Digital Signal Processing and its Applications
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Lecture 27 B
Design of FIR of length (2N+1) by the Truncation Method

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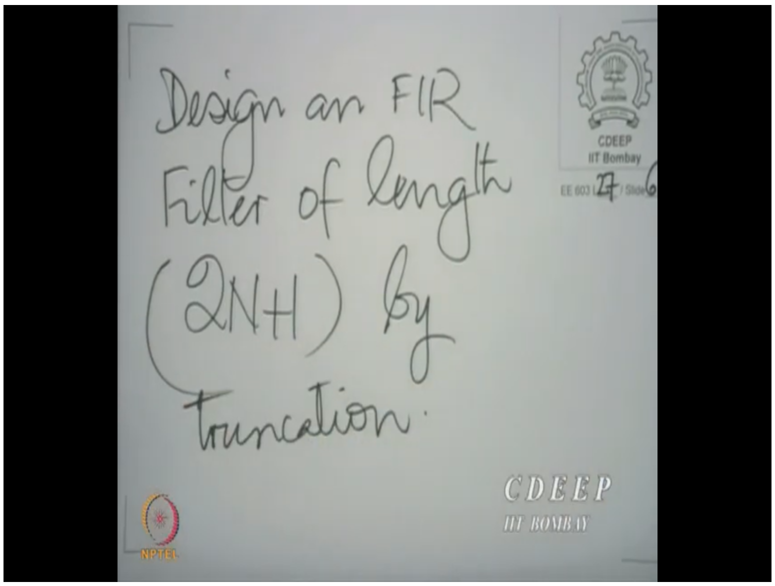

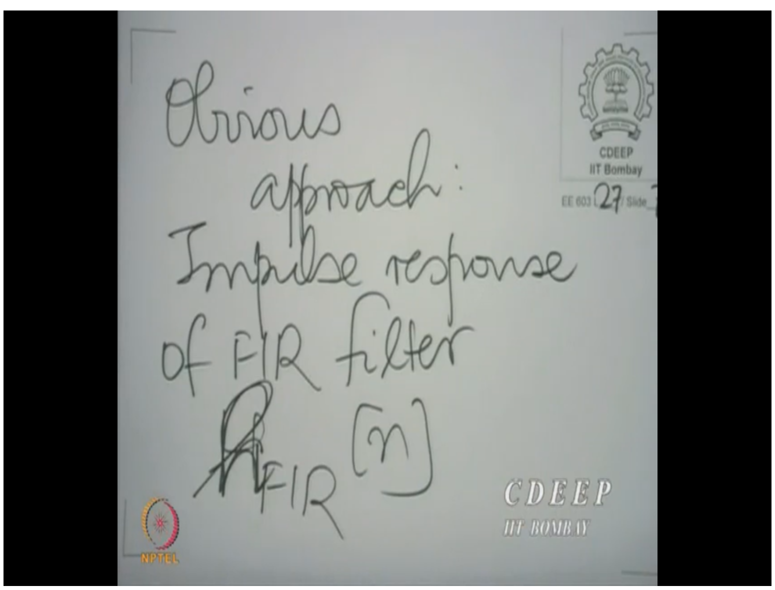

Design an FIR filter of length $(2N + 1)$ by truncation. Now, the obvious thing to do, impulse response of the FIR filter, let us call it $h_{FIR}[n]$. So, well I think we should call it h, so we will h because we are using H for the frequency responses.

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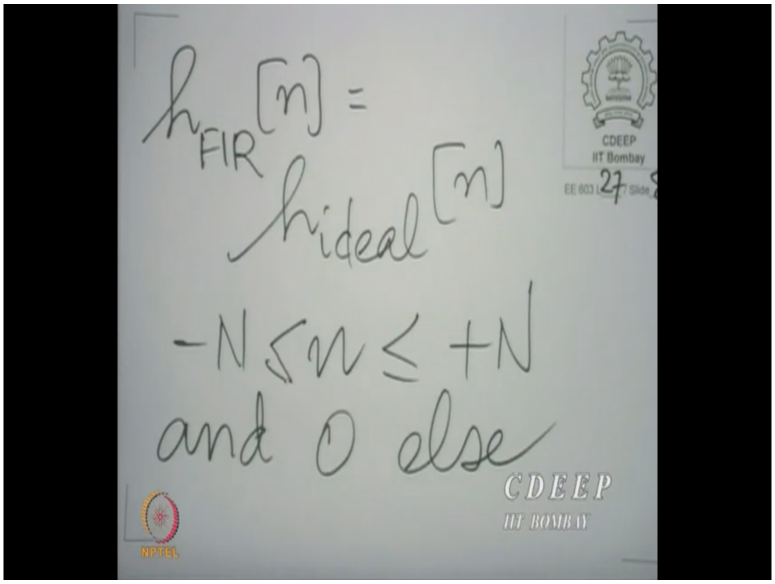

So, clearly, $h_{FIR}[n]$ is $h_{ideal}[n]$, for *n* between $-N$ and $+N$ and *0* else, is the obvious way to do it. Retain the samples from $-N$ to $+N$, throw away the rest. Now, this is equivalent to multiplying the ideal impulse response via the function which is *1* between $-N$ and $+N$, and *0* outside.

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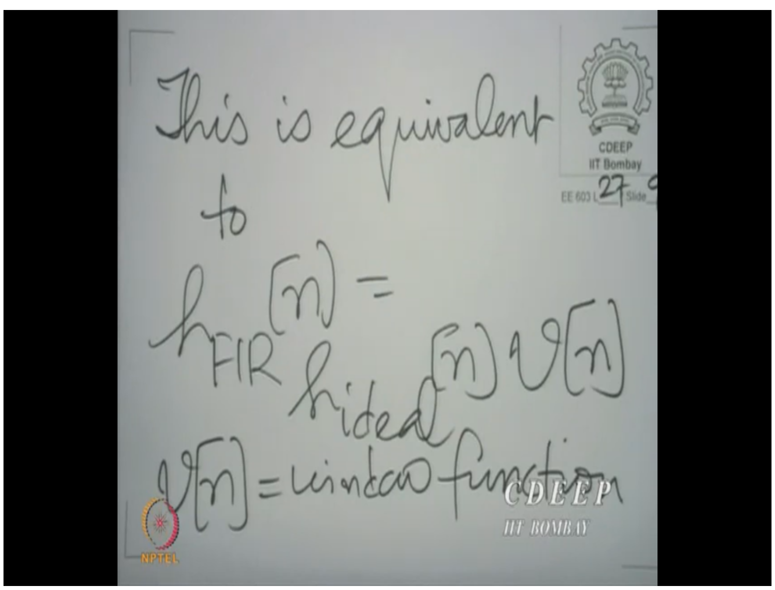

So, this is equivalent to $h_{FIR}[n] = h_{ideal}[n]v[n]$, $v[n]$ is called a window function. I used v instead of w for window, w is likely to be confused with ω and I do not want to use w for that reason. So, we will use v.

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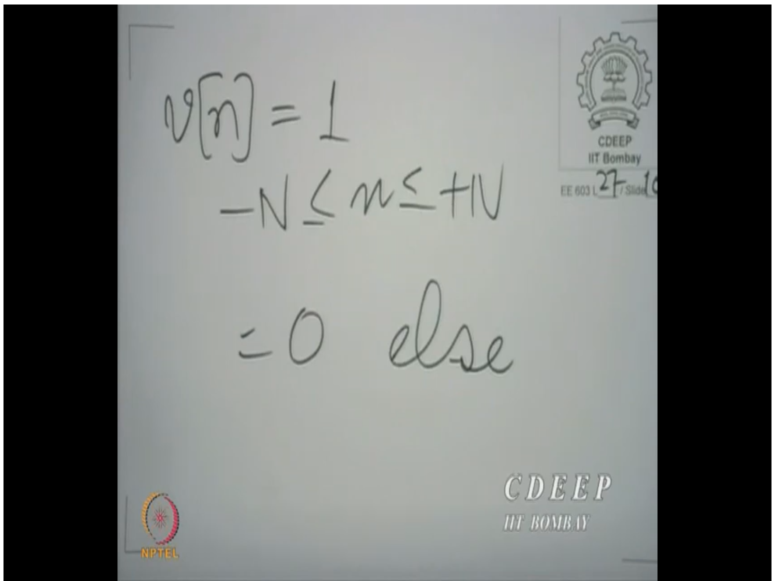

So, $v[n]$ in this case is equal to *1*, for n between $+\ N$ and $-\ N$ and, *0* else, that is obvious. So, multiplies by a function which is *1* in the range where you are retaining the samples and by *0* in the range where you are throwing away the sample, that is an obvious. Now, why this approach or why this perspective is useful to us is because it immediately gives us a clue how we can identify the scars that are produced in this truncation.

I said that when you cut something, I mean, that is to be expected. When you cut something, it is going to leave a scar. What scar does it leave on the frequency response, is what we now need to understand. And the obvious thing to do is to see what happens when you multiply two sequences in the time domain as interpreted in the frequency domain.

Now, we have done this when we discussed the discrete time fourier transform. At that time, we had just mentioned that this multiplication property would be useful when we talk about FIR filter design. But now we are actually coming down to the brass tacks of where this is useful.

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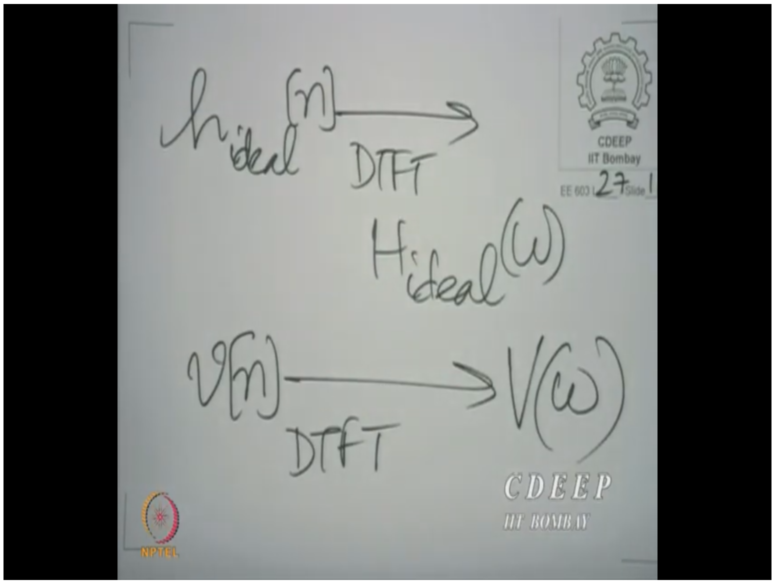

So, obviously, if you were to take the ideal response, $h_{ideal}[n]$ has the frequency response or the DTFT given by $H_{ideal}[\omega]$. $v[n]$ has a certain DTFT, which we will calculate in a minute and we will call it $V[\omega]$.

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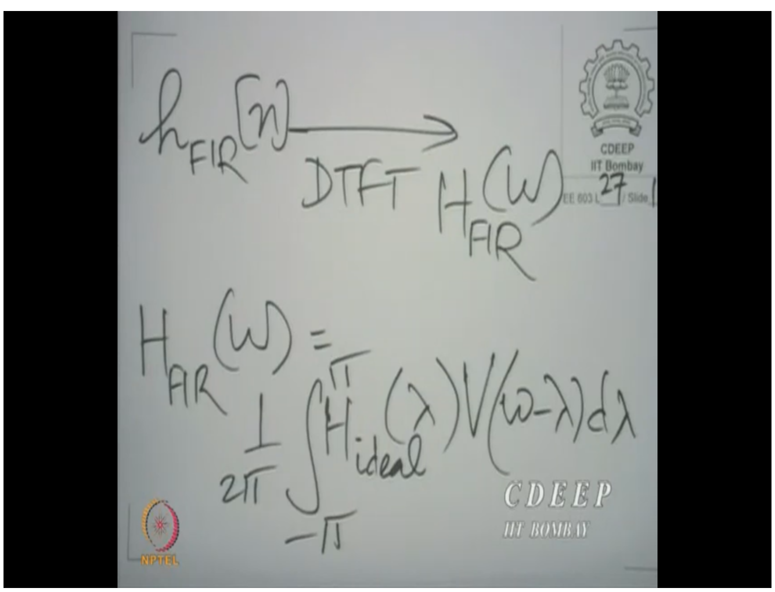

And obviously, the discrete time fourier transform of $h_{FIR}[n]$ which, we shall in a minute denote by $H_{FIR}[\omega]$, is given as follows, $H_{FIR}[\omega]$ is, as we know, $\frac{1}{2\pi}\int$ over a contiguous range of π, we could choose the principal interval $-\pi$ to π. Well, you know, we had a choice, we could either move V or we could move H_{ideal}.

Let us choose to move V, we will see why that is better. $H_{ideal}(\lambda)V(\omega - \lambda)d\lambda$, so it is like a convolution on the frequency axis. A restricted convolution evaluated between $-\pi$ and π, we have derived this property before. Now, let us calculate, to get a feel of how $V(\omega)$ looks likes, so let us actually calculate it.

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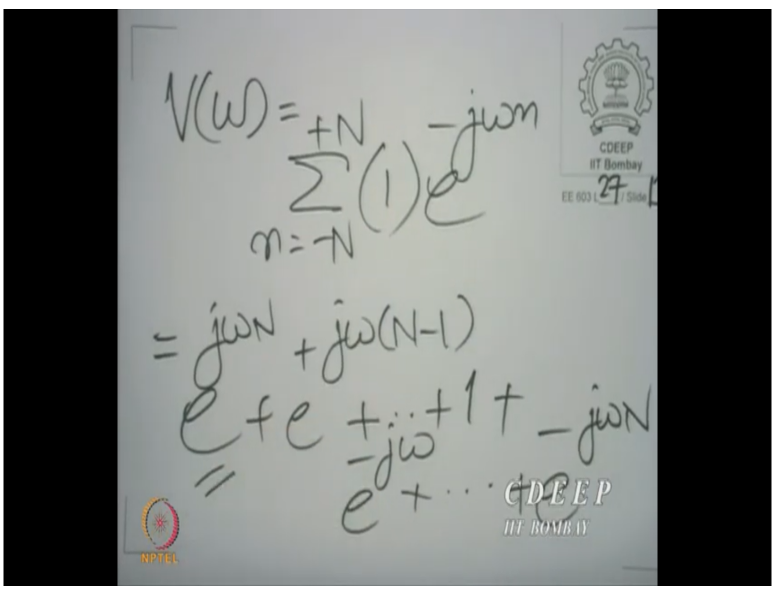

So, $V(\omega)$ looks like this. It is a discrete time fourier transform, $\sum_{n=-N}^{+N}(1)$, that is the value of $V[n]$ in that region and *0* outside, $e^{-j\omega n}$. Now, we can expand this. It is $e^{j\omega N} + e^{j\omega(N-1)} + ... + 1 + e^{j\omega} + ... + e^{-j\omega N}$.

And obviously, this is a geometric progression with the first term equal to $e^{j\omega N}$ and the common ratio equal to $e^{-j\omega}$. And therefore, we can calculate this discrete time fourier transform by using the sum of a finite geometric progression.

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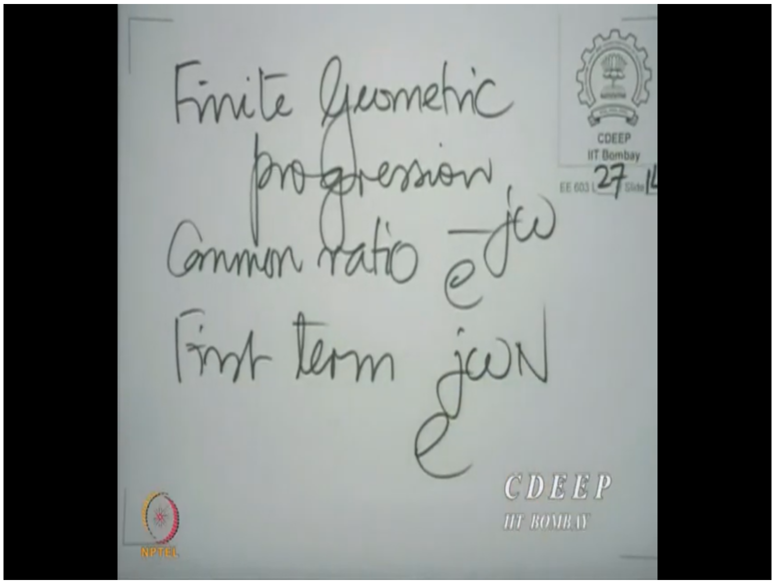

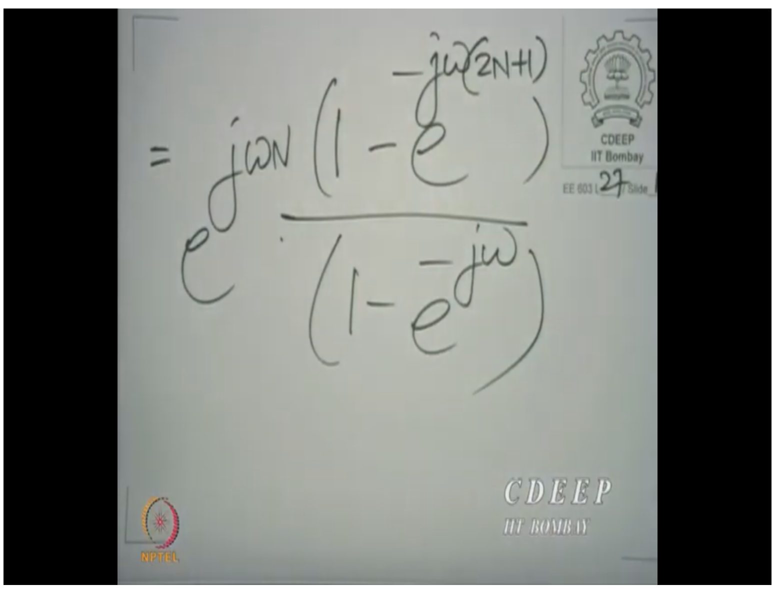

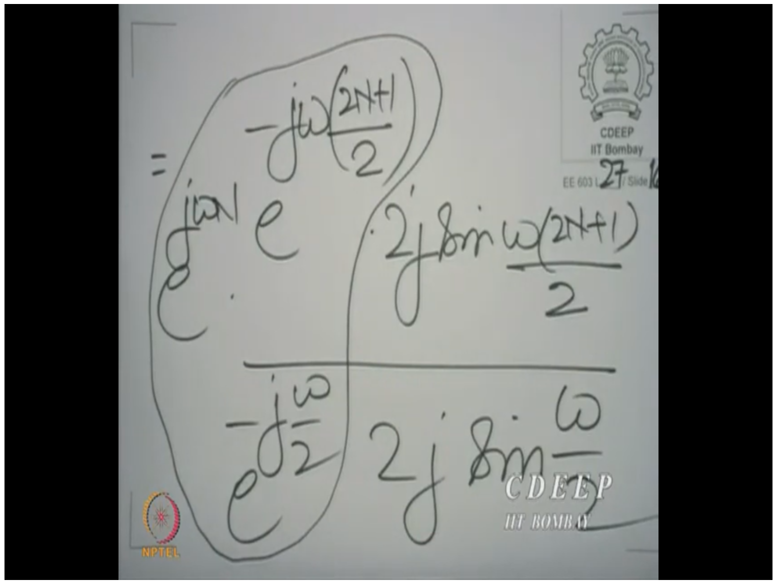

Common ratio, $e^{-j\omega}$, first term or leading term $e^{j\omega N}$. And that is $e^{j\omega N}\frac{\left(1-e^{-j\omega(2N+1)}\right)}{\left(1-e^{-j\omega}\right)}$, this is the sum. Now, we can do a little bit of work to get this in the form of a *sin* by a *sin*.

So, we can rewrite this as $\frac{e^{j\omega N}e^{-j\omega\left(\frac{2N+1}{2}\right)}2j\,sin\frac{\omega(2N+1)}{2}}{e^{-j\frac{\omega}{2}}2j\,sin\frac{\omega}{2}}$. We can do this by extracting *e* raised to power minus half of this in the numerator and *e* raised to power half of this argument common from the denominator. That is what we have done here, *e* raised to half this argument, common in the numerator and *e* raised to half this argument common in the denominator. And the rest of course, is *2 j sin* as you can see.

Now, it is not at all difficult to evaluate this part of the expression. All that we need to do is to add these indices, add these powers. So, it is $e^{j\omega N-j\omega\left(\frac{2N+1}{2}\right)-\left(-j\frac{\omega}{2}\right)}$, that is easy to do. And it is very easy to see that it adds to *0*. And therefore, this leaves you with just a *1*, a unitary factor and the *2j's* cancel.

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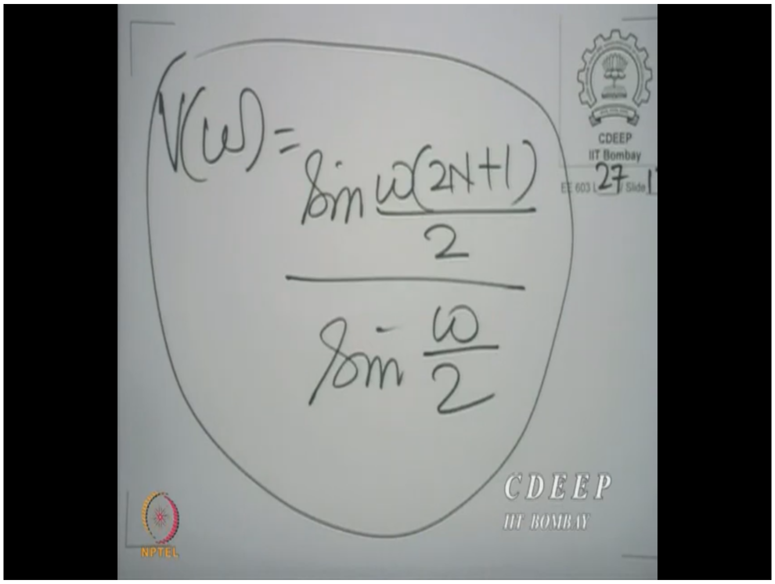

And therefore, what is left with us, is $V(\omega)$ turns out to be $\frac{sin\frac{\omega(2N+1)}{2}}{sin\frac{\omega}{2}}$, a very vey important expression indeed. We can sketch this, of course, both of them beginning with *0* at $\omega = 0$. As far as the denominator is concerned, it goes all the way up to $\omega = \pi$, that is $sin\frac{\pi}{2}$ which is *1*. So, it takes only a *1* quarter of a cycle between *0* and π. As far as the numerator is concerned, it takes $2N + 1$ quarter cycles. So, that is a little more difficult to visualize.

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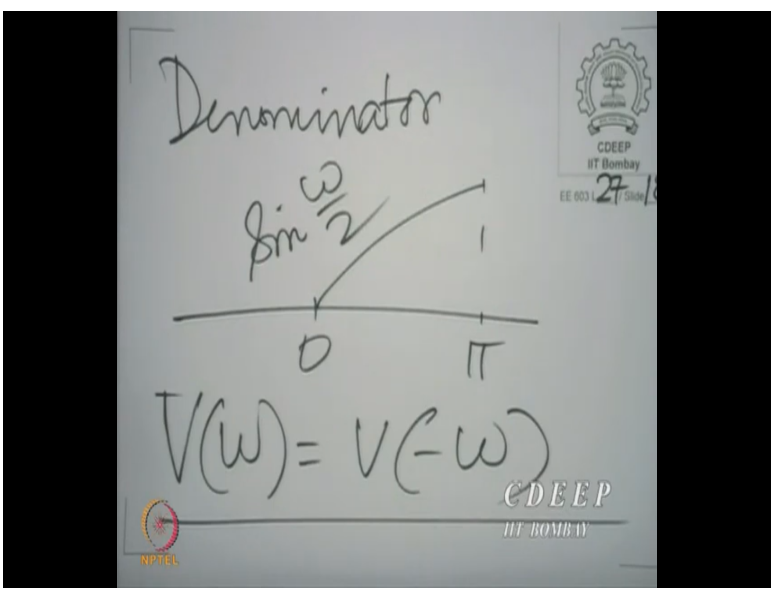

So, denominator is like this, of course, please remember, it is very easy to see $V(\omega) = V(-\omega)$, that is very easy to see. You also notice that this, this window function $v[n]$ is real and even. $v[n]$ and $v[-n]$ are the same. And of course $v[n]$ is real. A real and even sequence has a real and even discrete time fourier transform.

And that is what you of course observe here. This is fourier transform, this discrete time fourier transform is indeed real and even as expected. So, I need to work, I am quite satisfied working on the positive side of ω and then mirroring it on the negative.

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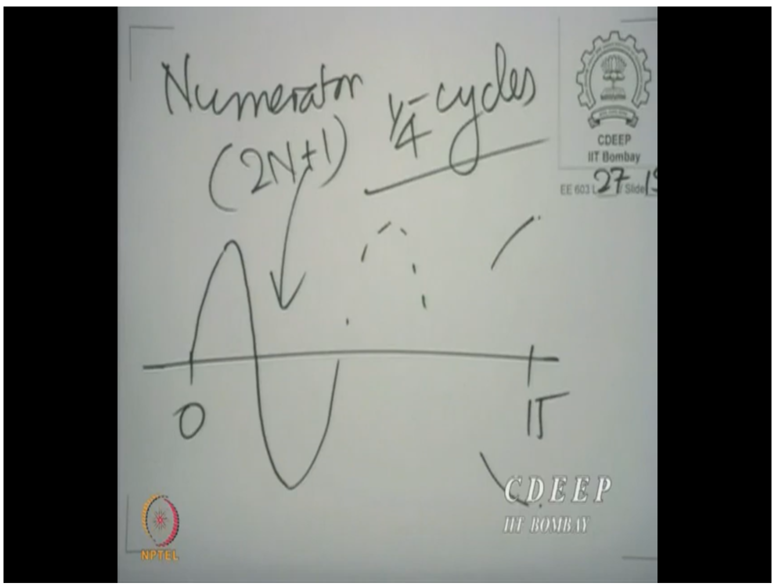

Now, as far as the numerator goes, it begins with *0* and then there are $2N + 1$ quarter cycles. One thing is guaranteed. Since the number of quarter cycles is odd, you are definitely going to end with either $+ 1$ or $- 1$. You are going to end either here or here. Now, it depends, if $N = 0$, you are going to end with plus 1. If $N = 1$, you are going to end with $sin\frac{3\pi}{2}$, which is $- 1$, again you go to $+ 1$.

So, it is all, you see as *N* goes from *0* onwards, *0, 1, 2*, you alternate between $+ 1$ and $- 1$ at the ending. So, you see, the beauty is that you are always going to end at $1 \div 1$or $- 1 \div 1$. So, as far as the beginning is concerned, we can find out. Now, the beginning means at $\omega = 0$, what happens to the, what happens to $V(\omega)$ at $\omega = 0$? Now, here again, we do not need to use the expression $sin\frac{\omega N}{2}$ and so on.

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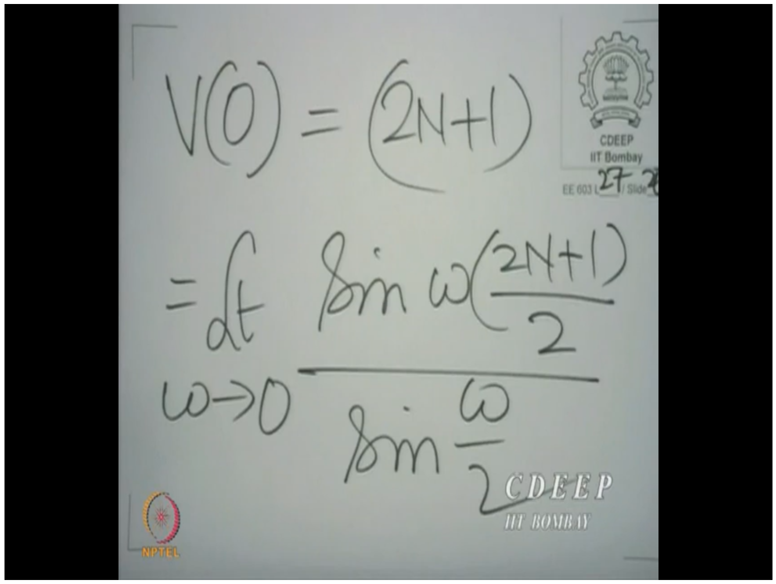

You see, V at 0 is obviously equal to $2N + 1$. Why? Because it is just a sum of $1's$. Now, you would always take the limit, and that limit is of course, it is equal of course to $\lim_{\omega \to 0} \frac{sin \frac{\omega(2N+1)}{2}}{sin \frac{\omega}{2}}$. You could do that. So, of course it is a continuous and analytic function.

But you do not need to do that. You could evaluate it directly at $\omega = 0$. So, it is very clear that there is going to be an oscillation, this window is going to, the window discrete time fourier transform is going to exhibit an oscillation.

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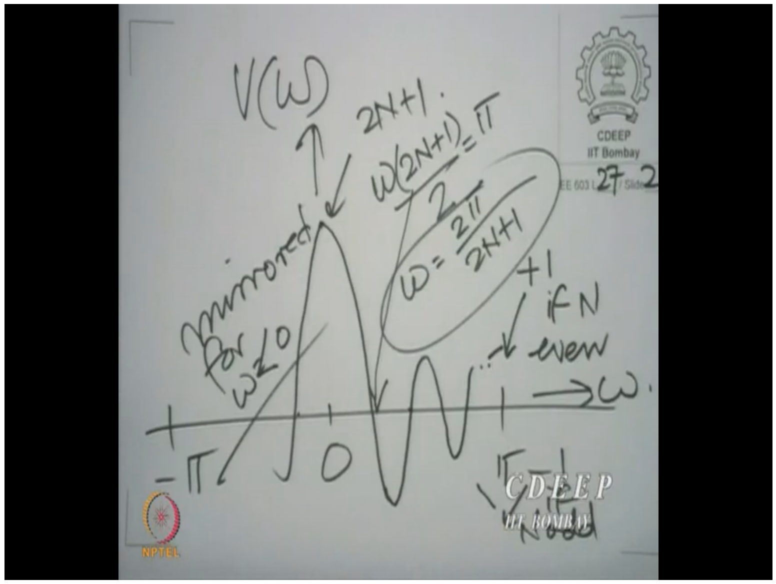

$V(\omega)$ is going to show a pattern like this, this time I go out all the way from $-\pi$to π. It is going to have its, what is called the main lobe and then it is going to have a decreasing character. So, it will either end this way or this way. Now, $+1$ if N even, and at -1 if N odd and mirrored on this side.

You see, these oscillations are going to decrease, because this begins from $2N+1$, they are going to decrease because the denominator decreases, the denominator increases, I am sorry. So, that is very obvious. You see, the denominator as you can see, is increasing from 0 to π. And the numerator is oscillatory and therefore the refraction would be decreasing in the amplitude of the oscillations.

And finally, it would reach either $+1$ or -1 here. This is the nature of the discrete time fourier transform of this window function. Now, there are two features that characterize, of course, you know where this first null would come. This first null would come where $\frac{\omega(2N+1)}{2}$ reaches the value of π or ω reaches the value $\frac{2\pi}{2N+1}$.

And the negative of that point gives you the other null, the other first null. So, it is very clear that the window discrete time fourier transform has what is called a main lobe, this is the main lobe and it has side lobes, this is the first principal side lobe and these are the other auxiliary side lobes. So, there is a principal side lobe and the word principle is meaningful because this is the most significant side lobe in terms of height.

That is to be expected because the denominator steadily decreases from, steadily increases or *1* by the denominator steadily decreases. The denominator $sin\frac{\omega}{2}$ steadily increases from *0* to π, so it is expected that the first side lobe is going to be the most significant, immediately after the main lobe.

So, you have a main lobe followed by side lobes. And of course, you know how wide these side lobes are and you know where the main lobe ends, that is not too difficult to work out. This, for example, would be the point where $\frac{\omega(2N+1)}{2}$ reaches the value 2π and therefore you can go on, in fact it is not too difficult to see that this would be twice this value, $\frac{4\pi}{2N+1}$.

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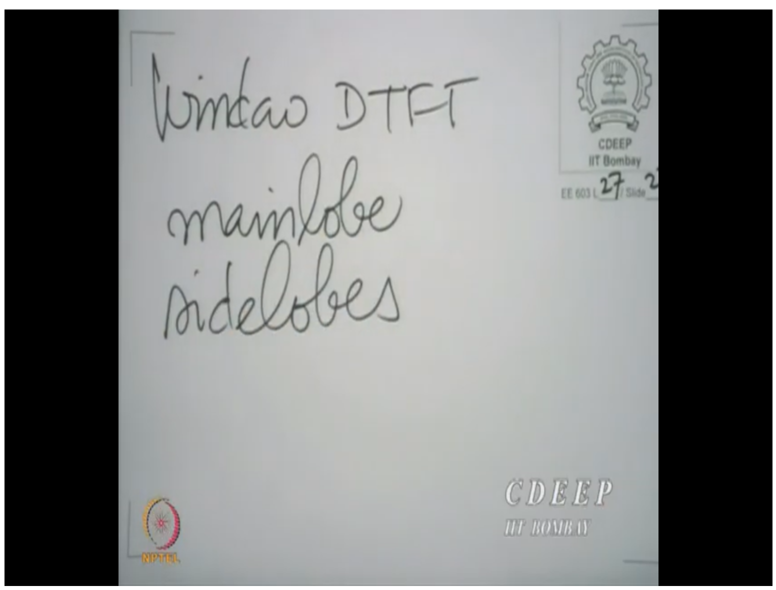

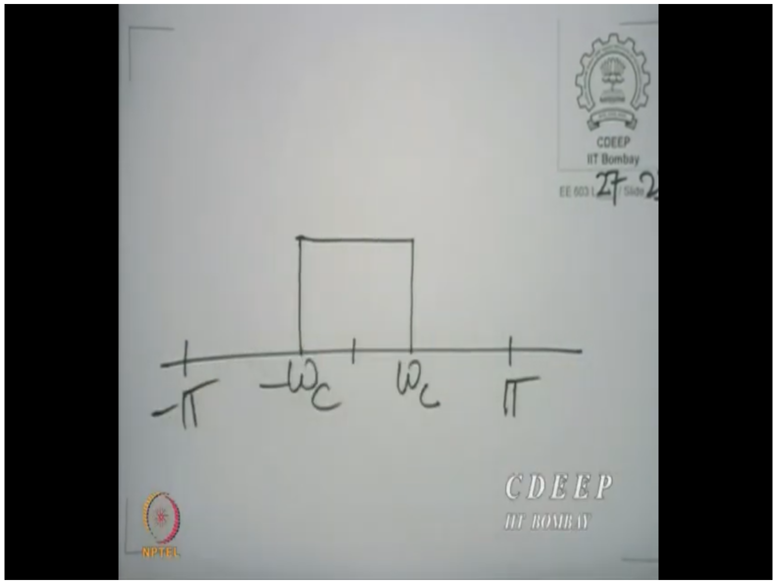

So, we have main lobes and side lobes, the window DTFT has a main lobe and side lobes. And we shall now see what role this main lobe and side lobe play in creating scars in the ideal frequency response. Let us, as we have decided to do, fix the ideal frequency response first. You have ω_c and $-\omega_c$, and π and $-$ π.

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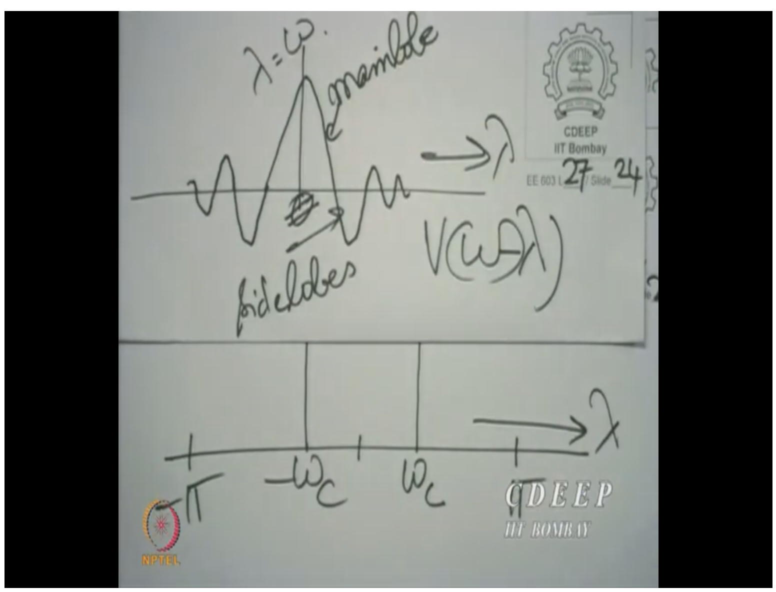

Now, what I shall do is to make the window spectrum and to do that, it is much easier for us to draw the window spectrum on a separate sheet of paper. We will exaggerate it by drawing the main lobe and side lobe in prominence and the others are kind of, they are of course there but we have drawn this. This is the main lobe and the side lobes.

Let us fold this. Let us place this here and now, let us move as we expect to do, you see now this is the λ axis, as you desire. Now, this we should, instead of calling it $V(\omega)$, we will call it $V(\omega - \lambda)$. And when we have λ here, then obviously this point is where $V(\omega - \lambda) = 0$ or $\lambda = \omega$, here.

So, that is origin *0* point is actually the $\lambda = \omega$ point now. It is moving. It that right? So, you see, now we need to move ω all the way beginning from $-\pi$ all the way up to $+\pi$ here. That is how we need to move it. And what we need to do at every step is to calculate the area under the product of these functions. You have the ideal response, you have this window response, you multiply them and then calculate the area under that product. And of course, then divide by 2π and so on.

The division by 2π is just a constant multiplication all over, so we will not pay so much of attention to that. We will just pay attention to how much is the area in the product of these functions. Now, if you look at it, when $\omega = -\pi$, as is the case here, what is the situation? The situation is that if the main lobe is small enough, and what you mean by small enough is that *N* is large enough. So, you have chosen not too small a filter length, if *N* is reasonable.

Now, what is reasonable, you have to actually test out. But if it is large enough, then the main lobe width is small enough and if it is small enough, so are the side lobe widths. So, what it really means is that around $\omega = \pi$, when this is the situation, it is only a few side lobes that are coming into this *1* part. You see, if you look at it, the product of these functions is simply that part of the window response which falls between $-\omega_c$ and $+\omega_c$, that is the product, rest of it is chopped anyway.

So, it is essentially, this window spectrum as much of it as overlaps between $-\omega_c$ and $+\omega_c$ integrated, that is the quantity at any value of ω. So, at $\omega = -\pi$, it is only a few far away side lobes which are coming into the pass band. And therefore, that area is small. Not only that, as I move this now, we are going to start moving this.

As we move this, by the way, the next main lobe, of course you will argue that this is a periodic function, we must not forget it is periodic. So, the next such main lobe is going to occur 2πaway. So, when this is at $-\pi$, the next main lobe is at π. So, anyway it is not going to bring in anything into the passband even so.

So, it is alright for us to look only at this set of main lobes and side lobes because the other one is 2π away, so it is not going to interfere. Anyway, now as you start bringing this from $\omega = -\pi$ towards $-\omega_c$, what is going to happen for some time, is that it is just these weak side lobes that are going to come into the passband.

And what is the consequence of these weak side lobes? Of course, the side lobes become stronger and stronger as ω approaches $-\omega_c$. So, the total area that is captured in the passband is going to be oscillatory because you have some side lobes that are negative, some side lobes that are positive.

So, there is going to be some negative area contribution and some positive area contribution. And initially those areas are going to be very small, later on those areas are going to grow in quantity. Now, as we come towards $\omega = -\omega_c$, it is the more important side lobe and finally the most important side lobe which is going to play the role in the area.

And afterwards, it is going to be the edge of, when you reach the edge of the main lobe here, then it is the main lobe which is going to start entering the passband and then we are going to have a totally different situation. Now, we shall take up from this point in the next lecture so see what happens and then come to a conclusion about the nature of the frequency response that results when we so truncate. Thank you.