Fuzzy Sets, Logic and Systems and Applications
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Lecture - 29
Fuzzy Relation

Welcome to lecture number 29 of Fuzzy Sets, Logic and Systems and Applications. So, in this lecture today we will discuss Fuzzy Relations. Before discussing fuzzy relations, let us first discuss the Cartesian product of fuzzy sets and then from this Cartesian product of fuzzy sets we will find the fuzzy relation set and accordingly you know we will move ahead.

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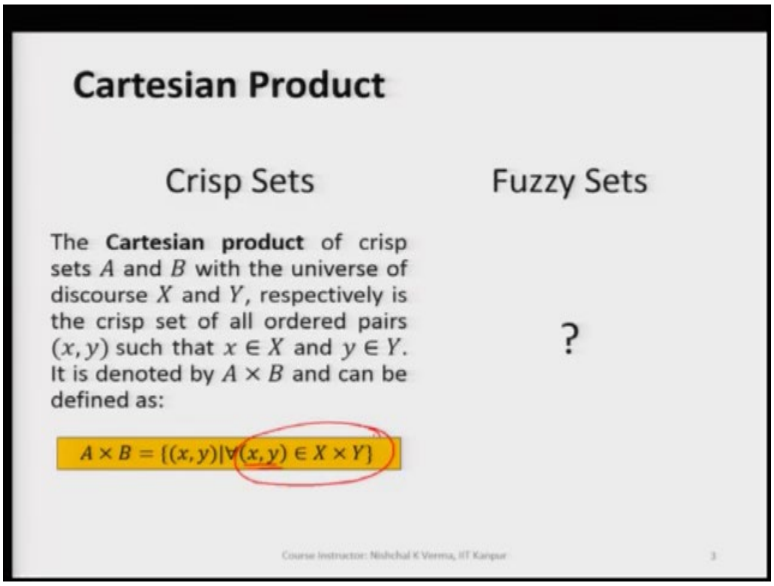

In the previous lecture we have seen the Cartesian product of a crisp sets. And, in the Cartesian product of a crisp sets we had a set $A \times B$ that is actually the collection of all the elements. And, these elements are the ordered pairs of the elements from A and B. Again, this pair must belong to the universe of discourse that is $X \times Y$.

So, we can speak it like this like the Cartesian product of crisp sets A and B with the universe of discourse X and Y respectively is the crisp set of all ordered pairs x, y, such that x belongs to X and y belongs to Y and it is denoted by $A \times B$. So, this all the ordered pairs X, Y must belong to the universe of discourse $X \times Y$ which is here.

So, this must be understood that all these ordered pairs x, y will be from the universe of discourse that has been created from X and Y by taking the Cartesian product. So, this was all about the Cartesian product of crisp sets.

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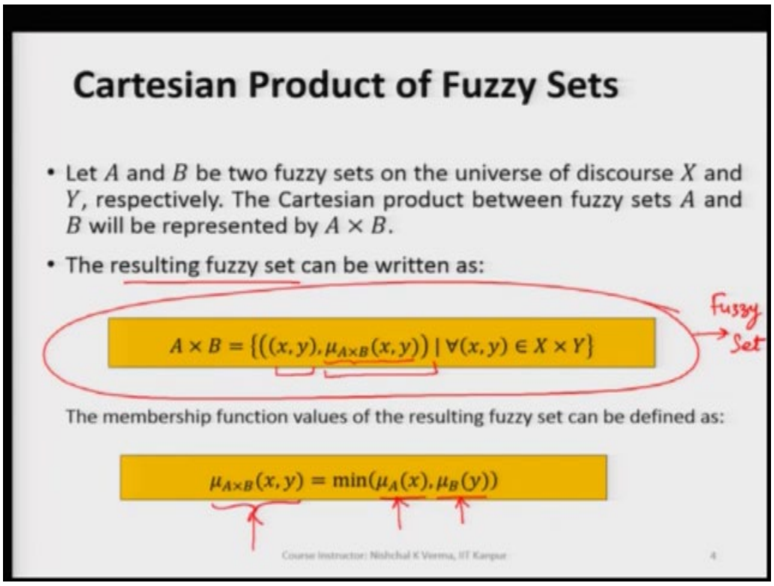

Now, let us see the Cartesian Product of Fuzzy Sets. So, if we have A and B two fuzzy sets on the universe of discourse X and Y respectively.

The Cartesian product between fuzzy sets A and fuzzy set B will be represented by $A \times B$ same as we have had in crisp set case. So, the resulting fuzzy set can be written as $A \times B$ and of course, this $A \times B$ is the resulting fuzzy set because this $A \times B$ is going to take the form of a fuzzy set. So, $A \times B$ can be written by this expression here please understand and we can write it like this like A cross B is equal to collection of all the ordered pairs x, y.

Along with you see here along with the membership value and this is the membership value corresponding to x, y, this is ordered pair corresponding to the ordered pair element. So, $\mu_{A \times B}(x, y)$ is important here. So, we see that we have another element here which is along with the ordered pair you see x, y and so you see x, y here which was there in case of crisp Cartesian product.

So, this you know in case of Cartesian product of crisp sets, we had only the ordered pairs means x y. Now, here we have along with x, y we have the $\mu_{A \times B}(x, y)$. So, this is because it is a fuzzy set and mu is important because all the ordered pairs will have it is corresponding membership values. So, that is why this $\mu_{A\times B}$ has been added along with the ordered pairs.

So, this is a very important point that has to be noted this we have already discussed when we were discussing the difference between a crisp set and the fuzzy sets. So, this is now clear and now the membership function values that means the membership values corresponding to the generic variable values arising from the x, y, that is I know this membership value depends on both the generic variable values x is for y.

So, you see here and this is going to be computed. So, that is very important here to note that $\mu_{A\times B}(x, y) = \min(\mu_A(x), \mu_B(y))$. So, since we already have a fuzzy set A and fuzzy set B. So, we will be getting this $\mu_A(x)$ from fuzzy set A and then here we will get the $\mu_B(y)$ from fuzzy set B. So, this is very important point that has to be noted.

And, this way by taking mean of these two values corresponding to the generic variable value x,y we get the $\mu_{A\times B}(x, y)$. So, this way we compute the membership value, of the corresponding generic variable values, that is x, y and this is nothing, but the ordered pair element of $A \times B$. So, with this we now know that how $A \times B$ is going to look like.

So, $A \times B$ is here the Cartesian product and this Cartesian product is going to be the fuzzy set again. So, Cartesian product of two fuzzy sets is again going to be a fuzzy set. And, that is why we have the element and it is membership value associated with it as a element of the $A \times B$ set.

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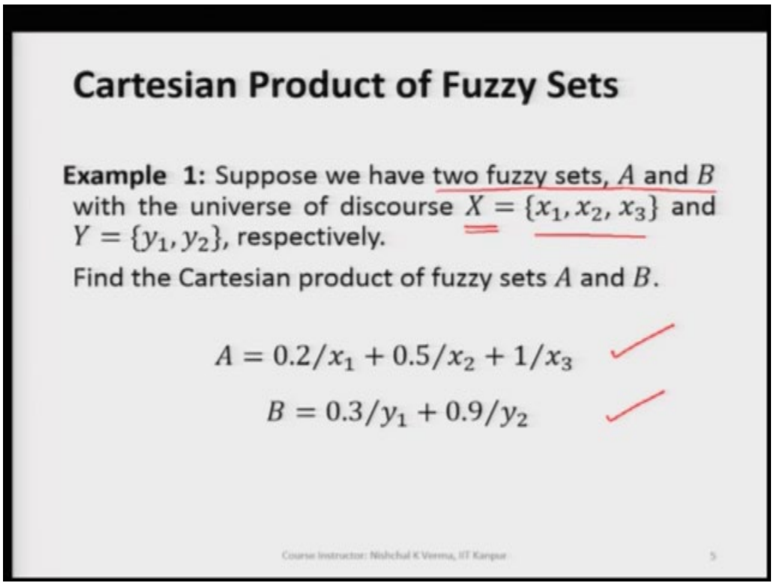

Let us take an example here to understand this thing better. So, if we take an example like this like here we have two fuzzy sets A and B. And, here we have the universe of discourse as X, which is equal to the x_1, x_2, x_3 means we have in the universe of discourse 3 elements x_1, x_2, x_3 and in the universe of discourse Y we have y_1 and y_2.

So when we have this the fuzzy set A B and the universe of discourse is already given. So, then let us find the Cartesian product of fuzzy sets A and B. So, we apply the expression that we have just discussed and substitute the values of fuzzy set A and fuzzy set B, which is given here. So, since A and B are given as $0.2 / x_1 + 0.5 /x_2 + 1 / x_3$.

So, this is fuzzy set A and here B is equal to $0.3 / y_1 + 0.9 / y_2$. So, this our fuzzy set B and please note that both of these fuzzy sets A and B are the discreet fuzzy sets.

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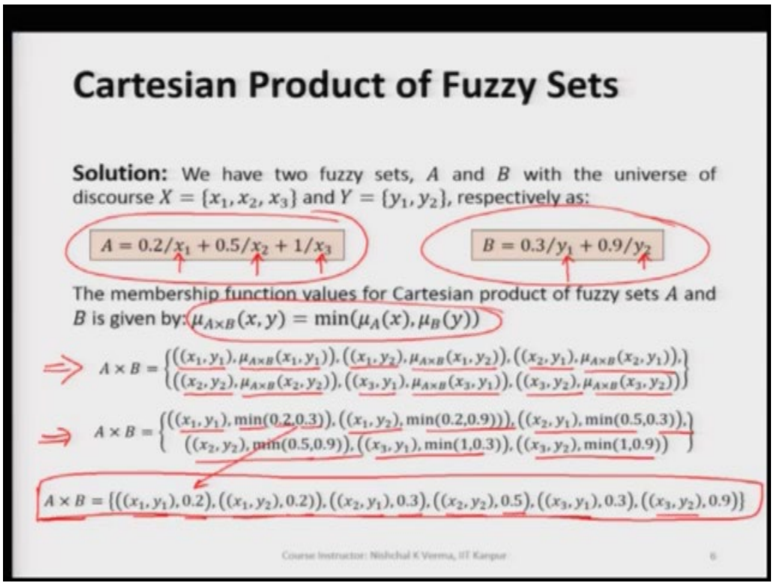

Now, let us quickly find $A \times\ B$. A is given here as a discreet fuzzy set and B is given as another discreet fuzzy set. So, let us find the $A \times\ B$ which is the Cartesian product. So, just apply the syntax of the set that we have discussed.

So, what we do here is we find all the ordered pairs of A and B elements. So, we find the ordered pair generic variable values from A and B. So, this way we get here since we have in fuzzy set A, we have x_1, x_2, x_3, you can see here x_1, x_2, x_3 and in B fuzzy set we have y_1, y_2. So, when we take the Cartesian product of A and B, that is A cross B, we are getting here as $x_1 y_1$ as one of the ordered pair elements.

And, then this ordered pair element will have it is associated membership value and I told you as to how you are going to get this value, by taking the min of the corresponding membership values from A and B corresponding to x_1 and y_1 respectively. So, here we are writing $\mu_{A\times B}(x_1, y_1)$. So, this is going to be one of the elements of the resulting fuzzy set.

And, similarly now we'll have $x_1\ y_2$, so we have $x_1 y_2$ and then the corresponding membership value. Similarly, we'll have $x_2 y_1$ and then the corresponding membership value. We will have $x_2 y_2$ then corresponding membership value, $x_3 y_1$ then the corresponding membership value, $x_3 y_2$ then the corresponding membership value. Since, we have in fuzzy set A x_1, x_2, x_3 in fuzzy set B y_1, y_2.

So, we have three elements in fuzzy set A, 2 elements in fuzzy set B. So, we will have total of 6 elements that is 3 into 2, that means 6 elements in the $A \times B$ set, that is the Cartesian product of fuzzy set A and fuzzy set B. So, then when we compute $\mu_{A\times B}(x_1, y_1)$, similarly$\mu_{A\times B}(x_1, y_2)$ all of these, that I have just mentioned as the membership values.

So, when we use the min criteria to find the respective membership values. So, when we use this you see here, that for x_1, y_1 we have to take the $\min(\mu(x_1), \mu(y_1))$. Similarly, all other corresponding membership values will be computing using the min criteria as I just mentioned here and it is also written here you can see. So, this way our $A \times B$ becomes the collection of all the elements.

And, these elements are nothing, but the ordered pair along with the it is corresponding membership values, which are calculated by using min criteria. So, for x_1, y_1 we have 0.2, because if we take the min here you can see that when we take min of 0.2 and 0.3 we get 0.2. Similarly, for x_1, y_2 we get 0.2 for x_2, y_1 we get 0.3, for x_2, y_2 we get 0.5, for x_3, y_1 we get 0.3, for x_3, y_1 we get 0.9 as the associated membership values. So, now, very easily we have found out the Cartesian product of two fuzzy sets. So, in this fuzzy set we have the ordered pair elements along with it is associated membership values.

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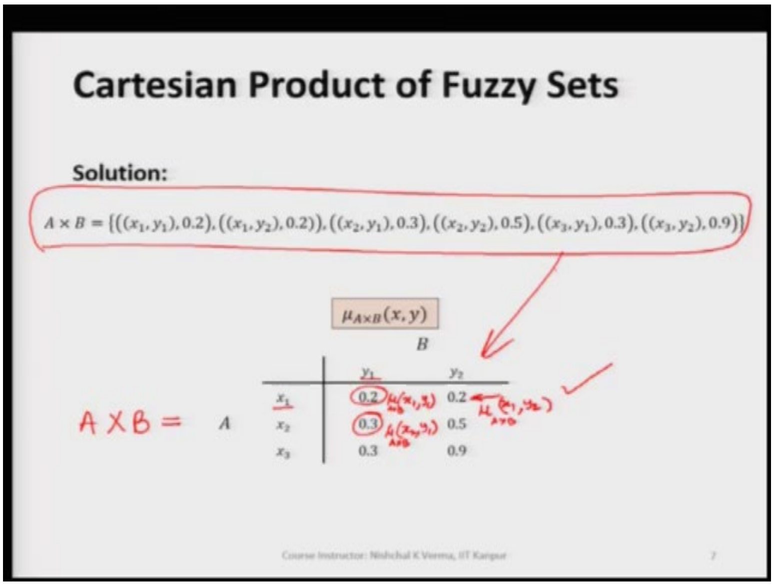

Now, the same can be written here by this equation and then this can also take the matrix form. So, in the matrix form, if we are interested in writing the same equation the Cartesian product we can use this matrix form here.

So, where we have the column elements as the generic variable values of the fuzzy set A and then we have the row as the elements the generic variable values of the fuzzy set B. And, when we have these two as row and columns values, then corresponding to x_1, y_1 we write here as the $\mu(x_1, y_1)$ like this and here we write the mu of x_2, y_1.

So, like that all the values are represented here. And of course this is nothing, but this is from the Cartesian product of A and B, that is $A \times B$. So, this is very easy to represent. And, similarly here also we have x_1, y_2 and this is corresponding to $\mu(x_1, y_2)$. Since, this is from the $A \times B$ set that is the Cartesian product we can also write $A \times B$ here also we can write $A \times B$.

So, this way we see as to how all these membership values associated with the combination of generic variable values from fuzzy set A and fuzzy set B that is x_1, x_2, x_3, y_1, y_2. So, we have listed here all the membership values as elements of the matrix of A cross B that is Cartesian product of A and B. So, this is another way of writing this equation here. So, the same is represented in this form.

And, this is actually easier to understand here by just looking at the elements we can understand. And, these values once again I am telling that these values are here the these values these elements of the matrix are nothing, but the associated membership values. And, these membership values are computed by taking the min of the corresponding membership values from fuzzy set A and fuzzy set B.

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So, once we now know the Cartesian product of two fuzzy sets A and B. Now, let us understand the fuzzy relations on the same lines as we have seen in the crisp relation and we will understand as to how we find the fuzzy relation from the Cartesian product.

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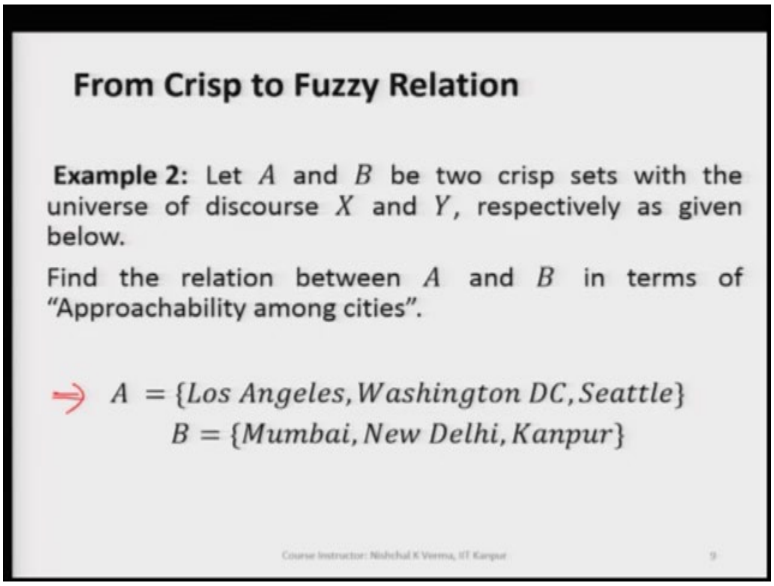

So, here let us first take A and B two crisp sets with the universe of discourse X and Y, respectively and here we are interested in finding the relation between A and B. In terms of the approachability among the cities, means if I am interested to move or approach from one city to another, what is the relation? Right.

So, here we have two crisp sets, the crisp set A here is the set of cities from US, that are Los Angeles, Washington DC, Seattle and then we have another crisp set B which is set of Indian cities like Mumbai, New Delhi, Kanpur. So, here we have two crisp sets A and B.

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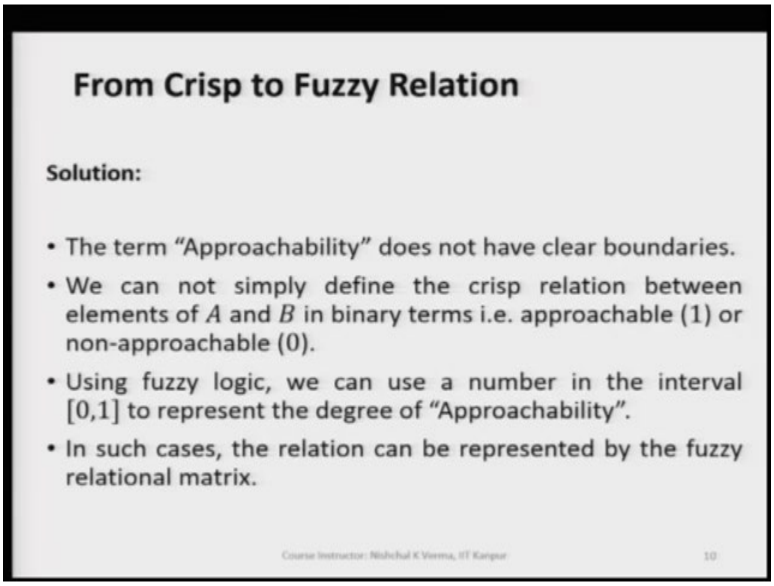

So, here the term Approachability does not have clear boundaries. So, here we cannot simply define the crisps relation, between elements of crisp set A and crisp set B in binary terms. That is approachable, 1 or non-approachable 0, means 1 for approachable and 0 for non-approachable. So, when we talk of approachability from one city to another. So, we cannot express this thing in 0 and 1.

So this is very clear that Boolean logic is not sufficient here to express the approachability among the cities. Now, here when we use fuzzy logic we can see that, this approachability can be expressed very easily and now let us see how it is done. So, in such cases the relation can be represented by fuzzy relational matrix when we fuzzy relation.

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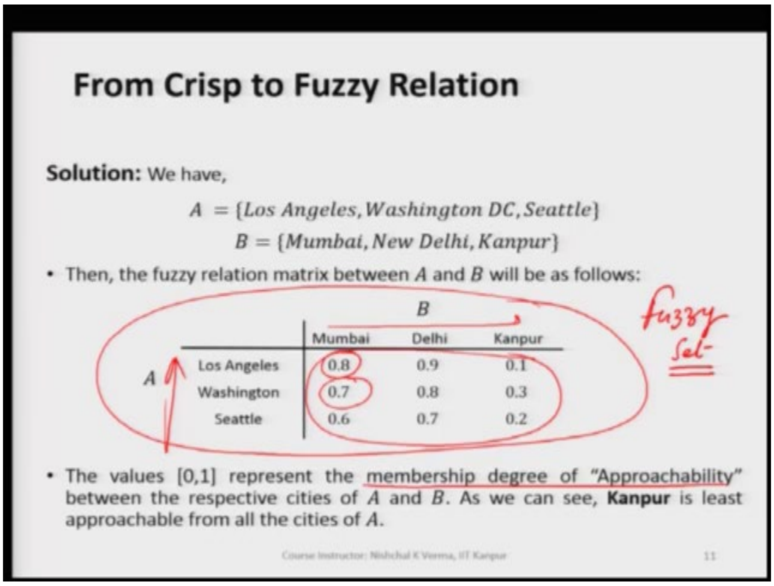

So, you see here as I have already mentioned that we have two sets the A and B. So, both the sets are crisp sets, but when we talk of approachability this cannot be very well expressed, but then when we use a fuzzy relation matrix as you have seen in the previous slides, that we have A as the set of cities and B another set of cities. So, the elements here are nothing, but the membership values.

The elements of this matrix the relational matrix nothing but the membership value is like $0.1, 0.7, 0.6$ in first column; $0.9, 0.8, 0.7$ the second column; $0.1, 0.3, 0.2$ in another column. So, this is here is nothing, but A fuzzy set. So, when we use fuzzy logic we can represent the approachability in a very convenient fashion. And, here what this represents is like if we are interested in traveling from the Los Angeles to Mumbai we have 0.8 as the degree of membership.

This means that you know it is approachability is very good that is 0.8 or if we want to compare with the other cities like Washington and Mumbai which is 0.7. So, this means that the traveling from Washington to Mumbai is not as good as traveling from the Los Angeles to Mumbai, when it comes to approachability.

So, here we have all these membership degrees as the degree of approachability see here. So, this is in a more convenient fashion we can represent when we take A and B both are the fuzzy sets. So, if we take only crisp sets, it is very difficult because this kind of things cannot be represented in Boolean logic.

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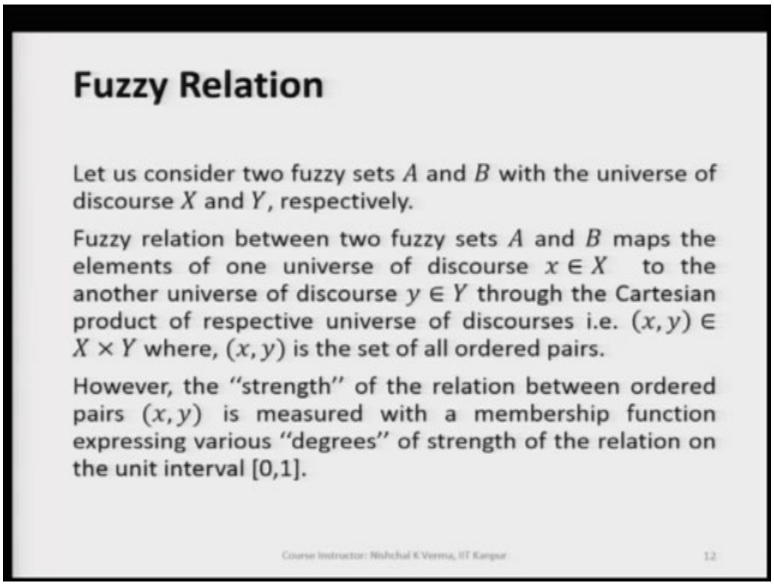

So, as I have already mentioned that when we talk of fuzzy relation normally, fuzzy relation is the Cartesian product as we have seen. So, the Cartesian product of two fuzzy sets result basically a fuzzy set again and this is nothing but the fuzzy relation unless otherwise some other condition is satisfied and if some other condition is satisfied, then the relation set based on certain conditions will be a subset of the Cartesian product of the two sets A and B.

And, of course, here this has to be noted that we have a universe of discourse which is again the Cartesian product of X and Y. So, we have a space that we have the Cartesian product space capital $X \times Y$ and this represents the you know the universe of discourse of the Cartesian product. And, since we have already used Q for representing the relation set like in crisp sets we have seen.

So here also we are using $Q(x, y)$. One more thing has to be noted here, that when we are going for Cartesian product.

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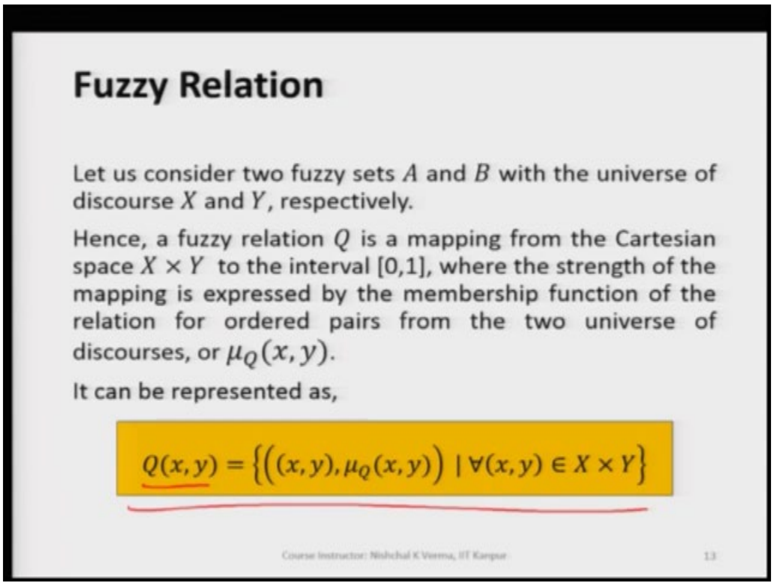

So, we need to understand here that Cartesian product fuzzy set here will have the increased dimension, like fuzzy set A which is defined in the universe of discourse X and the generic variable value here is x, whereas, fuzzy set B which is defined in the universe of discourse Y and here the generic variable value is y.

So, both these sets are defined in single dimension X and Y respectively, but when you combine the them together, when you take the Cartesian product of these two fuzzy sets. What we get here is a resulting fuzzy set, which is a surface, which is nothing, but it is a increased dimension it is a two dimensional X and Y, and then when we take a another dimension as the membership values of the corresponding X and Y then we get a 3-D representation.

So that is how it is normally called as the surface. So, surface is a 3-D surface is represented in 3 3 D space. So, that is why $Q(x, y)$ here is a fuzzy set. And, this is again if it is based on certain condition, we get this set if there is no condition then the complete ordered pair elements along with the membership values are included here in this set.

And, as I already mentioned that this fuzzy set is a 3-D is represented in 3-D space. So, two dimensions are for X and Y, generic variable values and then mu of X and Y will be the third dimension, which is here for representing the membership degree. So, that is why it is a 3-D representation.

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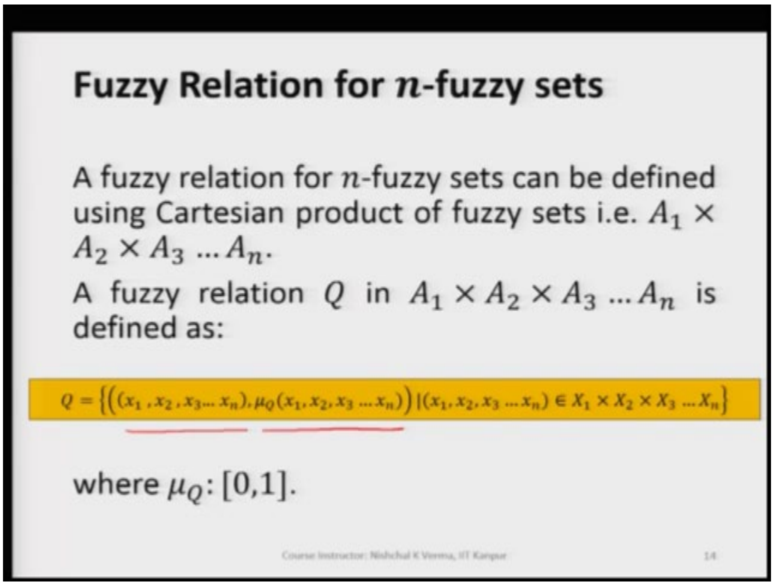

Now, like in crisp relation we have seen for n crisp sets the crisp relation for n or crisp relation of n crisp sets.

Here also let us see, how the fuzzy relation for n fuzzy sets look like. So, here also we have $A_1, A_2, A_3 \dots\dots A_n$, n number of fuzzy sets. So, let us take the Cartesian product of this n number of fuzzy sets. So, how will these look like? So, again like you have seen in the previous slide, that if we take the Cartesian product of two fuzzy sets we get the increased dimension here, also we get the dimension increase small n number of times.

So, here we have the ordered pairs as you see here x_1, x_2. So, this is not called ordered pair this is called basically n tuples, n tuples are here like $x_1, x_2, x_3 \dots\dots x_n, \mu_Q(x_1, x_2, x_3 \dots\dots x_n)$. And, so, that is how it looks like and then again this $x_1, x_2, x_3 \dots\dots x_n$ this will be from the Cartesian product of capital X_1, X_2, X_3 and so on up to capital X_n is space.

So, this has to be noted that here also we have a fuzzy set, which is arising out of the Cartesian product of n fuzzy sets. And, of course, the dimension here will be the n dimensional fuzzy set.

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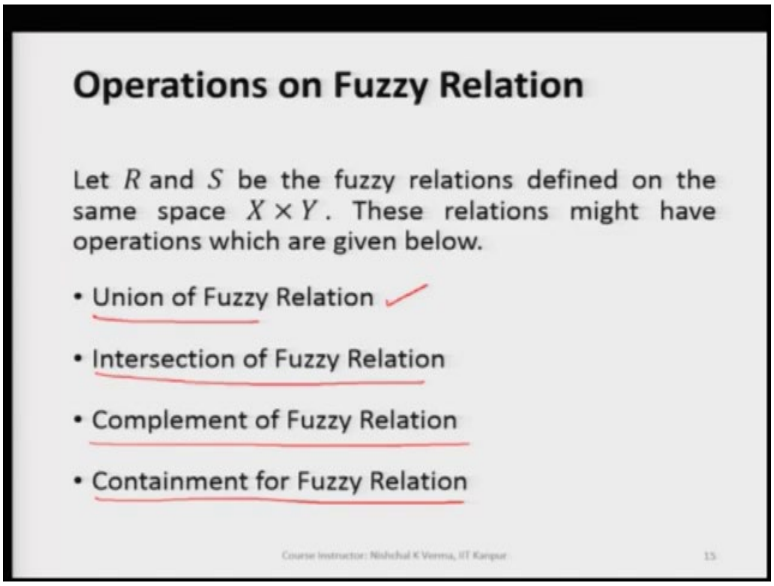

Now, let us move ahead and see some operations on fuzzy relation. Please understand fuzzy relation is a fuzzy set. Fuzzy relation is the fuzzy set in an increased dimension.

So, like any other fuzzy set the fuzzy relation is also fuzzy set. So, if this can be represented by R and S. So, and again if we define the space the universe of discourse as X crosswise. So, these relations will have the operations as given below here as the union of fuzzy relations.

So, we have the union of these fuzzy relations, and then intersection of the fuzzy relations, complement of the fuzzy relation, and then the containment of the fuzzy relation. So, with this we will stop here in this lecture.

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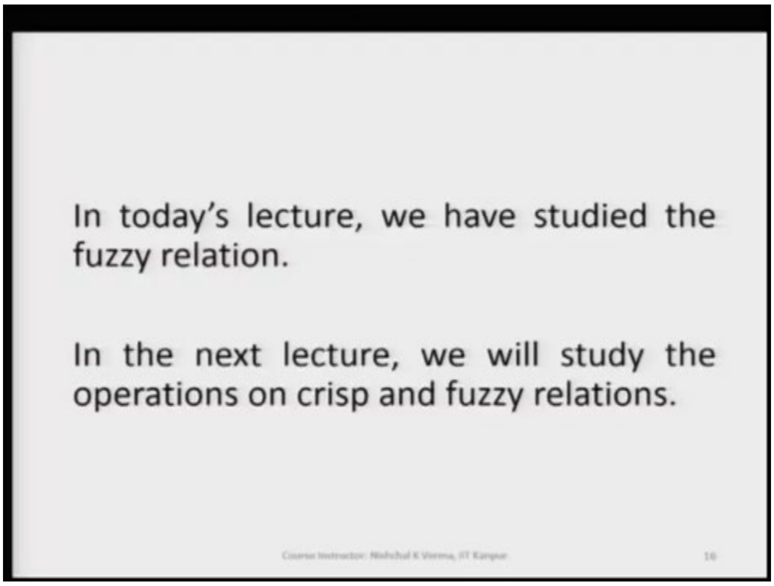

And, in the next lecture we will cover the these operations first on crisps and then on fuzzy relation sets.

Thank you.