that this is the problem statement. So, now, we can factorize the matrix H in the following way, u times σ times v Hermitian. So, the MIMO equation now becomes Y is equal to $H x$ plus n , now this can be written as $u \sigma v$ Hermitian times x plus $n x$ plus n . So, basically we have submit the factorized form for this, now here comes a very interesting two simple steps which make the MIMO problem statement very interesting one. So, if I define the vector x in the following manner, if I define the vector x to be equal to V times x tilde or in other words let me write it in a

slightly different way. I would now like to take this vector x pre multiplied by the matrix V and we will call that as \tilde{x} .

So, a modified input is going to be provided to the channel. So, in other words we are now going to feed \tilde{x} . So, I have \tilde{x} , \tilde{x} is going to be fed into the channel matrix H . So, by using the original equation I will now get modified output and the modified output we will call that as \tilde{y} , and \tilde{y} will be H times \tilde{x} plus n . So, some modified input and modified output to the same equal.

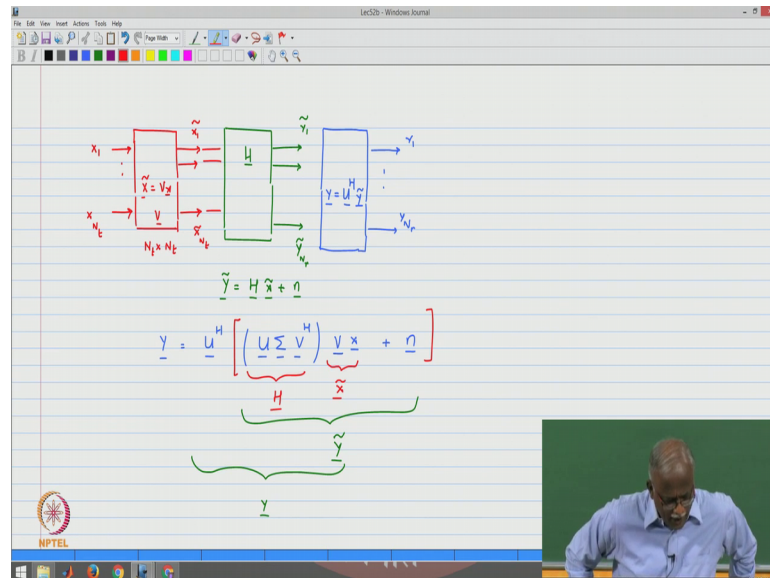
So, now substituting for \tilde{x} in inside this expression, what we find that this is equal to u times Σ times v Hermitian, \tilde{x} is V times x plus n . So, this is \tilde{y} . So, we know that since V is a unitary matrix v Hermitian times v is a identity matrix. So, therefore, we can simplify this expression and one more step that we do if we now say that y is equal to u Hermitian \tilde{y} . So, basically I do a transformation of the received vector, then this comes out to be u Hermitian u times Σ v Hermitian, v I have already dropped this is remaining is x plus v Hermitian u Hermitian times n , u Hermitian times n . So, that is the expression that we have.

Now, notice that u Hermitian u is also a unitary matrix. So, therefore, by unit matrix property we can that it becomes the identity matrix. So, this gives me a result which is y is equal to Σx plus u Hermitian times n . Now we can treat this as a vector n dash which since use a unitary matrix has got the same statistical properties as a n . So, basically this is a modified noise vector, but notice the rest of the equation is very very interesting, because the rest of the equation says that my input and output are related by a diagonal matrix Σ is a diagonal matrix. Notice that we have completely removed the presence of the inter antenna interference.

So, the effects of the in interaction between the different transmitted signals with the different received signals have been completely removed by means of the two steps. The first one was a transmitter encoding this was a pre coding at the transmitter, so, pre coding at the transmitter, where we did V times x , and then post processing at the receiver that that was obtained at the step; this is post processing at the receiver. Now both of these are very simple steps both of which are matrix operations based on matrices that are obtained from the singular value decomposition. So, this is post processing at the

receiver. So, a simple processing step at the transmitter a simple processing step at the receiver gives us a very nice and interesting result, where we have this expression.

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So, let us do two things one is first let us write down the process that we have done, the post the pre processing or the pre coding that we did at the transmitter, where we said that we will take \tilde{x} is equal to V times x . So, if you were to think of this as a transformation or x_1 through x_{N_t} , V is a this is the matrix V , N_t cross N_t that will produce for us the different inputs the modified inputs \tilde{x}_1 all the way to \tilde{x}_{N_t} . So, that is what was given to us.

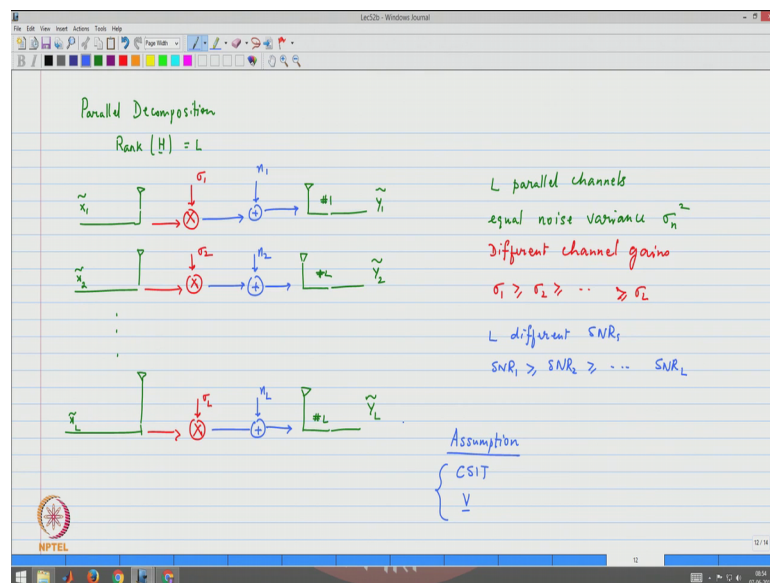
Now, this is passed through the channel this is a channel which is H . So, \tilde{y} is equal to H times \tilde{x} plus the noise that is represented by these channel elements. So, if you were to connect these the modified input passing through the channel, and the channel producing the outputs \tilde{y}_1 to \tilde{y}_{N_r} and then we have the post processing step at the receiver. Where take this and do the following transformation where we say that y is equal to U^H times \tilde{y} , U^H times \tilde{y} . So, if this is \tilde{y} then what we get here are the outputs y_1 through y_{N_r} and this is the expression that we have obtained

Again it is a very powerful result this is something that we will use repeatedly in our discussion. So, we will write it in the following form, y is equal to the post processing step U^H the channel, which is $U^H V^H$ this is the channel times the

pre coded transmitter which is V times x . So, this is the and of course, we have the noise element that is being added. So, this is the post processing step, this is the pre processing one. So, this becomes the pre coding x tilde, this is my channel H . So, x tilde passing through the channel H , produces the output Y tilde this whole thing is Y tilde and that we have do a post processing at the receiver which is u Hermitian times Y , that becomes the final step where this becomes equal to the vector y ok.

So, interesting way to visualize it, it is a transmission that is correspond that is from N_t transmit antennas to N_r received antennas that is the original structure, but we have split the transmission process in to pre coding a step, the MIMO channel followed by a post processing step, and this is a good way to visualize it. Now another interesting way to draw this figure.

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And I believe that gives us a lot of insight and what we would like to do is look at how we can represent this. So, the parallel decomposition of a MIMO channel parallel decomposition can be captured in the following form parallel decomposition. So, we have N_t transmit antennas and we have N_r transmit antennas, but effectively the number of channels the parallel channels that we have depends on the rank of the matrix.

Now, let the rank be equal to L ; the rank of H equal to L , now once we have done the parallel decomposition then what we can say is the parallel decomposition of the channel looks like this, there is there are only L transmit antennas because effectively there are L

parallel channels, we can call this as x_1 x_2 x_L . Now this will get transmitted by this antenna and effectively the channel looks like a gain for each antennas which is different.

So, for example, the first antenna experiences gain which is proportional to σ_1 . So, think of the gain as a multiplicative term which is represented by σ_1 this signal now has a multiplicative factor σ_2 , and the L th antenna has got a multiplicative gains which is given by σ_L . So, each of the this we are looking at the parallel transmissions. So, there are L parallel transmissions and the parallel transmission have got a diagonal gain which is σ_1 through σ_L , which is what we are trying to capture in our diagram here.

Now, of course, in any transmission system there will be the additive noise. So, this one we get a noise term n_1 second branch we get a noise term n_2 and the third and the last branch we get a noise term n_L . Now this comes out at the receiver and this is what is picked up by the received antennas. So, the number of the received antennas is also equal to L , this is received antenna number 1 number 2 and the last one is number L and the signals that we obtained are y_1 , y_2 and all the way to y_L . So, the parallel decomposition can be visualized in two ways.

One is the matrix representation that we have shown in this figure, the second way is to think of it as parallel channels where the transmitted signals x_1 through x_L are experiencing different gains and this noise terms notice that the noise terms are all additive white Gaussian noise terms with the same variance and the received signal. So, this if you were to describe it in equivalent form, consists of L parallel channels. Let us write it down there are L parallel channels the noise variance in each of these.

So, each of them has got equal noise variance noise variance each of the noise terms is σ_n^2 . So, each of them have got the same noise variance, but each of them have got different channel gains and if we have ordered our singular values such that $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_L$, then what we find is that we have a L parallel channels where each of the channels has got different signal to noise ratio.

Notice that σ_1 is a gain for the signal and on and on and channel one. So, there are L channels each of the L channels have different L different signal to noise ratios SNRs.

So, SNR of channel one will be better than or equal to SNR of channel two better than or equal to SNR of the channel L. So, this is what we have as our ordering of the channels.

So, to summarize we have a MIMO system, a MIMO system where we can represent the input output by means of a transfer matrix. We have explained the assumptions that we have made regarding the entries of this matrix. The next step was to the assumptions then we did the singular value decomposition; the singular value decomposition tells us that any rectangular matrix whether its real or complex in our case it is a complex valued matrix it can be decomposed in the form $u \sigma v^H$ Hermitian, and the expressions for the u v and σ were obtained based on that we showed what the structure of σ would be depending upon whether you had the number of rows greater than the number of columns or vice versa.

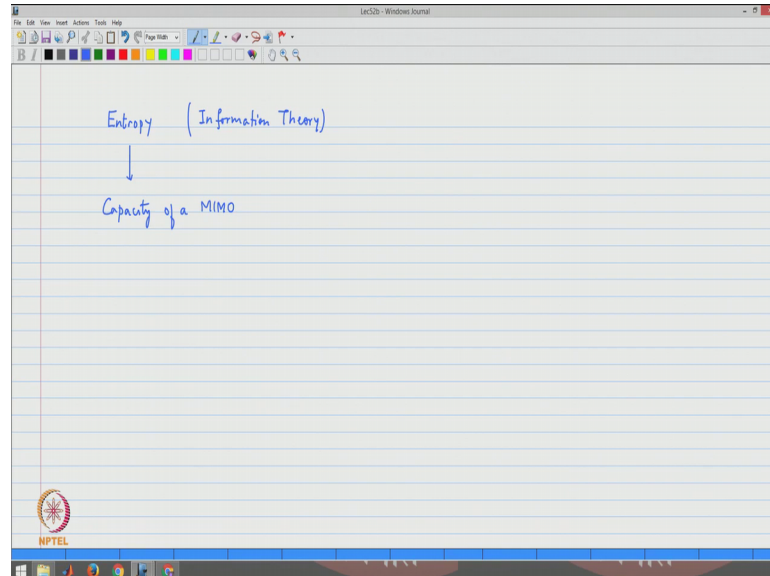
Then finally, we can write down the expressions, and we can show that the SVD decomposition can be captured very nicely using matrix form or in terms of a parallel decomposition for. Now notice that once you have done the parallel decomposition it is also easy for us because each of these channels has got a corresponding SNR, we can of course, calculate the channel capacities because we know that the capacity is related to the SNR of the channel. So, the beauty of the SVD process is that we it is now enabled us to do the separation of the channels at the in to parallel channels.

However keep in mind that one of the key assumptions that we have made in the obtain the parallel decomposition, again maybe we should write it down it is a very important assumption that we have made, and that is that the transmitter knows the channel CSIT or at least it knows what it needs to do to obtain the matrix V Hermitian. So, because that is what is required at the sorry it requires it requires knowledge of the matrix V so that it can do the pre coding. So, this under this assumption we have obtained a very nice result.

Now, we; obviously, have to ask question what is the transmitter does not know the matrix V which is based on the parallel singular value decomposition or does not have the CSIT. So obviously, we are interested in those situations where the this information is not known to the transmitter. So, we would like to get a understanding of what could be done in such a situation, when you do not have the ability to do the pre processing; what

would be the understanding of capacity in such a situation. So, in that regard we will now study or in the next chapter next lecture.

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We will be studying an important element which represents, which pertains to our understanding of the concept of entropy. This is the same as the entropy that we would discuss in information theory.

So, what we would like to do is get a understanding or review the basic concepts and see how the understanding of the concept of entropy can help us in understanding the capacity of a MIMO channel. So, capacity of a MIMO channel based on our understanding of the capacity or the concept of entropy, that will be the best contexts content of the next lecture. So, please do review the material that we have covered today and we will pick it up from here in the next lecture.

Thank you very much.