Advanced Microwave Guided Structures and Analysis

Professor. Bratin Ghosh
Department of E & ECE
Indian Institute of Technology, Kharagpur

Lecture No. 20

Relation between Wavenumbers, Radiation from an Electric Current Source

Welcome to this session of the lecture on the Relationship between the Wave Numbers in a homogeneous medium and from there we will then cover and discuss the Radiation from our Electric Current Source in a homogeneous medium. So, let us go to the lecture. So, let us look at the Helmholtz wave equation again. So, let us write it down again.

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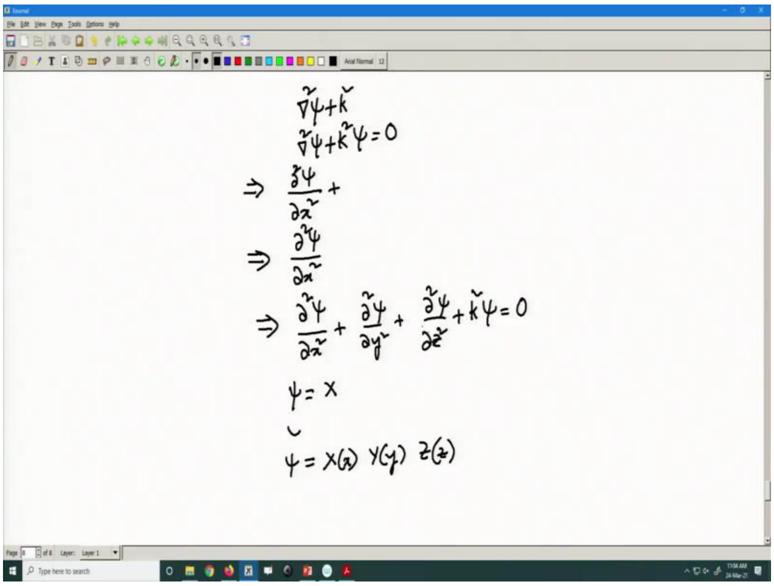

Now, writing this explicitly in terms of x y z variation, we have $\nabla^2\psi + k^2\psi = 0$, writing this explicitly in terms of x y and z variations, we have $\frac{\partial^2\psi}{\partial X^2} + \frac{\partial^2\psi}{\partial Y^2} + \frac{\partial^2\psi}{\partial Z^2} + k^2\psi = 0$.

Now, we invoke a well-known principle in the theory of partial differential equations, which is the separation of variables technique in order to solve this equation. So, we express psi as $\psi(x, y, z) = X(x)Y(y)Z(z)$

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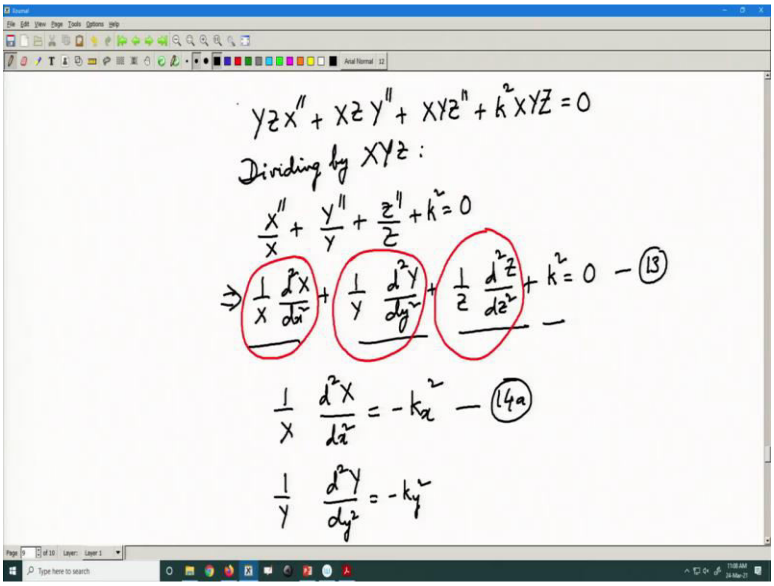

So, substituting this in the Helmholtz equation, we have $YZX'' + XZY'' + XYZ'' + k^2 XYZ = 0$, dividing by X Y Z, we have $\frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} + k^2 = 0$ or we can write this as, $\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} + \frac{1}{Z}\frac{d^2Z}{dz^2} + k^2 = 0$, we call this equation 13.

Now, since each coordinate value can be independently varied. Therefore, we can write:

$$\frac{1}{X}\frac{d^2X}{dx^2} = -k_x^2$$

$\frac{1}{Y}\frac{d^2Y}{dy^2} = -k_y^2$, and

$$\frac{1}{Z}\frac{d^2Z}{dz^2} = -k_z^2$$

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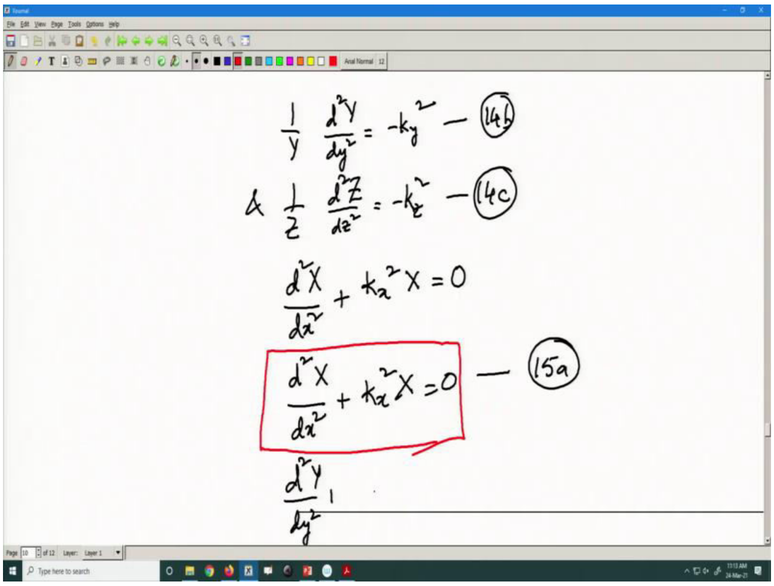

So, this we call equation number 14 a, this we call 14 b and this we call 14 c. So, rearranging equations 14 a, 14 b, 14 c yields the following equations:

$\frac{1}{X}\frac{d^2X}{dx^2} + k_x^2 = 0$, $\frac{1}{Y}\frac{d^2Y}{dy^2} + k_y^2 = 0$ and, $\frac{1}{Z}\frac{d^2Z}{dz^2} + k_z^2 = 0$

We call this to be 15 a.

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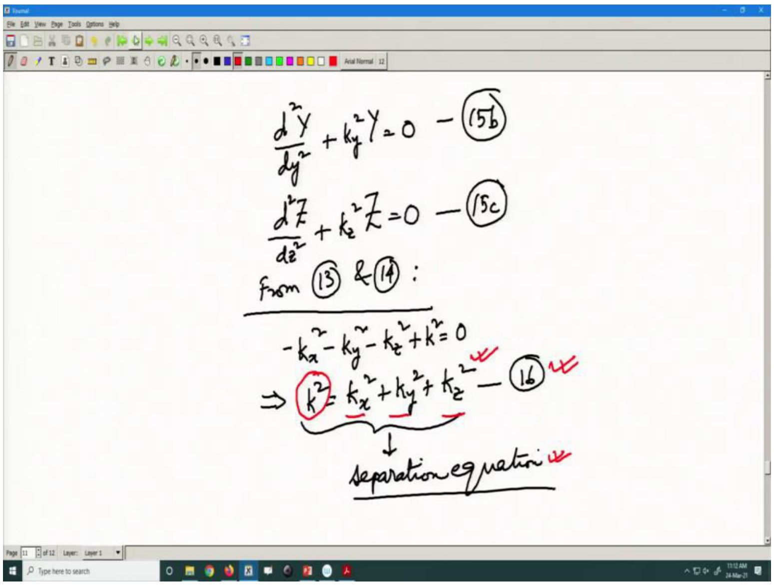

We call this 15 b and we call this 15 c. Also from 13 and 14, we have the relationship between Kx, Ky, Kz and K. So, we have $-k_x^2 - k_y^2 - k_z^2 + k^2 = 0$ or $k^2 = k_x^2 + k_y^2 + k_z^2$ we call it equation number 16. This is called the separation equation, it the relationship between the wave numbers Kx, Ky and, Kz along the three directions.

So, the power of equation 16 is that it is a remarkably simple equation, but it is remarkably also general equation which is applicable in any situation, because we have not made any kind of assumption in deriving equation 16. So, equation 16 will hold good under any circumstance and we will give the relationship between the wave numbers along the three directions and link it with the wave number K.

So, therefore, the separation equation is a simple but a very powerful tool in electromagnetic analysis and we will have the chance to meet this equation many times in the analysis of waveguides. Now, we look at equation 15 a, what kind of functions will be solutions to equation 15a, the solution to equation 15 a will be called harmonic functions.

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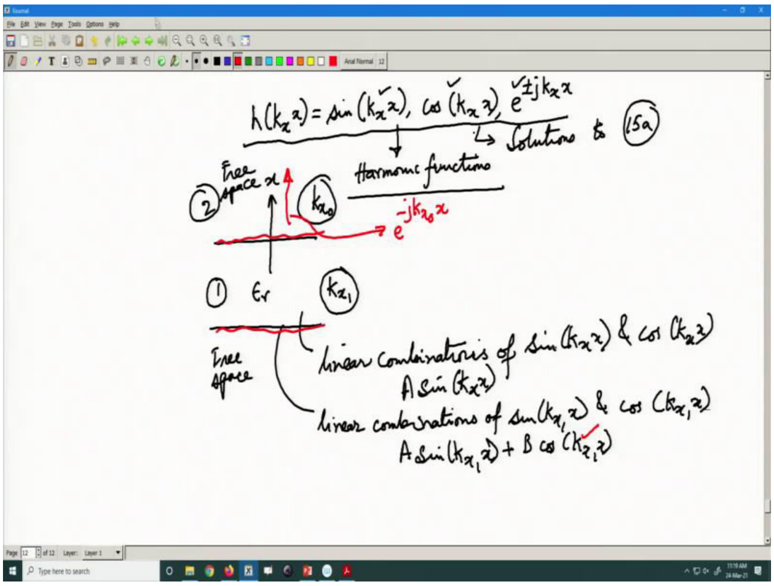

So, they are of the form $h(k_x x) = \sin(k_x x)\cos(k_x x)e^{\pm k_x x}$, they would satisfy equation number 15a and these are called harmonic functions. Now, the question is which kind of function, we should choose in a particular solution. The type of function we choose will depend on the boundary conditions, the type of media we are in.

For instance consider a problem like this, I have a dielectric slab, this is the x direction. So, this is medium 1, this is medium 2. So, there will be a reflection of waves between medium 1 and medium 2. So, there will be a reflection of waves inside medium 1. So, this medium has the wave number along the x direction as Kx1, this medium has the wave number Kx0, this is free space.

So, because of the reflections inside medium 1, the functional variation along the x direction will be expressed by a linear combination of $\sin(k_x x)$ and $\cos(k_x x)$. So, here we have linear combinations of $\sin(k_x x)$ and $\cos(k_x x)$.

So, it will be linear combinations of $\sin(k_{x1}x)$ and $\cos(k_{x1}x)$. So, we can write the functional variation of the psi function as $A\sin(k_{x1}x)+B\cos(k_{x1}x)$. However in this region, the wave is going forward, it is going forward and it is going to infinity.

So, here the functional variation will be expressed in terms of $e^{-k_x x}$, because there is no reflection here, so the wave is going forward. So, then it will depend on, as you see the type of boundary in which I am describing my psi function, these eigen values k_{x1}, will be found out by applying the boundary condition at this interface and at this interface, let me write free space here.

So, this is an example, which we will deal with more in the discussion of slab guides, how we find out the eigenvalues and how the harmonic functions are written, but this is just to elucidate, in a very sketchy manner the finding out of the eigenvalues and writing the appropriate harmonic function relative to a particular problem.

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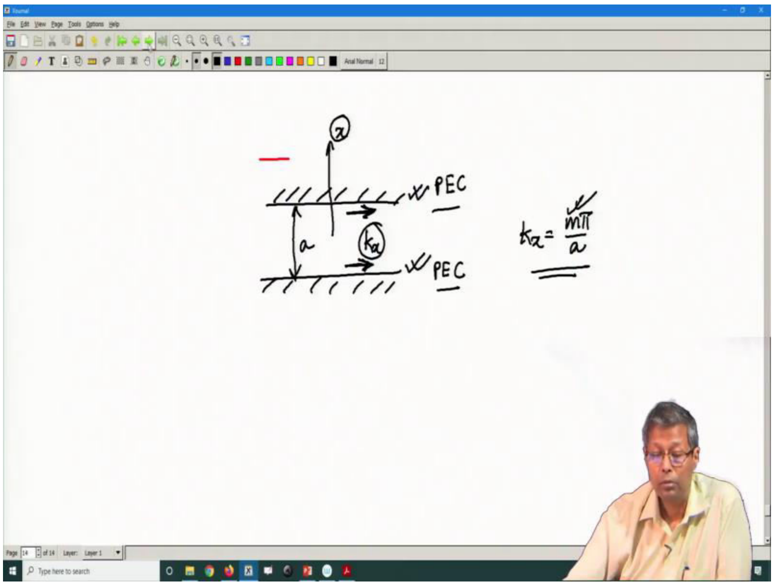

Similarly, if I were to consider the wave variation the same problem with the region bounded by two metallic plates, the variation will be either sine or cos but not both and will depend on which choice of potential function causes the tangential electric field to vanish at these two locations.

So, to satisfy the perfect electric boundary conditions, the choice of the potential functions is going to be dictated by the fact that which choice of potential functions which should be either signed or cos, but not both, which will cause the tangential electric fields at these two locations to finish.

Also, the eigenvalues Kx here will be discrete. So, we will come to this when we do the waveguide problems. So, if this distance is a then eigen values Kx will be given by $\frac{m\pi}{a}$, where m is a discrete integer.

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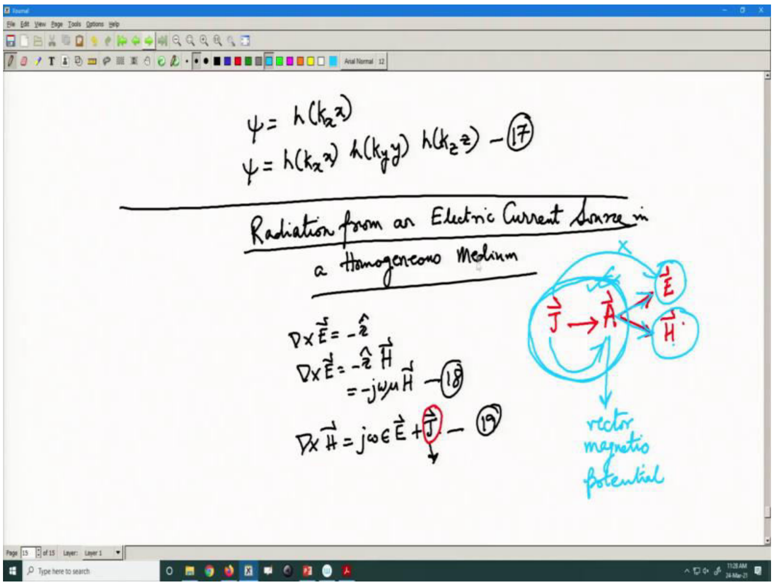

Now, from this we can write down the form of the potential function psi as $h(k_x x)h(k_y y)h(k_z z)$ so it should be equations 17. So, this will satisfy the Helmholtz equations, where the Kx Ky and Kz will satisfy equation number 16 which is the separation equation.

So, this is the way, the solution to the wave equation comes and it is expressed in terms of harmonic functions and these harmonic functions are appropriately chosen depending on the medium of interest, the boundary condition surrounding the medium of interest and whether there are internal reflections in the medium or whether the wave is going towards infinity.

So, we choose the appropriate harmonic function and the value of Kx, Ky and Kz are chosen by appropriate applications or boundary conditions, but Kx Ky and Kz will always obey the relationship $k^2 = k_x^2 + k_y^2 + k_z^2$, in the particular where K is the wave number in that particular medium.

So, next we deal with the problem of radiation from an electric current source in a homogeneous medium. Prior to this we discussed the source free problem. So, there was no source inside the medium, we did not ask the question that when we are talking of the potential function, when we are talking of the wave equation who excited the medium, this question was not asked.

Now, we graduate from there to the introduction of a current source in a homogeneous medium. Now, when we introduce the current source in a homogeneous medium, our wave equations particularly read like this. So, $\nabla \times \vec{E} = -z\vec{H}$ which is $\nabla \times \vec{E} = -j\omega\mu\vec{H}$, assuming that there is no magnetic current source, but only our electric current source.

So, we call this 18 and $\nabla \times \vec{H} = j\omega\varepsilon\vec{E} + \vec{J}$, we call this equation 19. So, j is the current source. So, in this case the same trick, in order to express the wave equations either in terms of $\vec{E}$ and $\vec{A}$ by direct substitution from one equation to the other will not hold good.

Therefore, we follow the strategy of finding out what we call a potential, finding out what we call our potential in this case a vector magnetic potential $\vec{A}$ which is linked to the current source j and from $\vec{A}$ we can find out the electric and magnetic fields. So, the direct computation of the electric and magnetic fields from the current source j is not possible.

Therefore, we first find out the vector magnetic potential $\vec{A}$, let us call this the vector magnetic potential. So, it is called the vector magnetic potential. So, from the current source j we find the vector magnetic potential and once the vector magnetic potential A is evaluated, we found out the electric and magnetic fields. Normally this step is the main step in the process, we will come to why that is the main step.

Once this step is formed, the electric and magnetic fields can be evaluated in a one shot process, it is a very easy step. So, these two steps take very little computation time, either the computation of electric field or magnetic field after you have computed the vector potential $\vec{A}$ is very, very simple, it is a one step process, but this process is the main process which will take a lot of mathematical resource and computation. This is akin to what we do in the scalar domain. This is akin to what we do in static electricity.

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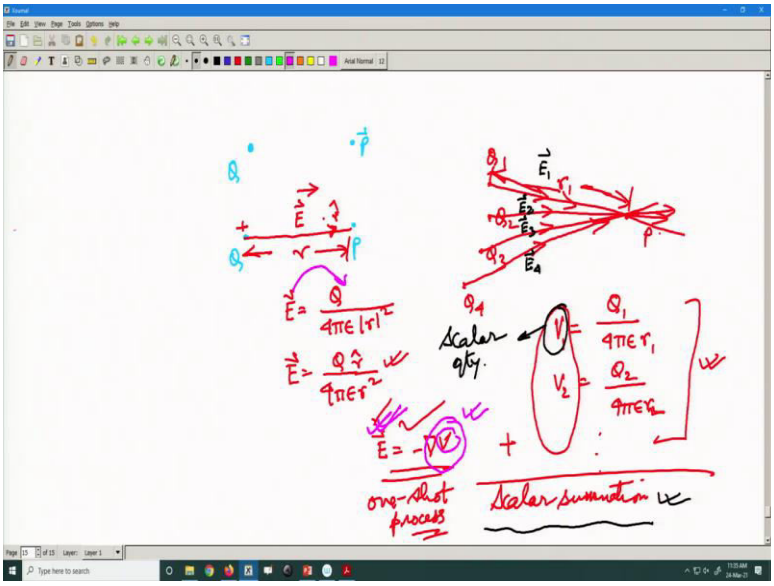

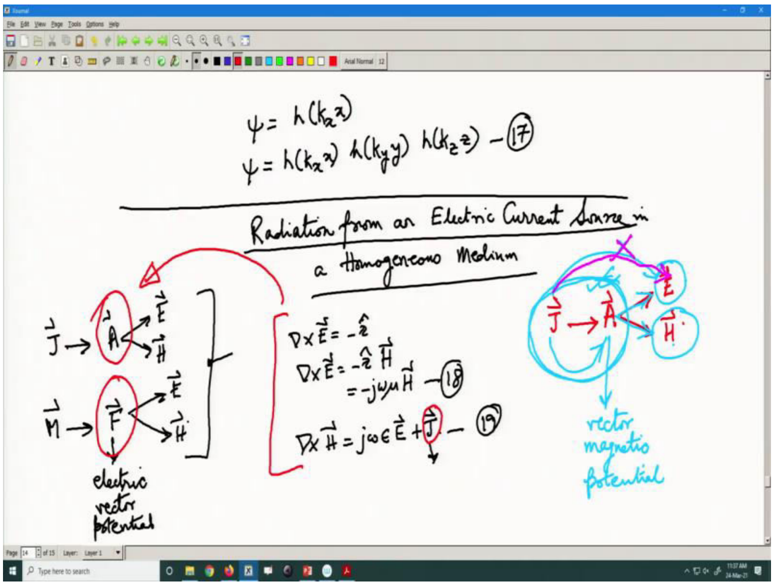

Given a point source Q and given an observation point P, the electric field at P, due to the point charge at Q is given simply by **E** equal to Q by 4 pi epsilon r square along the r vector.

So, this is the distance r and this is the unit vector along the r direction. So, this is if I have a positive charge, then this electric field will be along this direction. So, this is the expression for the electric field. However, the problem is, if there are more charges let us say Q1, Q2, Q3, Q4 and if I have to find out the electric field at point P due to all these charges, then I have to do many such greater additions.

So, this one, this one, this one, this one, the electric field at point P due to Q1 plus the electric field at point P due to Q2 plus the electric field at point P due to Q3 plus the electric field at point P due to Q4. So, this will be **E1** that will be **E2** that will be **E3** and this will be **E4**. So, you have to perform many vector additions and these vector addition, if the number of charges is very, very large, it becomes a very complex process.

So, in order to avoid these we follow the route that we compute the potential at point P due to one particular charge and that is given by Q by 4 pi epsilon r. Let us say for Q1, the potential V1 is 4 pi epsilon r1, where r1 is this distance from here to here, let us rub this point. So, this is r1. Let us put in working.

Similarly, we calculate the potential V2, Q2 by 4 pi epsilon r2 and so on, due to the other charges and then we perform the summation of this which is a scalar summation. So, this is a scalar summation because these potentials are scalar quantities. So, performing this scalar summation is not a problem at all, given an arbitrarily large number of charges.

So, after the scalar summation is performed, we find out **E** is equal to minus grad V in one shot process, it is a very quick process. By one shot process we mean, it is a direct quick process, the main resource is consumed here, which is however addition of some scalar potentials.

So, it hardly takes time for the computer and after this is found, we can directly find out the electric field, which is why we walk the potential route, we find the electric field through the the potential and do not attempt to walk this route, if the number of charges is arbitrarily large.

The scalar problem gets simplified by the introduction of this potential, the concept of potential, this artificial concept of potential, because Coulomb's law does not talk about the potential, we have brought in the concept of potential, we have defined the concept of potential at least for electromagnetics, in order to make my life easy in the computation of the electric field, due to an arbitrarily large number of charges.

So, we have created this term potential in order to simplify this mathematical burden of calculating the electric field at a point due to a number of static charges. So, if the process is simplified in static electricity, the process is completely possible in the case of time harmonic fields, which we are going to illustrate.

That means, that without walking this route, that means that this route, the direct computation of electric field is not at all possible, we always have to walk through the vector potential. In fact, if we have an electric current source, we find out $\vec{A}$ and from which we have found finding out **E** and **H**, as we detailed.

If we have a magnetic current source, we calculate the electric vector potential **F** and from there we find out **E** and **H**. So, these are the two distinct steps used to find out the electric and magnetic field for the case of the electric current source and the magnetic current source. But the important fact remains that we have to always go to the intermediary of the vector potentials $\vec{A}$ and **F.** Now, let us therefore, from here, find out how we lead to this kind of solution.

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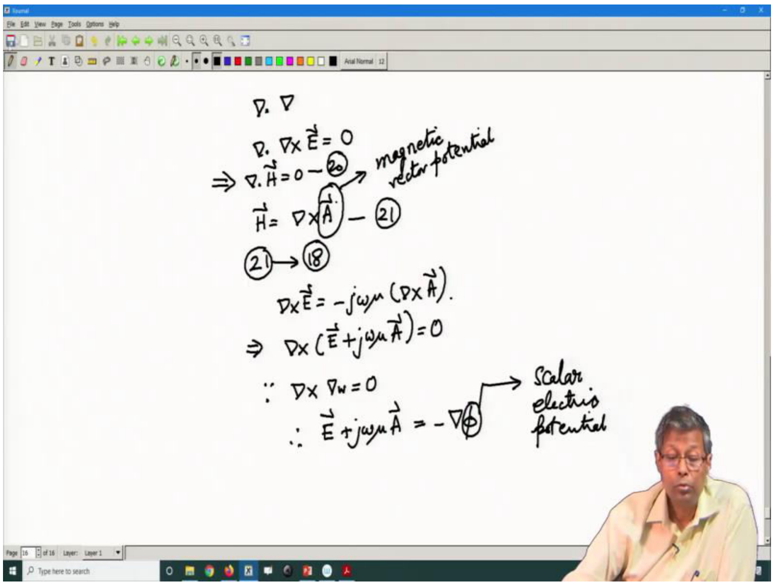

So, for that, we note that divergence of curl of **E** equal to 0, divergence of curl of any vector is 0 and therefore, divergence of **H** is 0, we can also argue the divergence of **H** is 0. Anyway So, therefore, **H** can be expressed as the curl of a vector, we call this vector the magnetic vector potential.

So, let us call this 20 and let us call this 21. Now we substitute 21 in 18, then we will get $\nabla \times \vec{E} + j\omega\mu(\nabla \times \vec{A}) = 0$. Now, since curl of grad of any scalar say w is 0, since this is true, therefore $\vec{E} + j\omega\mu\vec{A}$ can be expressed as the gradient of a scalar, we call this the scalar electric potential. So, this is called the scalar electric potential.

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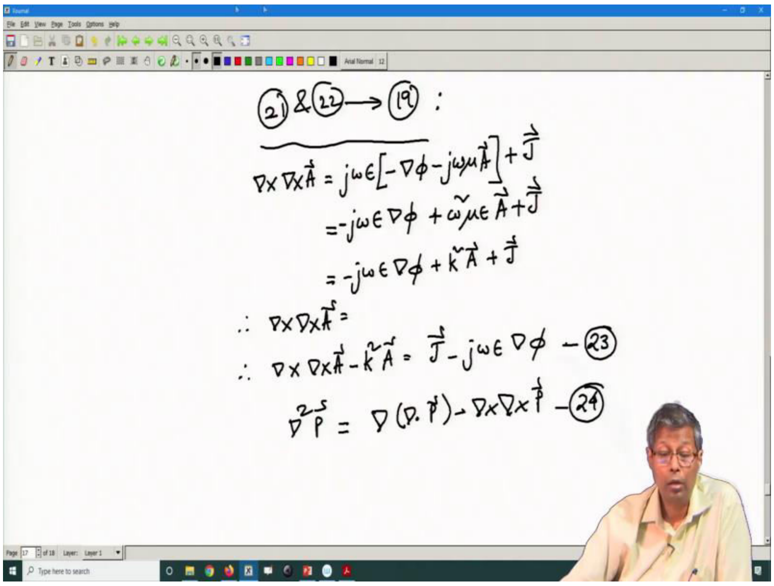

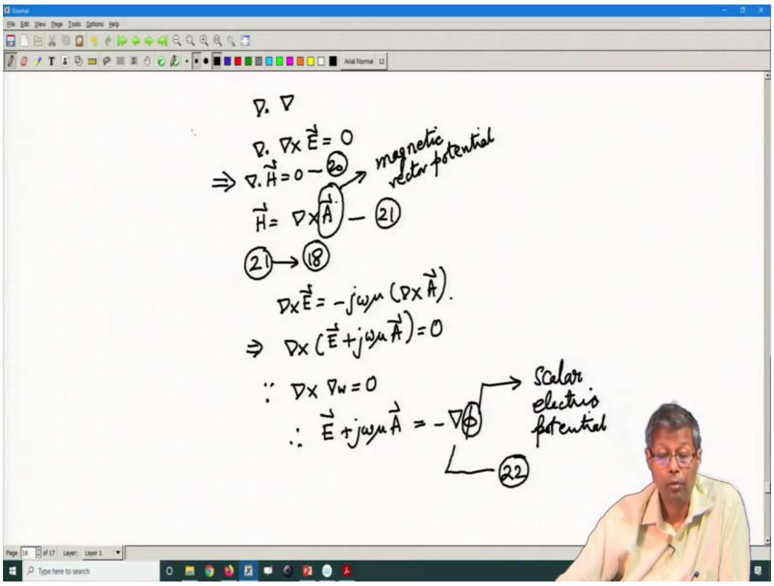

Now, we substitute 21 and 22 to 19 and when we do that we get $\nabla\times\nabla\times\vec{A}-k^2\vec{A}=-j\omega\varepsilon\nabla\phi+\vec{J}$

We call this 23, let us call this equation 22. Now, from a vector identity we know the same vector identity we use for the wave equation grad of divergence **P** minus curl curl **P**, this is a vector identity we met during the wave equation. So, we call this equation 24.

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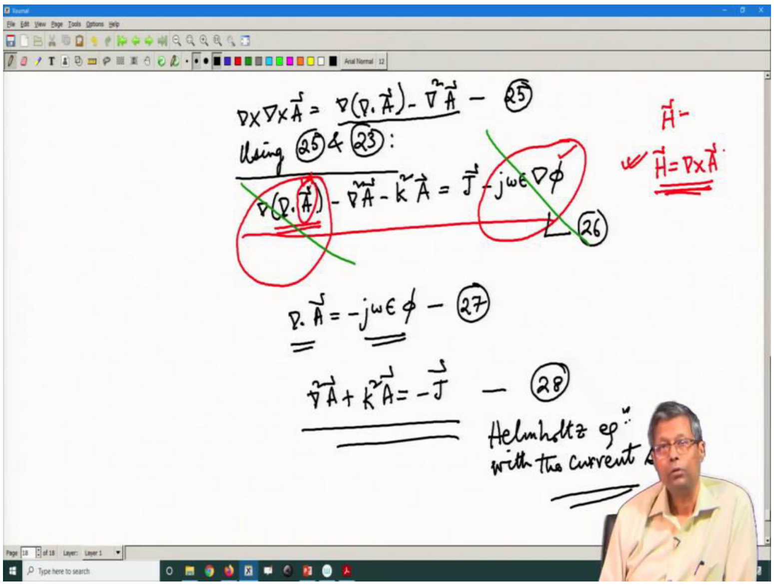

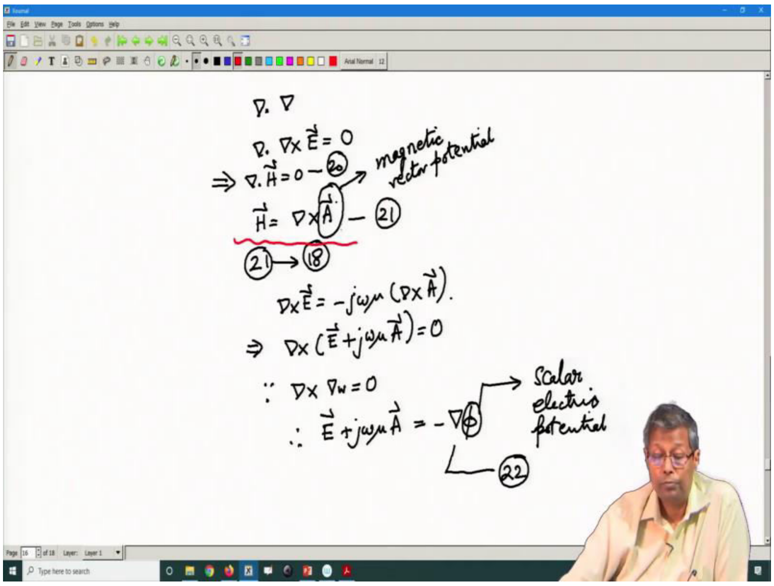

And then we can write down instead of the vector **P** substituting **P** for **A** curl curl **A** equal to grad of divergence **A** minus grad square **A** that is 25. So, using 25 and 23. Now, we get grad of divergence **A** minus grad square **A** minus K square **A** equal to **J** minus j omega epsilon grad of phi. So, this is by just substitution of this into the previous equation, which is 23.

Now, if you look at equation number 21, in this equation we have specified the curl of A, this is a rather difficult equation involving A and phi in a complex manner, it is a single equation involving **A** and phi. So, we cannot proceed unless we find out a relationship between **E** and phi.

Now, please remember that **A** just like the scalar potential V in Coulomb's law, we have defined the scalar potential in Coulomb's law, we have defined to suit our purpose. So, here also we have created this vector potential **A** in order to suit our purpose. So, we are free to attribute to it any property, which will help us to find out the electric and magnetic fields from the current source j.

In this context, we have already specified what will be the curl of **A,** we have defined that **H** in this context we have already defined that **H** is equal to curl of **A**, we have defined the curl of **A**, but the divergence of **A** has not yet been defined, the curl of **A** denotes the rotating tendency of **A**, it shows or it measures the rotating tendency of **A**, the divergence of **A** denotes the spread or the outward spread of **A**.

As the word divergence indicates, the outward flow of **A**, because the rotation and outward flow are two orthogonal characteristics. Once we have already defined the rotation of A, we are now free to define independently the divergence of **A**, I repeat that again, because the rotation and the divergence are mutually orthogonal characteristics.

Now, that we have defined the rotational characteristics of **A** through **H** is equal to curl of **A**, are free to define the divergence of **A** and we will define the divergence of **A** in a suitable manner and that suitable manner will be such that this term and this term simply cancel each other and what will be that definition therefore, will be divergence of **A** equal to minus j omega epsilon phi, that is 27, we call this 26.

So, once the divergence of **A** is defined in this way, immediately these two terms cancel and therefore, we are left with grad square **A** plus K square **A** equal to minus **j**. Which is equation number 28. So, this is the Helmholtz equation involving the current source. Helmholtz equation with current source, this is a very important equation in electromagnetics.

So, let us stop here we are going to start with the significance of Helmholtz equation in the next class and how it enables us to calculate the fields effectively. Thank you.