Course on Principles of Communication Systems – Part 1
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Lecture 43
Module 7
Lloyd-Max Quantization Algorithm, Iterative Computation of Optimal Quantization Levels and Intervals

Hello, welcome to another module in this massive open online course. So we are looking at the Lloyd-max quantization algorithm for the design of the optimal quantizer and we have said that the Lloyd-max quantization algorithm we compute the quantization intervals that is basically the boundaries of these quantization intervals and the corresponding levels or the quantization the levels in each quantization interval iteratively, alright.

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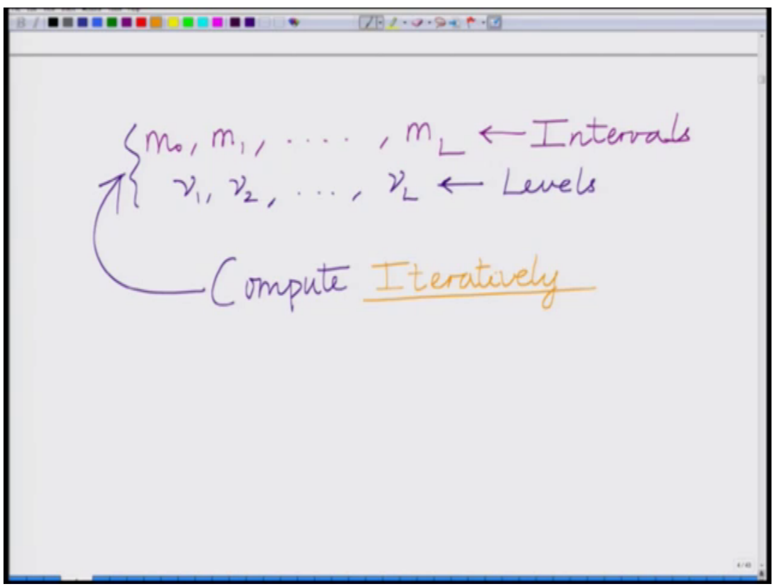

So we have set of quantization intervals which are characterized by the boundaries so these are your quantization intervals that is you have your m0, m1 up to mL this characterize the intervals, correct and also we have your nu1, nu2 so on up to nuL these characterize your levels and we compute them iteratively, okay compute iteratively as part of the Lloyd-max quantization algorithm, okay so that is what we are doing. Now let us start with computation of the quantization alright so let us start by illustrating how to compute the quantization level nuk corresponding to the quantization interval mk minus 1 mk.

That is given the quantization interval mk minus 1 to mk what is the corresponding optimal quantization level nuk, right to which sample will be to which the sample will be quantized to if it lies in this particular interval, okay.

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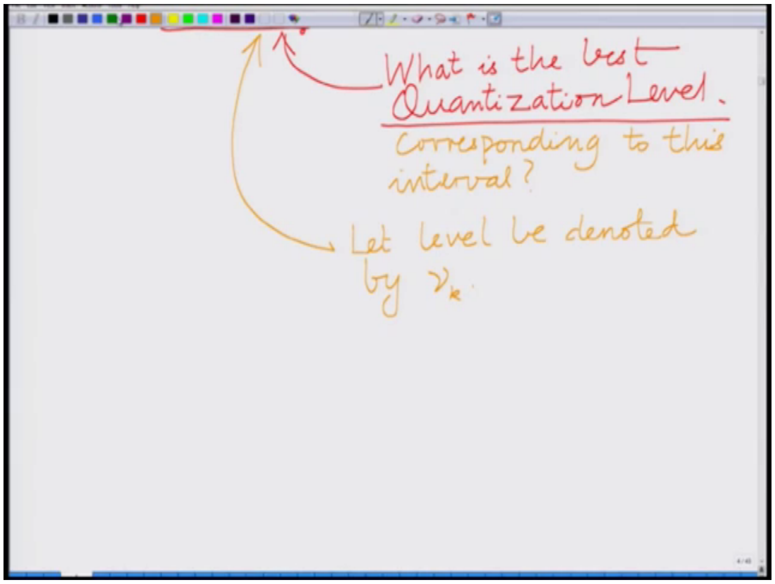

So let us consider quantization interval so consider this quantization interval consider this interval now you would like to ask the question what is the optimal or what is the best quantization level corresponding to this interval what is the best quantization level corresponding to this interval, okay.

What is the best quantization level corresponding to this interval, okay let that quantization level be denoted by nuk, alright let the level be denoted by nuk let the quantization level be denoted by nuk, okay. Now how do we choose this optimal quantization level nuk? Obviously we are interested in the lowest quantization error that is we are interested in minimizing the power or the variance of the quantization error. So we should choose the quantization level nuk in such a way that is it minimizes the quantization error but quantization error is random quantity therefore we should choose it in such a way that it minimizes the variance of the quantization error, okay.

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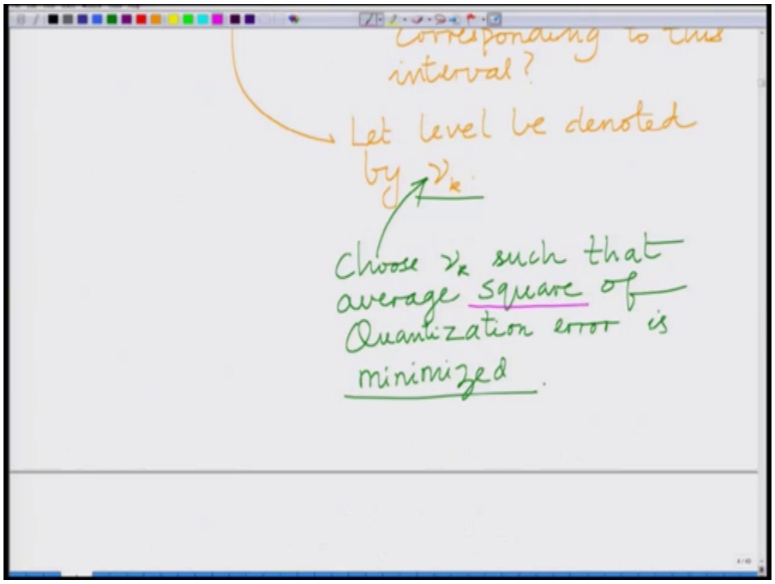

So choose nuk such that so basically choose nuk such that average, right average quantization error average square of quantization error let us put it that way square of quantization error is average square of quantization error is minimized. And further notice that we are choosing the square of the quantization error not just the quantization we are not simply minimizing quantization error the reason for this is as follows you can have a high quantization error and if it is both positive and negative with equal value equal probability then the average quantization error will still be 0, right.

So (ev) so if you minimize simply the average quantization error it is not guaranteed to minimize the (exa) absolute value of the quantization error, alright. To make sure that not just that the quantization error on an average goes to 0 but the magnitude of the quantization error is also small we minimize the square of the quantization error rather than simply the quantization error.

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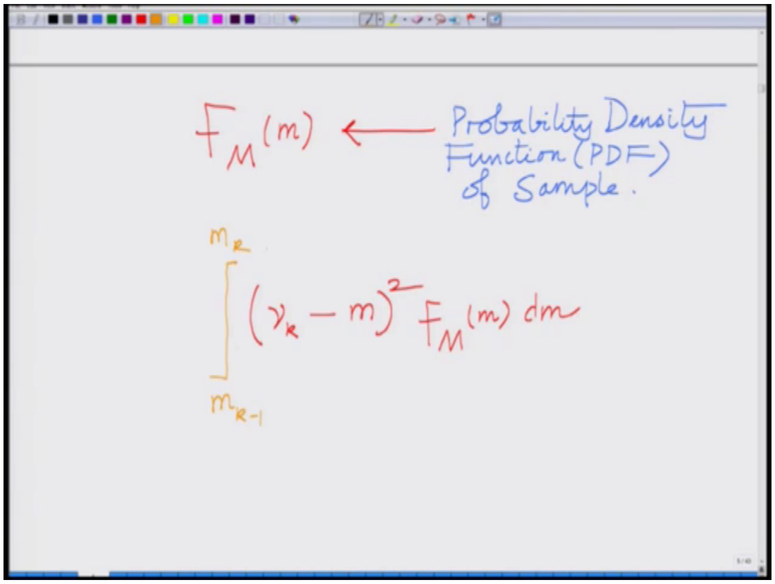

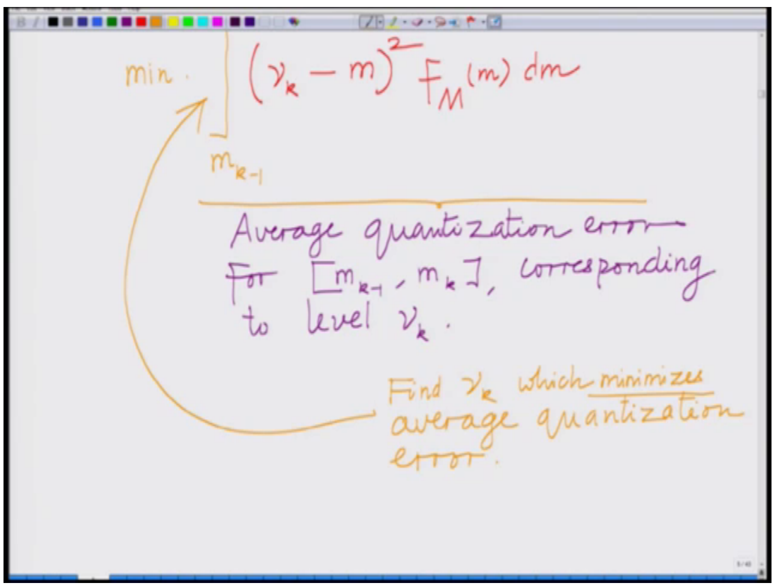

And we know that we are given that Fm(m) now for this we know that Fm(m) we know that this is the probability density function of the sample, okay.

Now therefore now what is the quantization what is the square of quantization error is basically nuk if the sample is m quantization error is nuk minus m nuk minus m square is the square of the quantization error now I have to weight by the probability density function and integrate it in this interval now what is my average I will integrate it now to compute the average over this interval mk minus 1 to mk I have to integrate it between the limits mk minus 1 to mk and now this gives me the average quantization error corresponding to threshold nuk in the interval mk minus 1 to mk, now let us write this is your average quantization error for the interval mk corresponding to the level nuk and what we would like to do we would like to find nuk which basically minimizes this, alright.

So find nuk which minimizes this quantization error or which minimizes the average quantization error, alright. Now find nuk find a nuk which minimizes the average quantization error, alright. So now we have the (quanta) average we have the means this is also known as the mean square, right the average value of the square of the quantization error the mean square quantization error corresponding to that interval mk minus 1 to mk alright in the interval mk minus 1 to mk corresponding to that particular threshold nuk.

Now I have to minimize this with respect to nuk, therefore naturally I can differentiate this with respect to nuk as set equal to 0 that will give me the value of nuk for which the quantization error or this mean square quantization error is minimum, okay.

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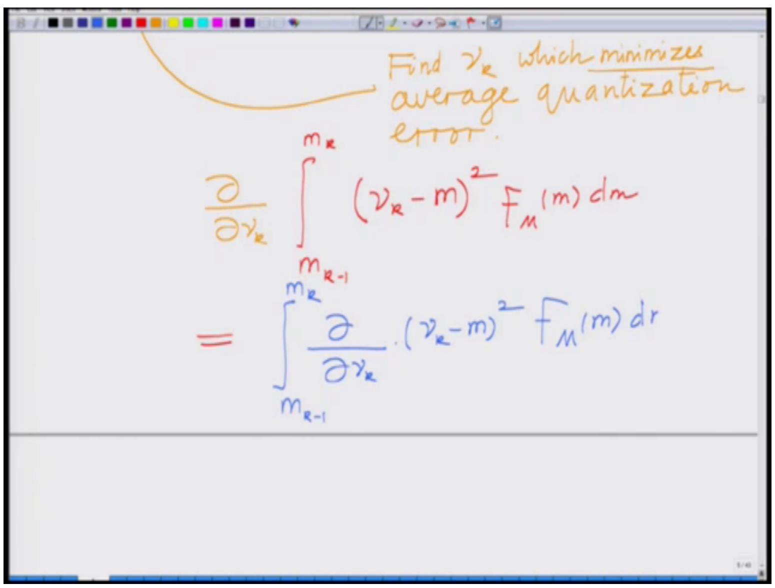


$$\int_{m_{k-1}} \frac{\partial}{\partial \nu_k}$$

$$= \int_{m_{k-1}}^{m_k} 2(\nu_k - m) \cdot f_M(m)\, dm = 0$$

To find ν_k for which Mean Squared Error is minimized.

Which Mean Squared Error is minimized.

$$\int_{m_{k-1}}^{m_k} \cancel{2}(\nu_k - m)\, f_M(m)\, dm = 0$$

$$\Rightarrow \int_{m_{k-1}}^{m_k} \nu_k\, f_M(m)\, dm = \int_{m_{k-1}}^{m_k} m\, f_M(m)\, dm$$

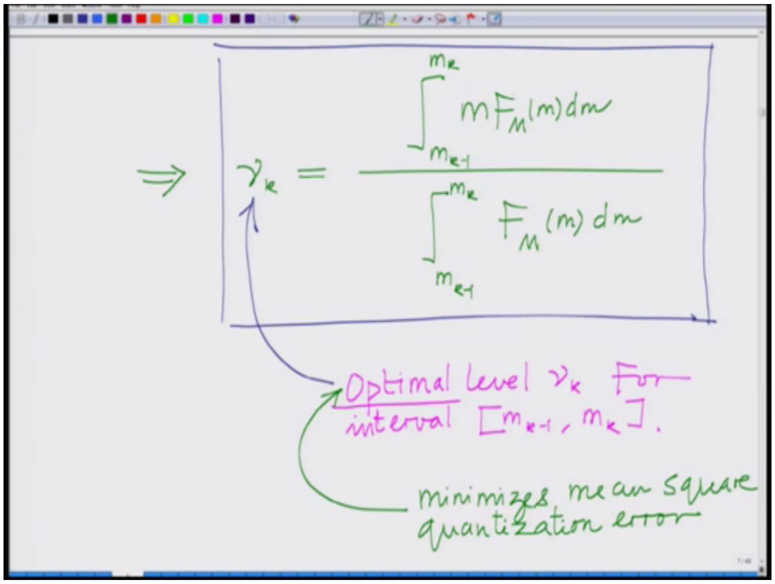
$$\Rightarrow \int_{m_{k-1}}^{m_k} \nu_k F_M(m)\, dm = \int_{m_{k-1}} m F_M(m)\, dm$$

Does NOT depend on m

$$\Rightarrow \nu_k \int_{m_{k-1}}^{m_k} F_M(m)\, dm = \int_{m_{k-1}}^{m_k} m F_M($$

$$\Rightarrow \nu_k \int_{m_{k-1}} F_M(m)\, dm = \int_{m_{k-1}} m F_M(m)\, dm$$

$$\Rightarrow \boxed{\nu_k = \frac{\int_{m_{k-1}}^{m_k} m F_M(m)\, dm}{\int_{m_{k-1}}^{m_k} F_M(m)\, dm}}$$



So therefore now to find the minimum I simply differentiate this differentiate this quantity with respect to nuk mk minus 1 to mk nuk minus well m whole square Fm(dm) remember I am using probability density function to compute the average, now take the derivative inside now this is equal to well integral mk minus 1 to mk dou partial with respect with respect to nuk nuk minus m whole square well Fm(m).

Now observe that this only this portion nu k minus m whole square depends on nu k and derivative with respect to that is fairly straight forward this is mk minus 1 to mk dou the derivative with of nuk minus m whole square is twice nuk minus m that is it Fm(m) dm. Now what we have to do is we have to integrate equate this to 0 and this is equal to 0 this we are doing to find the find nuk for which square error or mean squared error find error for which mean squared error is minimized, okay.

And therefore this equating to 0 is only for the purpose of that. Now we get an equation so now of course we can cancel the 2 so the equation that we have is mk minus 1 to mk twice nuk minus m Fm(m) dm equal to 0 now this implies if we solve this equation this implies well of course the 2 cancels mk minus 1 to mk well nuk Fm(m) dm this is equal to integral mk minus 1 to mk well Fm(m) dm. Now nuk is a constant, okay now observe that nuk is not a constant it does not depend on m so it will come outside so this does not depend on m nuk it does not depend on the sample value m, okay.

So the integral (())(12:29) with respect to m, therefore nuk is a constant with respect to m so it come out of the integral, okay and that will lead to the simplification this implies now you see something interesting this implies nuk integral mk minus 1 to mk Fm(m) dm is equal to mk minus 1 to mk m Fm(m) dm and now we have an elegant expression for nuk and this implies well nuk equals integral mk minus 1 to mk m Fm(m) dm divided by mk minus 1 to mk Fm(m) dm, okay.

And this is basically your this is basically the optimal value of the therefore this is now the optimal value for this is the optimal value nuk for the interval mk minus 1 to mk which minimize optimal in what sense optimal in the sense that it minimizes the mean square quantization error, alright optimal in the sense that minimizes the mean square quantization error, okay we can see that optimal in the sense that minimizes and you can see this formula if you look at it which gives optimal quantization level nuk if you look at it it is simply a integral mk minus what mk m Fm(m) that is weighting average of the probability density function in that interval divided by the total mass that is mk minus total probability mass, right integral mk minus 1, right? Integral mk minus 1 mk Fm(m) dm basically it is probability density function in integrated between mk minus 1 to mk which characterizes the total probability that the sample lies in that interval from a physics perspective you can think of it as the if it Fm if the probability density function corresponds to a distribution of mass you can think of it as the total mass, right characterized in that the probability mass that is characterized in the interval.

And the numerator integral mk minus 1 to mk m Fm(m) dm that gives the location, right that is basically location of the average, right average of the the distribution of the mass. Therefore the average divided by the total mass that gives the location of the center of gravity, right in a physics perspective this particular mass distribution this is similar to a physical interpretation in terms of so this quantization level basically is similar to the center of gravity corresponding to that interval mk minus 1 to mk if Fm if this probability density function Fm(m) can be thought of as a probability as the distribution of mass of an object, okay.

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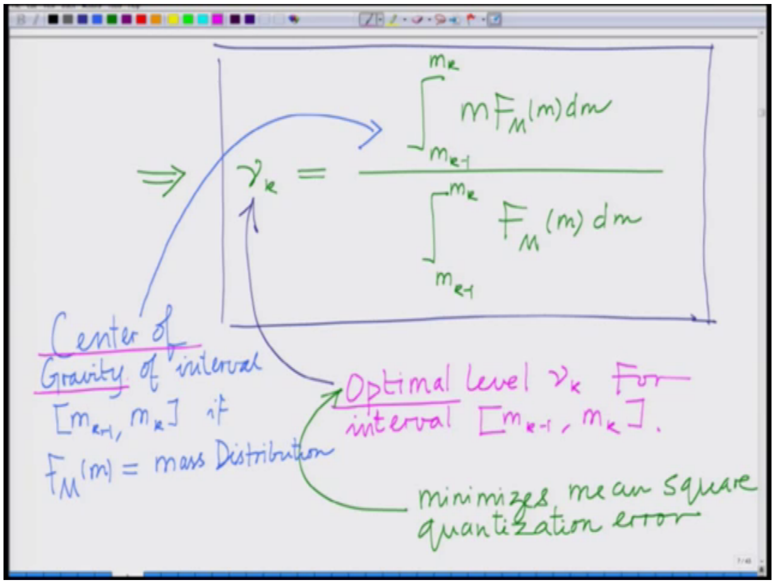

So you can see that analog this is nothing but the center of gravity if it is basically the center of gravity you can think of it as analog as if analog from physics you can think of it as a center of gravity corresponding to that interval.

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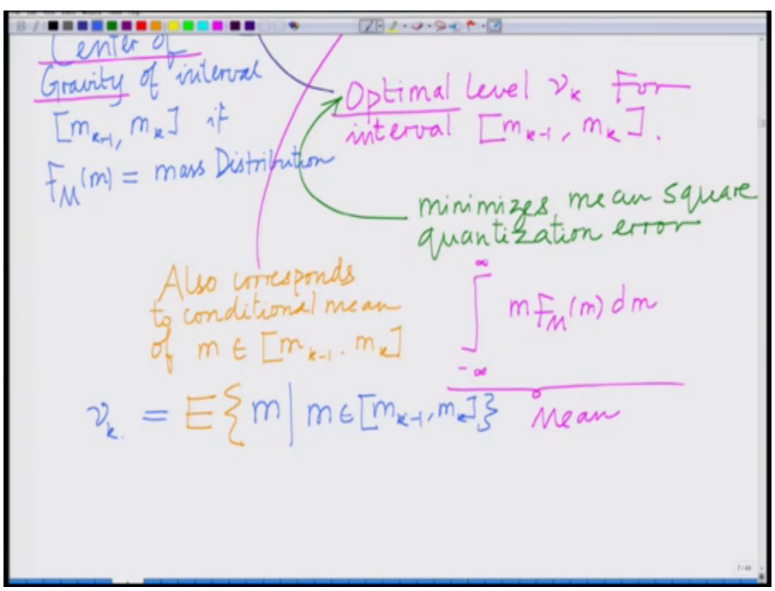

Now this also you can show that corresponds to the conditional mean already we know that what is the mean of the probability density function, okay the mean of a probability density function is again simply minus infinity to infinity this is the mean, okay.

Now this expression nuk that is integral mk minus 1 to mk m Fm(m) dm divided by integral mk minus what mk Fm(dm) this corresponds to the conditional mean of the signal sample belong that is condition on the signal sample belonging to this interval mk minus 1 to m, alright one can also note that so this also corresponds to conditional mean this also corresponds to the conditional mean of the signal sample we condition in the fact that the signals condition on the event that the signal sample belongs to this interval mk minus m.

So you can think of it that this is basically your expected value of m conditioned on the fact that m belongs to mk minus 1 comma mk this is what this is yours nu k this is the conditional mean conditioned on the fact conditioned on the event that m the signal sample lies in this quantization interval mk minus (())(19:03), alright. So that gives us a nice elegant expression to compute the optimal quantization level nuk corresponding to this quantization interval mk minus 1 to mk.

Now we have the other problem that is given the quantization levels nu0, nu1 that is nu1, nu2 up to nuL how to compute the quantization intervals m1, m0, m1 up to mL, okay.

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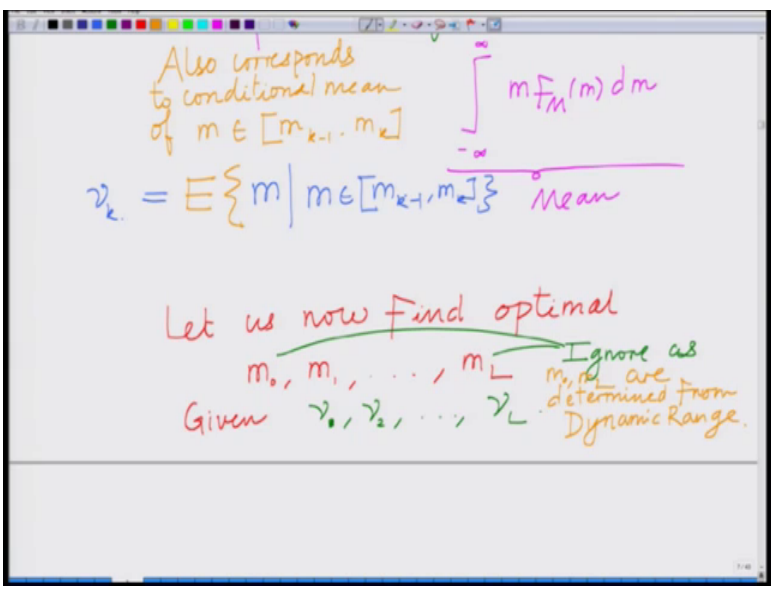

So now let us look at other problem that is now let us look at the other problem, okay now let us find let us now find optimal well the other problem m0, m1 up to mL given nu0, nu1 up to or nu1, nu2 up to nuL, okay. Now in this m0 and mL which are the first and last points first and last boundaries we can ignore them because m0 mL basically are determined based on the dynamic range of the quantizer.

So these we can ignore, so the first and start ignore as m0, mL are determined from the dynamic range these are determined from the dynamic range, alright so these we are going to ignore okay because these are fixed for a particular quantizer as we said the initial and the final values that is a total interval of quantizer the total interval of the quantization is determined by the total range of the quantizer that is determined based on the dynamic range of the signal and that has to be chosen appropriately, okay.

So this we are going to leave, so we are going to focus on how to determine the intermediate boundaries that is we are given this total interval over which the signal can lie (())(21:23) which the signal sample can lie, alright the signal sample can belong can lie in this interval m0 to mL this is known as the dynamic range of the quantizer we are going to determine where to place this intermediate boundaries that is m1, m2 up to mL by (())(21:37).

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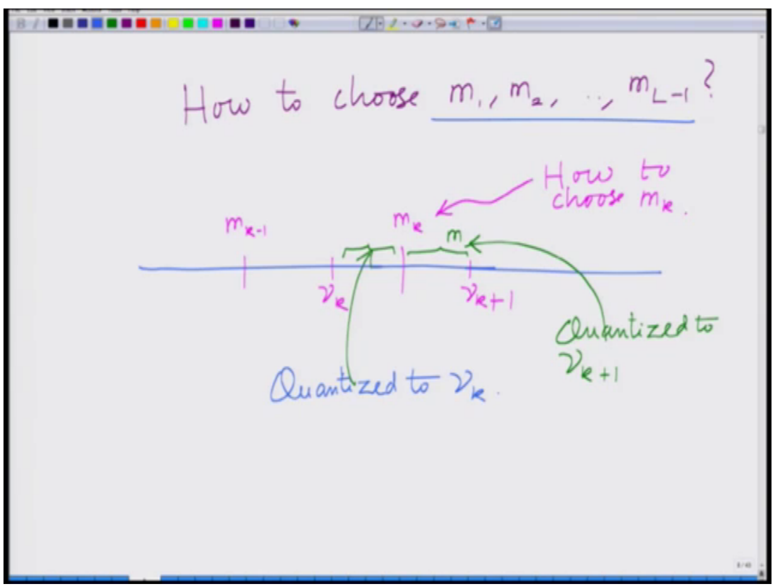

So how to choose it is really how to choose m1, m2 up to so we are really asking the question how to choose m1, m2 up to mL minus 1. And now, that is basically let us consider this scenario that is let us consider again the same scenario mk minus 1 to mk so now the scenario is different we are given well we are given nuk we are given nuk plus 1 okay alright and we are going to choose how to determine mk, okay. So we know that nuk is a quantization level corresponding to mk minus 1 to mk.

So given mk so given nuk nuk plus 1 how to choose mk , okay and this can be done as follows remember we want to choose mk now mk has this property if x is greater than mk now mk is the (quanti) now mk minus 1 to mk is the quantization interval, okay. So if x lies to the right of mk it is quantized to nuk plus 1 if x lies to the left of mk it is quantized to (nu) nuk, okay so let us note that property so if lies in this, okay this region quantized to nuk plus 1 while this region that is if sample m lies in this region this is quantized to nuk, okay.

So therefore we have to choose the this threshold, right we have to choose this the threshold in between this nuk and nuk plus 1 such that the quantization error is minimized.

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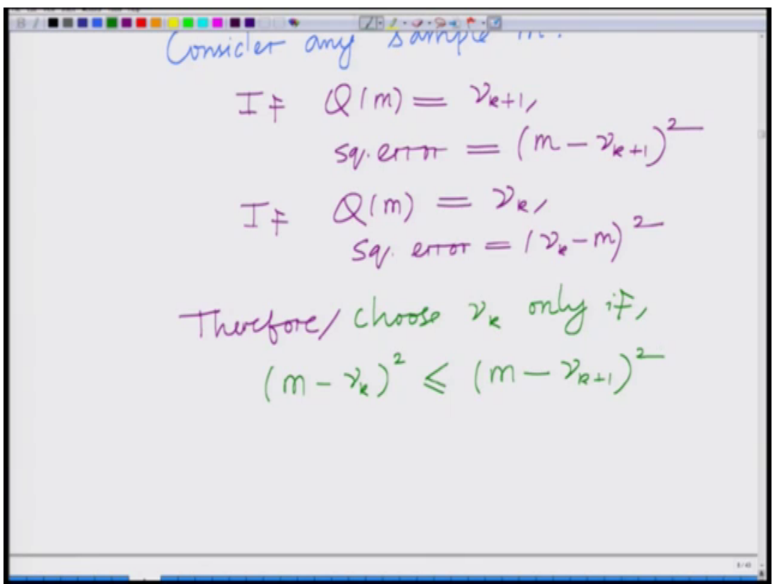

Now, if now let us look at this for now let us look at this, correct if Q(x) equal to, okay if let us consider any point x let us consider any point x, okay let us consider any point x or consider any sample x consider any sample m, okay. Now if m is quantized to nuk plus 1 that is if Q(m) equals nuk plus 1 then error or square error equals well square error equals x minus nuk plus 1 square.

Similarly if Q(m) equal to nuk on the other hand if Q(m) is equal to nuk the square error is nuk minus x whole square that is if x or I am sorry let us put this as m, okay. If Q(m) is quantized to nuk plus 1, okay then the error is m minus nuk plus 1 whole square, on the other hand if it is quantized to nuk then the error is m minus nuk whole square that is the simple fact. Now, we have to choose nuk or nuk plus 1, alright which ever minimizes the square of the quantization error.

Therefore we choose nuk that is we choose nuk that is choose nuk therefore choose nuk only if m minus nuk whole square is less than or equal to well nuk minus m or n minus nuk whole square is the same thing m minus nuk plus 1 whole square, okay that is the important observation that is your choosing nuk only if the square of the error m minus nuk square is less than or equal to m minus nuk plus 1 square that is the square of the error quantization error corresponding to choosing nuk plus 1.

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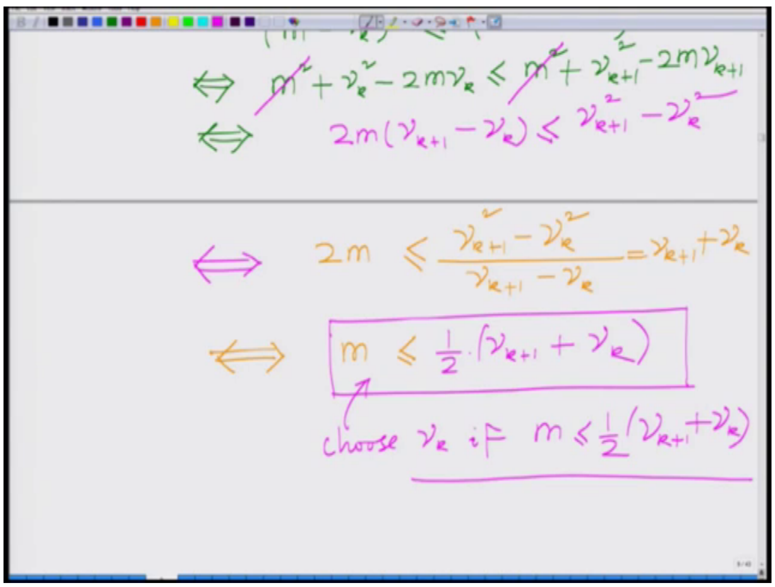

And now simplifying this implies and is implied by m square plus nuk square minus 2 m nuk less than or equal to m square plus nuk plus 1 square minus 2 m nuk plus 1 which implies and is implied by of course m squares go away which implies and is implied by well 2 m nuk plus 1 minus nuk less than or equal to nuk plus 1 square minus nuk square which is again implied and implied by note that nuk plus 1 minus nuk is positive to m less than or equal to nuk plus 1 square minus nuk square divided by nuk plus 1 minus nuk you can verify that this is simply nuk plus 1 plus nuk which is basically implied and is implied by m less than or equal to half which is basically implied implies and is implied by m is less than or equal to half nuk plus 1 plus nuk.

And remember this is choose nuk, so choose nuk if m less than or equal to half nuk plus 1 plus nuk, okay so that is the whole point that is we choose nuk if m is less than or equal to half nuk plus 1plus nuk. And therefore automatically the threshold becomes that is mk, alright automatically the threshold becomes mk which is half nuk plus 1 nuk plus nuk plus 1, okay? So if m is lying to the left of this threshold you are choosing well nuk if m is lying to the right of this threshold you are choosing nuk plus 1 and therefore the threshold that is mk the threshold is nothing but mk that is the boundary of that interval is half nuk plus 1 (())(30:03).

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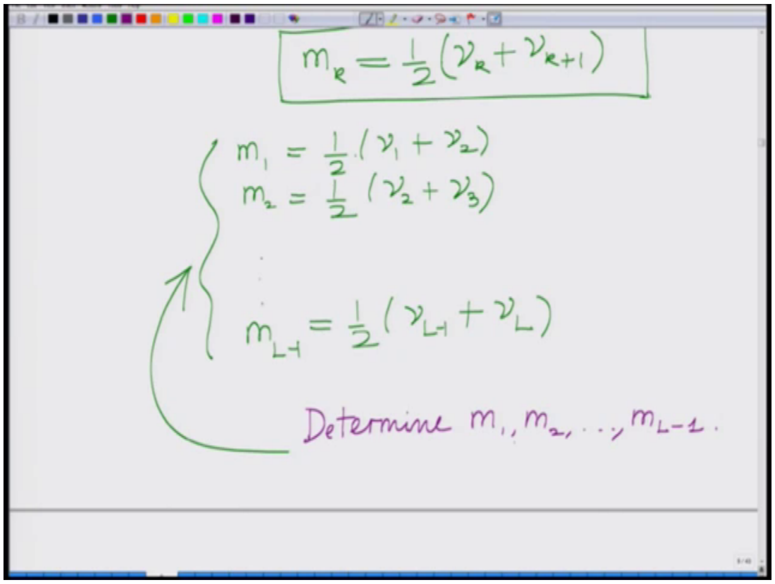

Therefore now therefore this summarizes therefore, the threshold or basically boundary of interval is well mk equals half nuk plus nuk plus 1, alright. So now given this quantization levels you can find the optimal boundaries of the quantization intervals, for instance m1 equals half nu1 plus nu2, m2 equals half nu2 plus nu3 so on and so forth mL minus 1 equals half nuL or nuL minus 1 plus nuL, okay these are determined these determine m1, m2 up to L minus 1.

So these determine your m1, m2, mL minus 1 we said that m0 and mL we are going to determine them from the dynamic range of the signal so we are not varied about that, we are determining the intermediate threshold (cor) corresponding to the various (())(32:09). So m0 to m1 that is one interval m1 to m2 that is next so on and so forth mL minus 1 to mL that is the last interval, okay now from the quantization levels we have determined the intervals.

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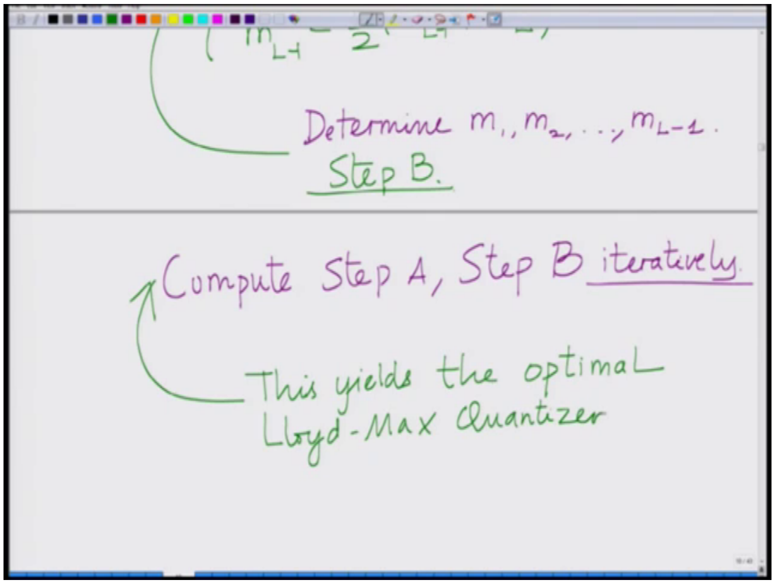

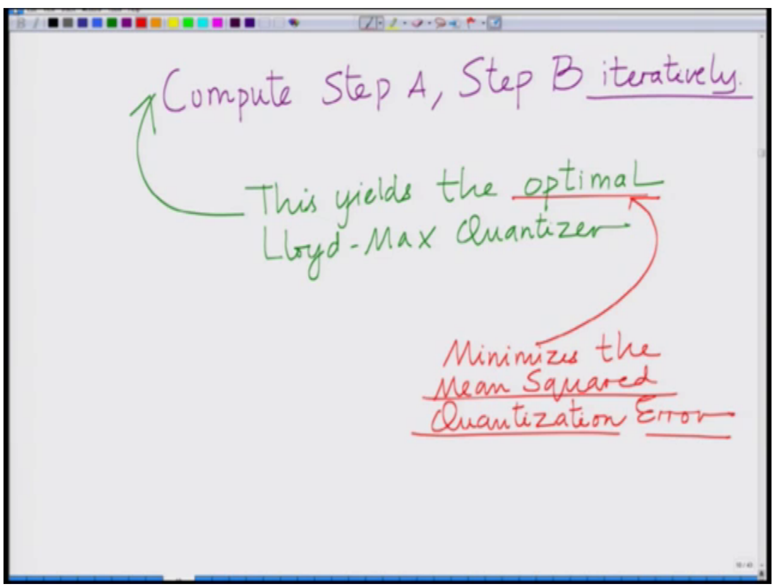

Now again from the intervals one can go back and determine the optimal quantization levels nu1, nu2, nuL that is step 1, okay so this is step 2 alright if you remember this is basically your step 2, okay if you go all the way back and look at our Lloyd-max quantizer design you will recall that basically the Lloyd-max I am sorry this is step B, alright step A is to determine the (qua) given the quantization intervals determines the levels that is what we have done earlier, alright the levels can be determined as the center of gravity of each interval treating the probability density function of the sample m as the (prob) as the mass as the distribution of the mass, okay that is step A.

Now step B is basically given the quantization levels determine the intervals and we have determined the (())(33:12) simply as the mid-point of each nuk plus 1 that is half of nuk plus nuk plus nuk plus 1 is nothing but the mid-point between nuk and nuk (())(33:19) that determines the boundary mk, alright so that is step B compute step A and step B iteratively, okay that gives the optimal quantizer, okay so we have basically this is your step B now compute step A step B iteratively multiple iterations this gives the optimal quantizer this yields the optimal Lloyd-max quantizer.

This yields the optimal Lloyd-max quantizer again optimal in what sense optimal in the sense that it minimizes the mean square minimizes the mean square quantization error so that is what we are getting, okay. So compute step A and step B iteratively that yields the optimal Lloyd-max quatizer which minimizes the mean square quantization (())(35:17) that gives us how to determine the optimal quantizer, remember previously we had only seen a very simplified version of the quantizer that is a uniform quantizer which need not which is a very simple which is simple to implement but (ob) (ob) obviously need not be optimal optimal in the sense that the quantization error (())(35:33) very large. So the Lloyd-max algorithm gives an iterative a convenient iterative algorithm to determine the optimal quantizer optimal quantizer in the sense the quantizer in the sense that the quantization intervals and the quantization levels which minimize the mean square quantization error, alright.

So we will stop this module stop here and look at other aspects in subsequent modules thank you very much.