Applied Linear Algebra
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Week 03
Invertible maps, Isomorphism, Operators

Hello and welcome to this Week 3 of lectures in this linear algebra course. So far, in the previous week, we looked at linear maps in a big way. Very importantly, we looked at the connections between linear maps and matrices and how we can represent a linear map with a matrix once you choose a basis, and how that helps etc. And, we also saw some other properties of linear maps like injectivity, surjectivity. We looked at properties of the null space and range space etc. All of that is very important to understand what a linear map is. So now we will make a bit more of a progress in that direction. In particular, we will consider invertible maps. So what are these invertible linear maps? These are special maps. And once a map becomes invertible, a lot of interesting things can be said about the spaces which have invertible maps between them. Isomorphism. An invertible map is also called an isomorphism. And there are the special types of linear maps which are called operators and all these we will define in this class. It's mostly likely to be a definition oriented lecture, but there will be some interesting properties and important ideas in this area that we'll come across in this lecture as well. So let us get started.

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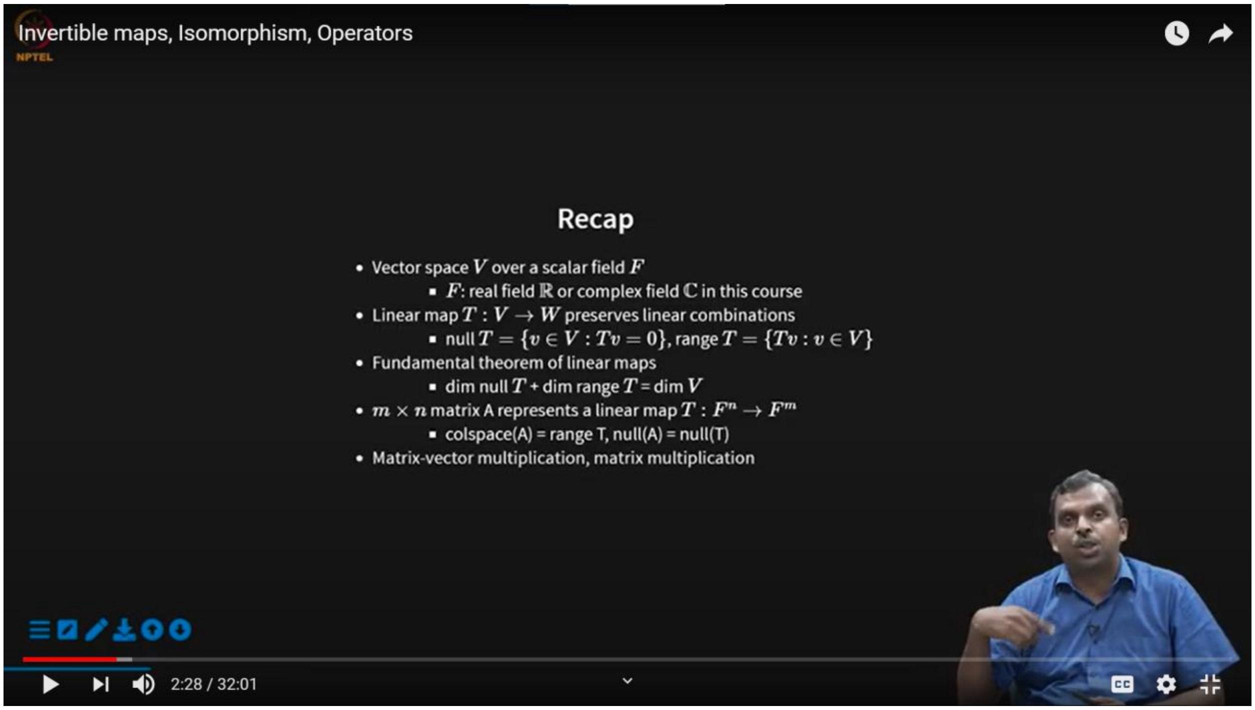

Okay, quick recap. As before, the usual notation, we'll keep using the same thing. The vector space V over a field $\mathbb{F}$. Linear map T will go from V to W. It will have some properties. Null and range, the fundamental theorem of linear maps will play a very crucial role as usual. Always, it keeps showing up. The fact that the dimension of V equals the dimension of the null space plus the dimension of the range space when V is finite dimensional. This is very important. And then we saw this connection between mxn matrix and a linear map from $\mathbb{F}^n$ to $\mathbb{F}^m$. And then we saw these column space and null, and all these connections between the matrix and the linear map. And finally we saw how this matrix vector product and matrix multiplication are naturally defined in terms of, you know, operations on the linear map, right? So the matrix vector product is simply evaluation of the linear map itself in the matrix domain, and matrix multiplication corresponds to composition of linear maps. When you interpret that as a matrix, that's what you get, okay? So this is a quick recap.

So let us proceed by looking at invertibility, isomorphism, operators and all that in this lecture, okay? So we will begin with the definition. When do we say a linear map is invertible, okay? So the definition is not very hard if you think about it. When you talk of a function being invertible... Function usually takes input to output. If there is another function which can take you from output to input in a proper faithful way, then you have an invertible function, right? So that's the basic idea behind invertibility. Function takes you from input to output. You should have a clear-cut, precise correct way of coming back from the output to the input. Somebody tells you - hey, this output I got by evaluating that function, you should be able to uniquely find what input it corresponded to, okay? So if you can do something like that, you have an invertible map... And there is a more precise way of stating it, which is what is given in this definition here, right? So a linear map $T: V \to W$ is said to be invertible if there is a linear map again... We want a linear map to be the inverse of a linear map okay, of course, from W to V, from the output side to the input side. And such a linear map will be called the inverse of T, okay? What are the two conditions we want? We want S composed with T, or if you first hit a vector with T and then again with S, then you should get the identity map, right? And the same should be true for T composed with S. It should be the identity map on W, okay?

So maybe a picture is in order here to sort of illustrate this, what I mean here. So we will do our famous ellipse pictures, okay? So you go from V to W. If you have an invertible map which takes an input to an output... This is T, right? If you go from V, then this S should be a map in the other direction which takes you from output to input, that's the thing to keep in mind. So you have to have exactly like that, okay? A particular input goes to the particular output, okay, under this linear map T. This inverse linear map S should bring you back and the reason why you say ST and TS have to be identity is - it has to work either way, right? So you have to be able to go back and come back to the right thing. You have to keep closing the loop. So that's when you say it's invertible, okay? So this is a very precise, particular definition and you can see why this, you know. S composed with T, T composed with S... If you have a vector in V, you hit it with T first and then with S, you will get an identity in V. Likewise if you start with W and then you do S and then T

again, you should come back to the identity map. The same point in W. So this is a very... You know, invertible maps are very, sort of precisely defined, okay? So every point here should go to every point there and then the output should come back to the, you know, the inverse should bring the output back into the input, okay? So that is the definition, hopefully it's clear to you. I can give you… We'll see some examples as we go along, okay?

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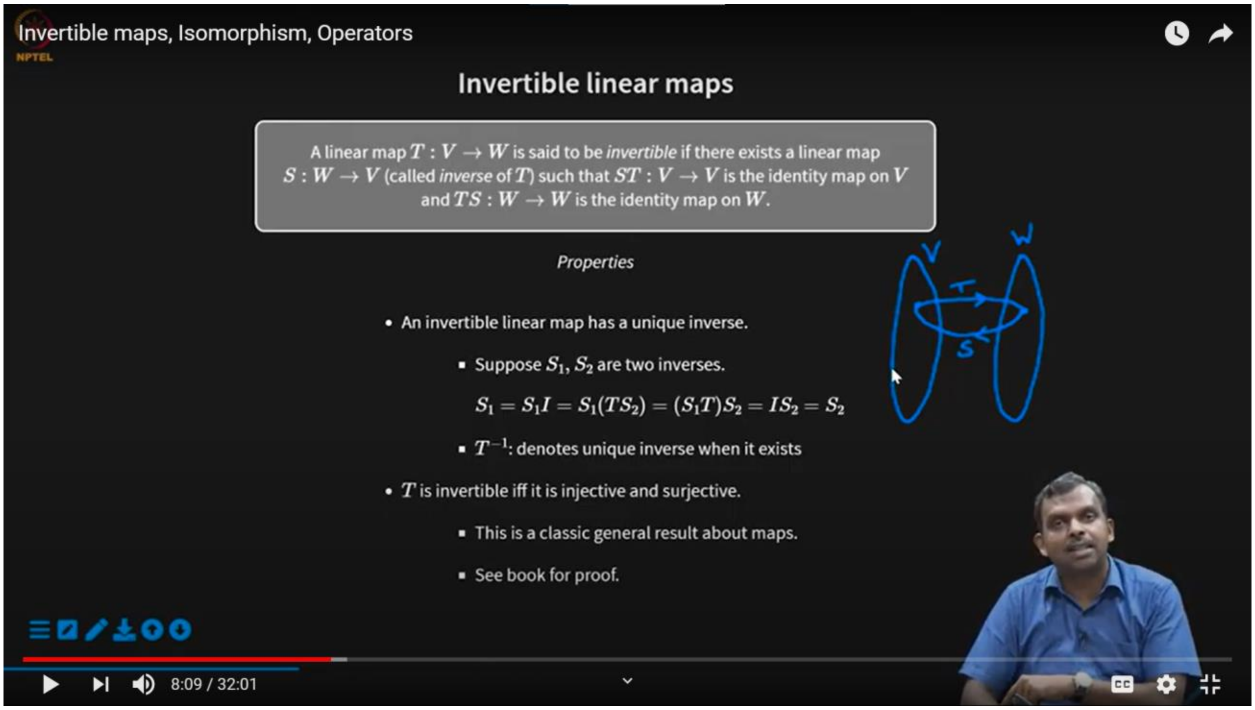

So even before we, you know, see examples and, you know, try to get a feel for what this is, there are some nice properties, a couple of important properties which will help, you know, simplify some of these examples and thinking and all that. So the first thing is, if you know the map is invertible, then its inverse is unique, okay? And that comes about because of the associativity of this matrix multiplication or the associativity of the composition, right? And the proof is very easy. It's written down here. You can see how it works. S_1 you can write if... So the proof basically works by contradiction, okay? So you, what you say is - how do you show that the invertible linear map has a unique inverse? You go ahead and assume maybe there are two inverses, right? S_1 and S_2. And then using the properties of the inverse, you have to show S_1 has to be equal to S_2, okay? So that's the strategy for the proof. And you can see how it proceeds. It's quite simple. It uses associativity in a fundamental way, okay? So $S_1 = S_1 I$, and instead of I, I can replace it with TS_2, which I know is also I. And then you put a bracket here, and then, you know, you use associativity to sort of put the bracket in the other way, right? So $S_1 T$. And then $S_1 T$ also you know is I, and then you are done, okay? So you see the way the definition is important, why you want both TS and ST to be identity. Only then you will have unique inverses. Otherwise you can have multiple inverses and it just becomes a bit messy, okay? So if you want a unique inverse? You better enforce these conditions, okay?

So whenever we know that inverse exists for a map T, we will denote the inverse as T^{-1}, okay? So sort of like the, you know, multiplicative inverse in a usual number domain. So you put T^{-1} to indicate the inverse, okay? So that's important. Now the other important property about invertible maps is - when the map is invertible, it is both injective and surjective, okay? So you can see it's one-to-one and everything at the output has to be covered because the map going the other way has to take you back to the input, right? So both of them have to be true. And it is also true the other way, okay? If you have a map which is both injective and surjective, then it is invertible, okay? So both of these are true. I am not going to do the proof in this lecture. There is a detailed proof in the book. And also I should point out that just as functions and maps, this is a very fundamental property of functions and maps, okay? So injectivity and surjectivity is an if and only if condition for invertibility, okay? So this is something important to know, okay? So basic properties we have seen for invertible linear maps, and maybe we will see some basic examples, very simple ways to come up with examples.

In fact you can easily come up with very many examples either way, okay? Both which are invertible and non-invertible. Actually non-invertible is the easiest to come up with, right? So for instance if you know if a map is not injective then you know it's not invertible, okay? So any transform whose null space is not exactly zero, right? Any transform whose null space is not exactly zero will automatically not be invertible, okay? So it's easy to come up with a non-invertible thing. So you just pick a, you know, transform which has a non trivial null space, then you know it's not invertible, okay? Same thing is true with the maps that are not surjective. If the map is not surjective, if the range is not going to occupy the entire vector space, then you know it is not going to be invertible, okay? So those are conditions you can easily eliminate, okay? But there are interesting possibilities particularly when you go to infinite dimensions. So if you go to polynomials, which is infinite dimensions, then all sorts of interesting cases end up happening. So for instance I have shown here two examples - one is multiplication by x^2 , okay? So we saw this linear map before, okay? Given a polynomial, you multiply by x^2, you get an output. I mean everything sort of gets, every term gets multiplied by x^2. You can see some interesting thing - it is one-to-one. We saw that it is one-to-one, right? So if you take a polynomial, multiply by x^2, you will only get one other polynomial, right? So there is no way to get that in multiple ways. So it's one-to-one. But it's not surjective. We saw it’s not surjective, right? So because the polynomials with no constant polynomials are not there, linear polynomials are not... All those things are not there, so it is not surjective, okay? So it becomes non-invertible. So you have a case in the infinite dimension situation where the linear map is injective but not surjective. So it is not invertible.

So a similar thing can be worked out in another way. You can define this left shift in polynomials. What is left shift? If you want to think of what is left shift... So basically if you have $a_0 + a_1x + a_2x^2 + \cdots$, a left shift will sort of take, you know... All the coefficients will shift to the left. What will happen to a_0? You simply kill it, right? So you just shift it out into nothing. So you get $a_1 + a_2x + \cdots$ okay? So that is the, that is the mapping. So you can see a couple of interesting properties here. First property is - it is not injective, right? It's not one-to-one, okay? A whole bunch of polynomials which vary only in the constant, right? So a0 alone changes. They will all map to the same output, okay? So it's actually many-to-one, right? But it is surjective, right? It's clearly surjective. Any polynomial I have I can always go find another polynomial whose left shift will be this, right? So I simply have to push this to that side and add whatever constant you want you will get it. So here is an example where a map is not injective but it is surjective. So it becomes non-invertible, okay?

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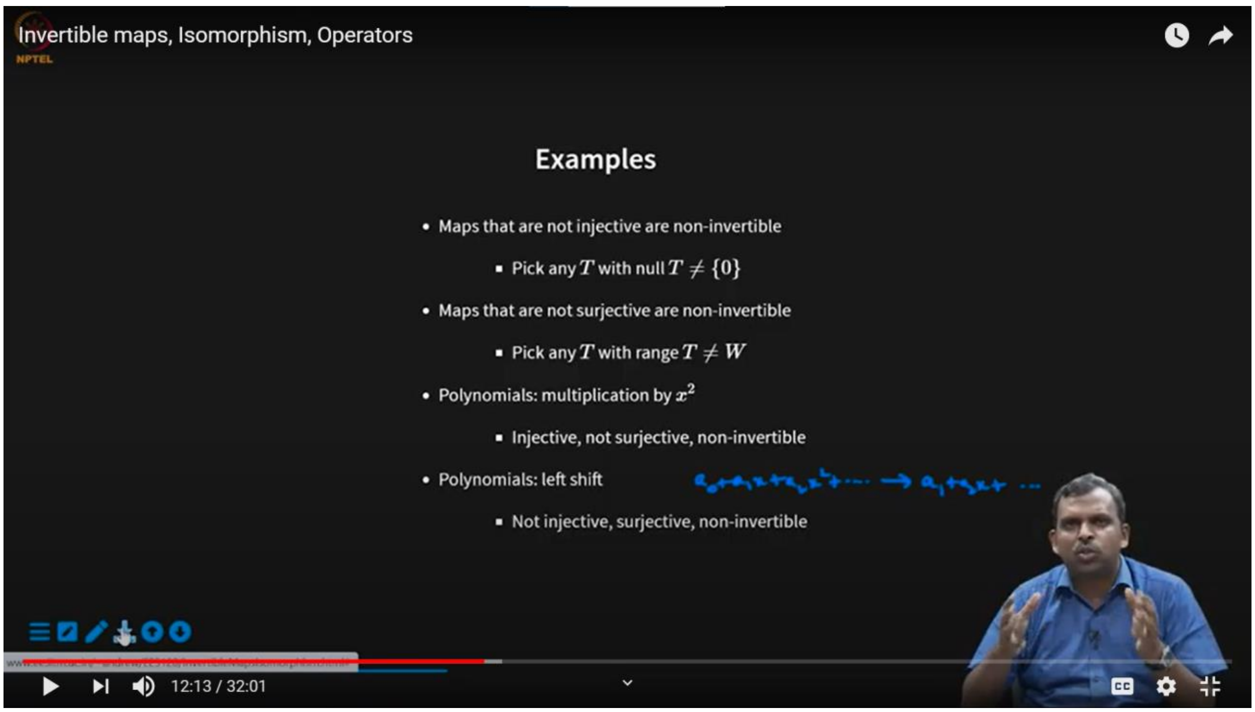

So all these interesting situations are possible in infinite dimensions. In finite dimensions, something very interesting will happen. So you will see, we will see that result later on, but these are examples. So if you want to come up with examples of invertible matrices, please wait till the end of the lecture, I will give you a clear way of how to do it. But right now you can think of various ways in which you can create it. It is most useful to think in terms of matrices. How do you construct a matrix which represents an invertible operator, invertible linear map is something we can think about. So maybe we'll see a couple of more properties and then come up with some concrete ideas on how to construct these things. It's actually very easy, once we do that it'll be clear. But for now let's hold on for a couple of more slides, at least we know how to come up with non-invertible examples, right? So non-invertible examples are easy to come up with. Maybe invertible examples just wait for a little while, we will come up with more of them, okay? At least one very standard example for an invertible map is the identity, right? So if you have V to V, the identity map is clearly invertible. So at least one example we have. More examples we'll come up with as we go along, okay?

So now what are these isomorphisms? So this is a term that is quite often thrown around in math. And a little while people say two things are isomorphic. So isomorphism usually means, you know, they are the same, okay? In some sense they are the same. So an invertible linear map, in the area of vector spaces, an invertible linear map is called an isomorphism, okay? Whenever you can find an invertible linear map from one vector space to another vector space, you've found an isomorphism between the two vector spaces and these two vector spaces are said to be isomorphic, okay? So isomorphic basically means similar structure, similar in form, similar in shape, like that, okay? And you can see why, right? So if you have an invertible linear map, a map which preserves linear combinations, and it's also invertible, so it doesn't matter whether you do any linear combination based calculation on V or any linear combination based calculation on W, you can do it in whichever place you want and you can always go back, right? As long as you only did linear combination based operations, right? It doesn't matter whether you did it in V or it did it in W, you can always use this invertible linear map that you have to either go from here to there or there to here, and do the calculations anywhere, okay? So in that sense these are isomorphic. So isomorphism is very nice in that fashion. Of course it is only for the, you know, linear combination operation. If you do some other crazy operation it may not be preserved, okay? But this isomorphism is for this operation, okay? So quite often in mathematics as we build up from a basic definition, you want to have isomorphism defined immediately. Because then that tells you similarity in structure. So maybe you do not have to deal with all the, you know, anything which has an invertible map between them, it's not so interesting, okay? So you want to study something more which is beyond that, okay? All right, so that's isomorphic.

So in fact, quite a few vector spaces are isomorphic, okay? So here's this wonderful example. It's very easy to come up with. Two finite dimensional vector spaces over the same field $\mathbb{F}$ are isomorphic if and only if they have the same dimension, okay? So it's a fantastic result. A simple result which tells you when two finite dimensional vector spaces are isomorphic, okay? All you have to do is check this only number, this single number that you have to check. You have to check whether the dimension of one vector space is equal to the dimension of the other vector space. The moment they become equal, they are isomorphic. If they are not equal, they are not isomorphic. You've finished, right? So the same thing sort of holds for subspaces also, right? So think about it. Subspaces are also, you know, vector spaces like that. So if you have a vector space, you have two subspaces, you can also talk about isomorphism between the two subspaces. When are two subspaces isomorphic? When they have the same dimension, okay? So you can always find an invertible map from one subspace to the other as long as they have the same dimension. Otherwise you can't, okay?

The proof is actually very simple. It is written down here. I'll go through it very quickly, you can read it and understand it a little bit more. See, this is an if and only if statement. Any time you have an if and only if statement, you have to prove both directions - from a to b and b to a, right? So what is the one direction? Supposing V and W are isomorphic, then I know that there is an invertible transform $T: V \to W$, okay? So what are the properties of invertible transform? Its null space should have dimension zero because it's injective. Its range space should have dimension W because it is surjective, okay? Now you use the fundamental theorem. You see $\dim V$ equals $\dim W$, that's it okay? So it's a very easy proof to show that if it's isomorphic then the two vector spaces have to have the same dimension. It is not very hard to see, okay?

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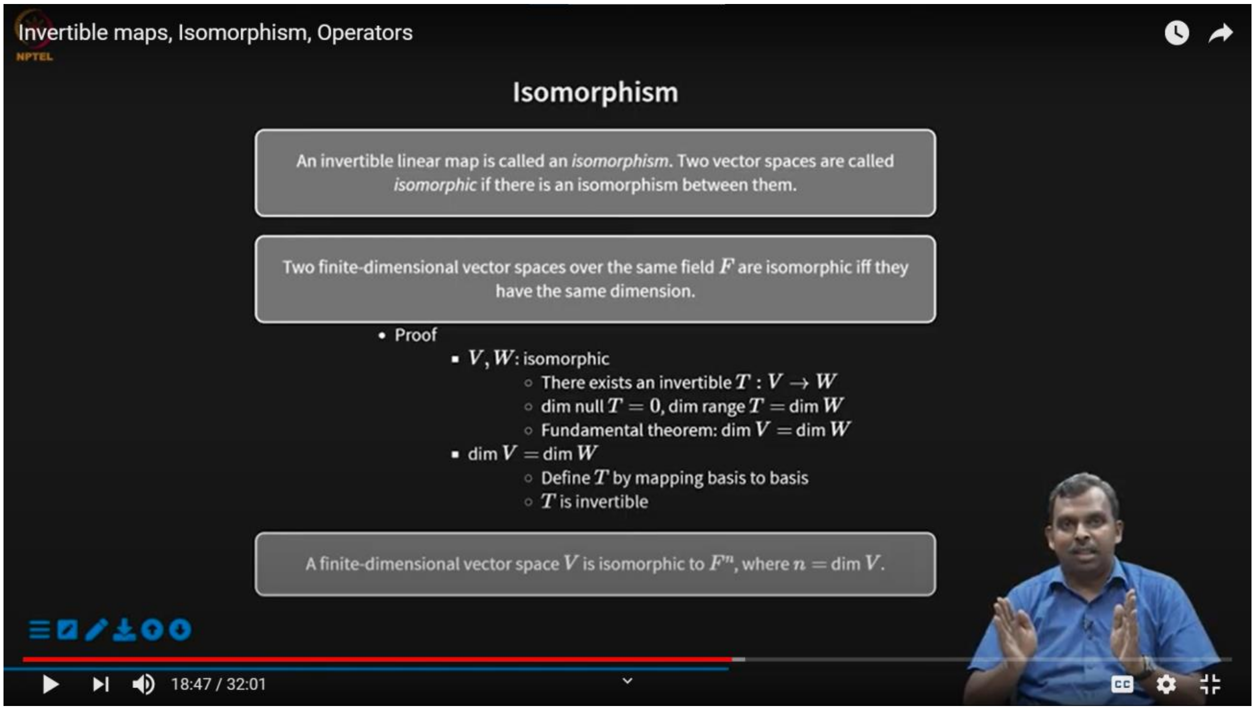

What if $\dim V = \dim W$, how do you go the other way, how do you show it's isomorphic? Then you just do a basis based thing, okay? So you find a basis for V, find a basis for W. Both the sets will have the same number of vectors, right? So you define a linear transform which maps the first basis element here to the first basis vector here, second basis vector here to the second basis vector, like that. If you map, right, you have an isomorphism, okay? You can show it's injective, you can show it's surjective just by definition, okay? So I am skipping the details of that. So this is an interesting result. So any two finite dimensional vector spaces with the same dimension are isomorphic, okay? So you can extend this a little bit and see that if you have a finite dimensional vector space... Somebody defines a complicated vector space but you know it's finite dimensional. Once you know its finite dimensional, it becomes isomorphic to $\mathbb{F}^n$ where $\mathbb{F}$ is the scalar field, right? And n is the dimension. That's it. Doesn't matter how complicated you define your vector space. As long as you know it's a vector space, you know it's a vector space over a field $\mathbb{F}$ and it has dimension n, it has to be isomorphic to $\mathbb{F}^n$, okay? So people tend to make a case that $\mathbb{F}^n$ is the only vector space that you really need to study, why do you need to study this abstract notion of vector space etc. The reason is, you know, subspaces show up, you know, when you want to deal with all these subspaces of different dimensions, you know, it's good to think of them abstractly and then make some statements about it. This will show up again and again, okay? So $\mathbb{F}^n$ is very important. Not to say it's not important, it's important to study it. But once you fix an n, you still have to worry about all its numerous subspaces and they all have different dimensions and then they may not be isomorphic to each other. So you have to study the abstract notion also. But know that $\mathbb{F}^n$ is a very important finite dimensional vector space, okay? So that is something about isomorphism.

So this tells you clearly when invertible maps can exist, right? So if you have a map from one subspace to another subspace and the dimensions are not the same, then you clearly know already that it cannot be invertible, right? Finished. So the dimension controls this invertibility in a very very important way. So dimension is very important. First thing to check is the dimension. If the dimensions are not the same, you cannot have an invertible map, okay? So only when the dimensions are the same you go in and check for injectivity, surjectivity and other conditions, okay? So this is a good thing to know. Okay. Now we have seen this important relationship between linear maps and matrices. We saw that once you define a linear map, okay, $T: V \to W$, if you fix a basis for V, fix a basis for W, you get a matrix representing that linear map, okay? We have seen that. Now we also know that the space of all linear maps is a vector space by itself, right? I can define addition of linear maps, I can define scalar multiplication of linear maps. So it becomes a vector space. Now the space of matrices, okay, when you have dimension m to n, right? $\mathbb{F}^{m,n}$, that is also a vector space. We have seen that, we know how to add matrices, we know how to do scalar multiplication of matrices and there is like a very close connection here. And in fact it turns out these two vector spaces are isomorphic, okay? So these are things we hinted at before. But these two vector spaces are isomorphic, the vector space of all linear transformations from V to W where dimension of V is n and dimension of W is m is isomorphic to $\mathbb{F}^{m,n}$, the space of $m \times n$ matrices. These two are isomorphic and the isomorphism goes in a very simple, obvious way. You define a basis. Once you define a basis, you can define the isomorphism. What is that? You find, simply find the matrix corresponding to the basis, okay? How do you go from linear transform to matrix? You define a basis for V, basis for W, compute the matrix corresponding to the linear transform. You go to the matrix. You also know how you can go from matrix to linear transform, right? So linear map is very easy to do. You know you can pick a basis, whatever basis you want, and then each column represents the map in the other direction, right? So it's easy to go from one to the other. So there is a map like this, you can show that the map is linear. It's injective, it's surjective. Those proofs I'm skipping, you can do this. So this connection between linear maps and matrices is not just some computational thing, it is tightly coupled. It is an isomorphism in some sense, okay? So both of these are the same. So once you know it's isomorphism, then you can now compute the dimension, right? So what? Because I know it's an isomorphism, these two should have the same dimension, right? So from there you can quite easily conclude that the dimension of the set of all linear maps, the vector space of all linear maps from V to W is simply equal to $\dim V \times \dim W$, okay? Why? Because it is isomorphic to $\mathbb{F}^{m,n}$ and $\mathbb{F}^{m,n}$ clearly has dimension mn, right? So we know we can find a linearly independent space there, set there which spans the space. So you are done, okay? So this is a nice result. I mean maybe the, you know, the beauty of it is not immediately apparent. It is sort of simple in some sense. But this sort of firms up our connection between linear maps and matrices, okay? So they are the same thing structurally, okay? In fact even under multiplication they are the same, right? Composition, though this result directly doesn't talk about it, the composition is also preserved by multiplication, okay? So that's isomorphism of linear maps and matrices. Okay.

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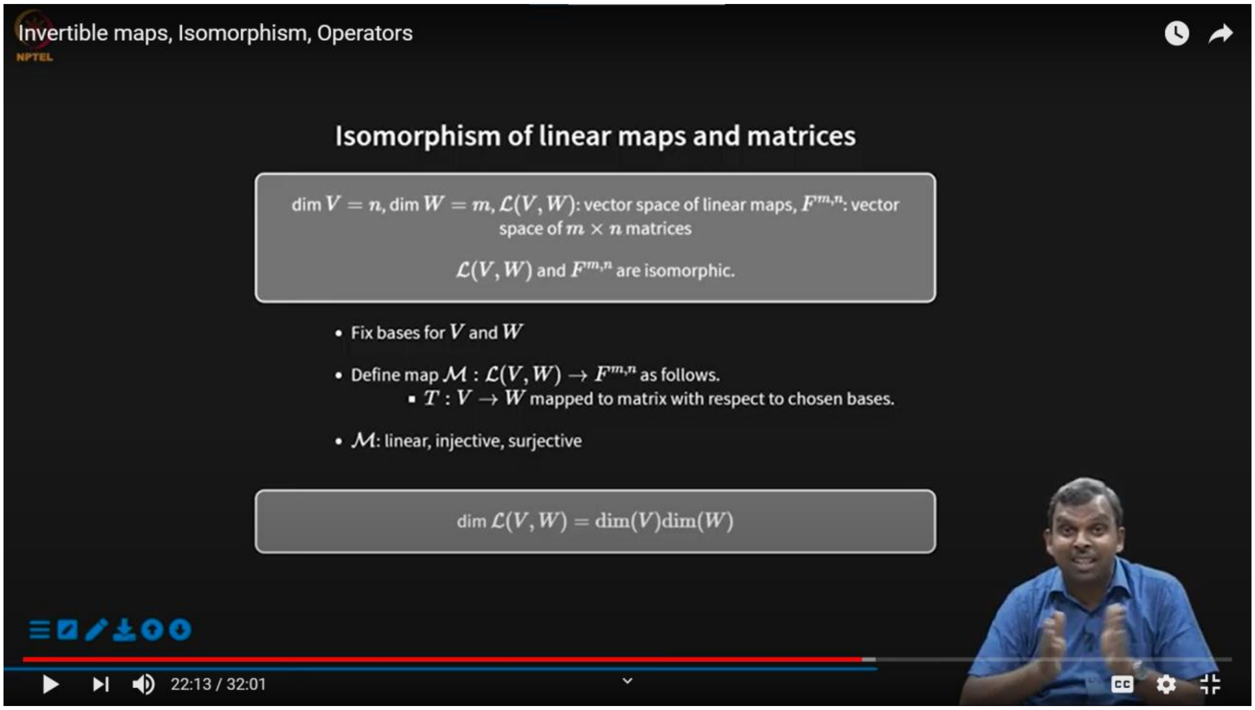

So the next topic that we want to do in this lecture is define operators, okay? They are very simple to define. A linear map from a vector space to itself is called an operator. It's given a special name and it's also especially significant. So most of the linear maps we study will be operators, they will work from vector space to itself. The input will be from one vector space, output will also be into the same vector space, okay? So this is the most interesting linear map of all because it has a lot of wonderful properties. We will study it over and over again. So we give it a special name, okay? We say it is an operator, okay? The set of all operators is $L(V,V)$, right? Maps from $V \to V$, so you can shorten that and simply call it $L(V)$. If you say $L(V)$, it is a set of operators on V, okay? And like I said, it is put in the slide here also - operators are easily the most important linear maps.

Lots of interesting connections they have and particularly invertible operators. What are invertible operators now? Operators which are invertible, right? So an operator is a linear map. If it is invertible, you become invertible. So now operators will correspond to square matrices, right? So you have both input and output having the same dimension. So square matrices and invertible square matrices play a very important role, okay?

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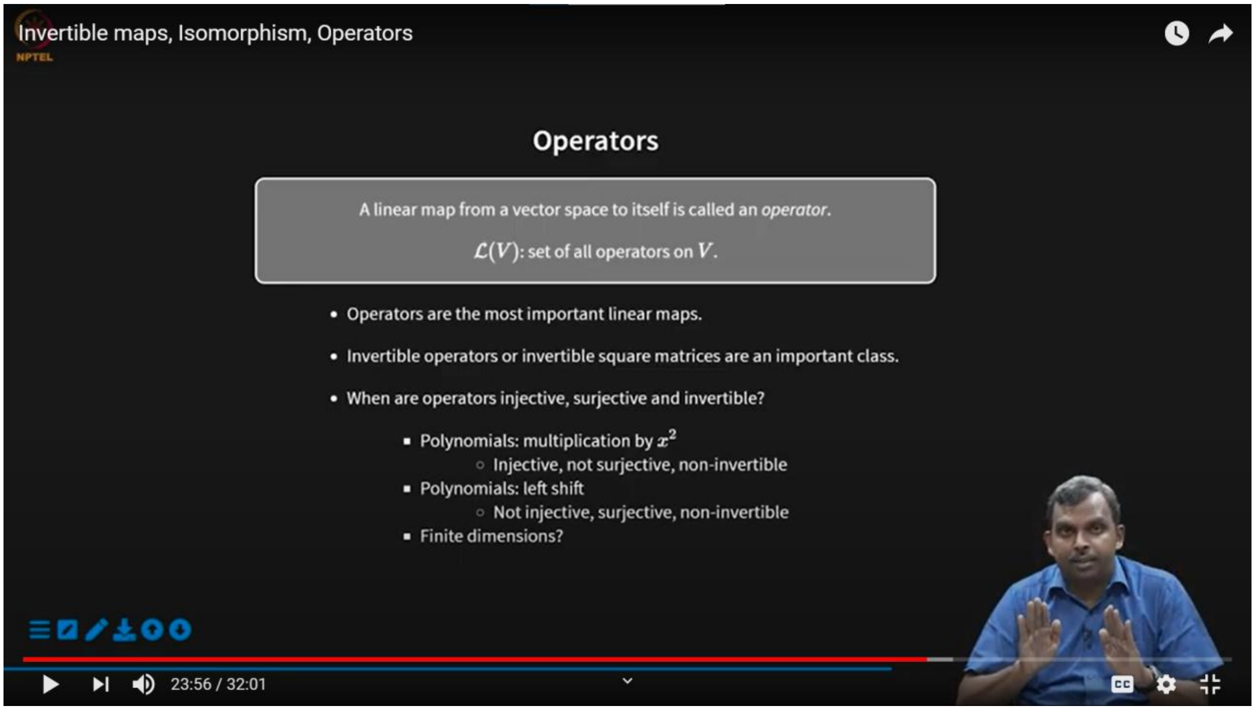

So a natural question is - supposing you have a finite dimensional space, okay and you are looking at operators, right? What can one say about injectivity, surjectivity, invertibility, okay? We have already seen this, two examples before. If you have infinite dimensions, then all sorts of strange things can happen. You can have injectivity without invertibility, you can have surjectivity without invertibility. What about finite dimensions? It seems like in finite dimensions, something should not be, you know, you should not have so much flexibility, you only have finite dimensions. And if you say it's one-to-one it looks almost like, you know, it should be invertible, right? So if everything, you cannot have one without the other. Seems like a result, okay? And it turns out it is true, okay? So in finite dimensional vector spaces, here is this wonderful result. This result says in finite dimensional vector spaces, the three properties we studied are equivalent for operators. If you look at operators in finite dimensional vector spaces, the three properties we study in terms of classifying the maps, right, injectivity, surjectivity invertibility - they are all equivalent. What do I mean when I say a bunch of things, properties are equivalent? Any one implies the other, okay? So when I say a, b, c are equivalent, a implies b, a implies c, b implies a, b implies c, c implies a, c implies b. Everything. So any one you check, everything is true, okay? And the way to prove these things when you say, list a bunch of conditions and say they are all equivalent, the way to prove it is the following. Usually you show a implies b and then you show b implies c and then you show c implies a, okay? Once you show this, it's enough. You don't have to show all the possible other combinations, right? Why is that? Because, you know, you can sort of complete the loop, right? So a implies b, b implies c, c implies a. So c will imply b also, you can just go on like that, okay? So that is the way to prove these implications.

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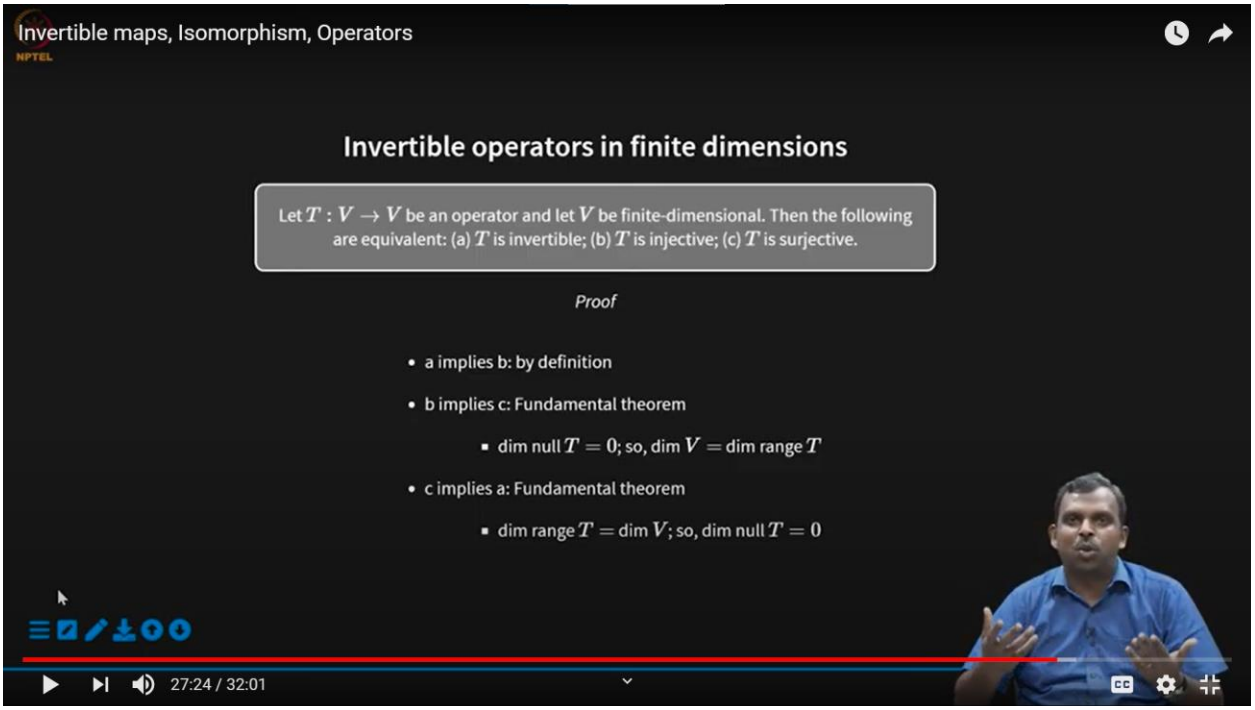

The first implication, a implies b, is sort of direct. If it is invertible, it’s injective, right? So I do not need to restrict myself to anything for that. So clearly that is true, okay? What about b implies c? We will rely on the fundamental theorem again. You know T is injective, so dimension of null of T is zero, so you simply use the, you know, fundamental theorem. So dimension of V equals dimension of range of T, and remember T is an operator, okay? So which means it went $V \to V$, so its range occupies the full space V, so you get surjectivity, okay? What about c implies a? Remember in c, I am told that T is surjective, okay? And I want to show T is invertible, okay? What is the definition? I already know if an operator is surjective and injective, then it is invertible, right? So I already know it’s surjective. So the only thing I really have to show is T is injective also, okay? So for c implies a, if you use the fundamental theorem, I know the dimension of range of T equals dimension of V, so we use the fundamental theorem. You get dimension of null of T equals 0, so you get that the map is injective also. So once you have surjectivity and injectivity, it is also invertible, okay? So a implies b is true, b implies c is true, c implies a is true. And you are done, okay? So that's nice to know, okay? When you are dealing with operators in finite dimensional vector spaces, you just talk about invertibility and non-invertibility, you do not have to worry about injectivity, surjectivity all those things, right? So that's why you will see invertibility playing a, you know, central role in all of these vector spaces. So people don't talk too much about injectivity, surjectivity, they are not interesting in finite dimensional vector spaces. Just invertibility is good enough, okay?

Okay, so this slide sort of summarizes how to come up with these examples, particularly with respect to matrices, right? So we know linear maps and matrices are one and the same thing and some sort of an isomorphism. And supposing I give you a matrix, okay? How do you find out whether it's invertible or not, okay? The first thing is - it has to be a square matrix, right? So you just rule out the cases when it's not a square matrix, okay? If it is not a square matrix, if $m > n$, number of rows is larger, then the matrix, you know it cannot, matrix cannot represent a surjective map, okay? So we know immediately it is out, okay? Same thing with the number of rows. If it is lesser than the number of columns, okay, it becomes like a broad matrix, then it cannot be injective, we know that. That's also true. So it cannot represent an invertible map. So non-square matrices are out. They're not, they're not going to be invertible, right? So we know that already. If they are square, then what should happen? The dimension of the range should be equal to the entire dimension, like the size of the matrix, right? If you have an $n \times n$ matrix, okay, so the square, if you have an $n \times n$ matrix, dimension of column space should be equal to n, right? So how many vectors do we have in the column? We have n vectors in the column, n columns, right? So there are n vectors in the matrix. Let me rephrase that. I have an $n \times n$ matrix, so there are n columns, the column space is the span of those n vectors, right? So if those n vectors are linearly independent, right? Then the dimension of the column space will be n, isn't it? If they become linearly dependent, then the spanning set, you know, linearly independent spanning set will be lesser. It will not be full rank, right? So a simple condition I need for a square matrix to be invertible is - it should have full rank, full column rank, right? All the column vectors, the n columns should be linearly independent. If they are linearly dependent, then the column space is spanned by a smaller set less than n, so then the dimension of the column space does not become n, it's not surjective and it's not invertible, okay? So this condition is very interesting. So given a column, given a matrix, how do you find if it’s invertible? Simply check if the columns are linearly independent or not, okay? So that you know how to do. We can do Gaussian Elimination to check it, okay? So this way you can come up with a lot of invertible matrices, right? So you just take any basis, right, what is in fact - if you say, if you have n columns and they're all linearly independent, then the n columns form a basis for V, right? You take any basis for V, put it as the columns, then you get an invertible matrix, okay? So that is a nice thing to know as well, okay? n columns should be linearly independent, okay? The n columns, they are in n dimensional space, you know they are linearly independent. That implies n columns form a basis for $\mathbb{F}^n$, so this gives you sort of a recipe to come up with invertible matrices. How do you come up with invertible matrices? Simply take any basis for the vector space V and then you know how to do that. Gaussian Elimination will help you. If you have to extend some basis to a basis for V, you can use Gaussian Elimination etc. okay?

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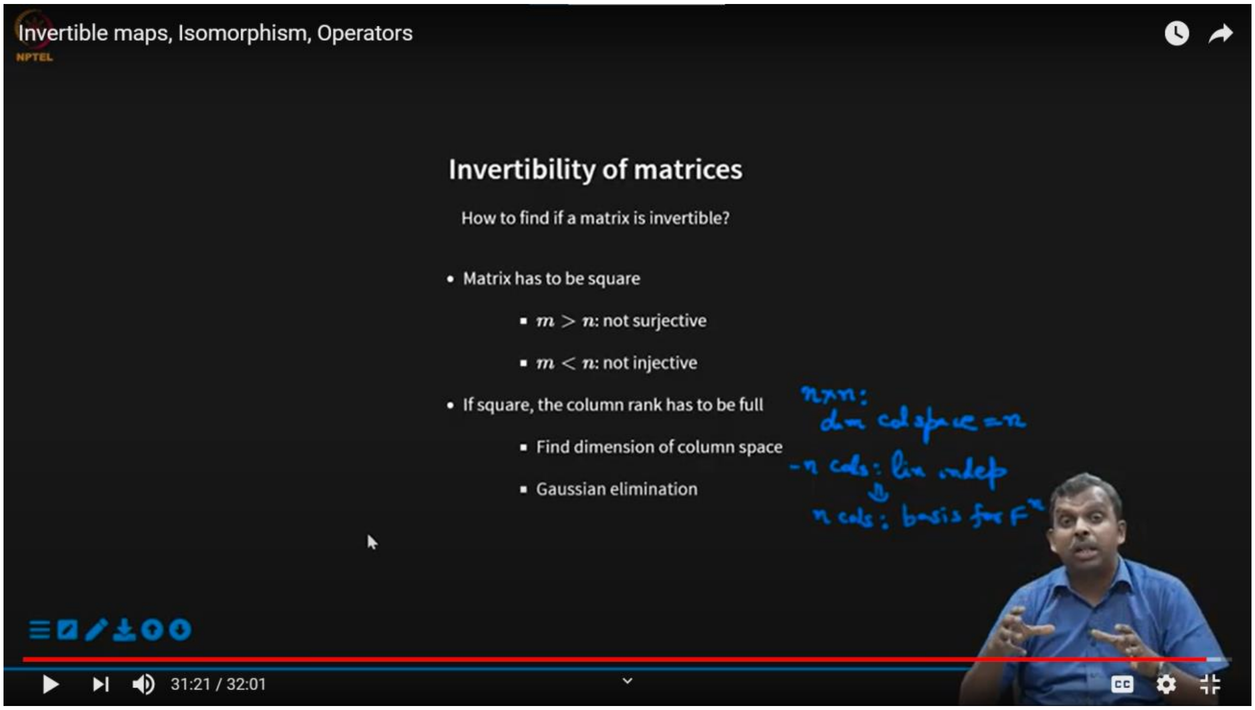

So that is the end of the lecture. We saw quite a few important ideas or maybe mostly definition oriented, but these ideas will help firm up a lot of things. First thing we saw is invertible linear maps and invertible operators. So these are very very important. They play a central role in a lot of constructs in linear algebra. And then we saw that when you have an invertible map, things become isomorphic. So you can work in either one space or the other. As long as they are connected by an invertible map, you can go back and forth and you will be fine, okay? So those are the central ideas. We'll proceed and see how to use invertibility of operators. And operators in particular for doing something, right? So we have defined all these things. What can we do with it? What can we do with invertible operators? We will see that in the next lecture. Thank you.