Applied Linear Algebra
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Week 06
Properties of Eigenvalues

Hello and welcome. In this lecture we are going to look at a few properties of eigenvalues with respect to, you know, the matrix and the linear map for which you're computing these eigenvalues, what they mean in terms of, say, rank of the linear map or, you know, some other properties of the linear map. Are they related to that? Some interesting properties like that. And some connections between eigenvalues for different types of matrices. Say A and A^{-1} if it's invertible. A and A^T. Are there connections? Are there interesting things to look at? So these are basic properties. Let's quickly go through them. Many of them are easy to prove but nevertheless it's good to see them once and understand.

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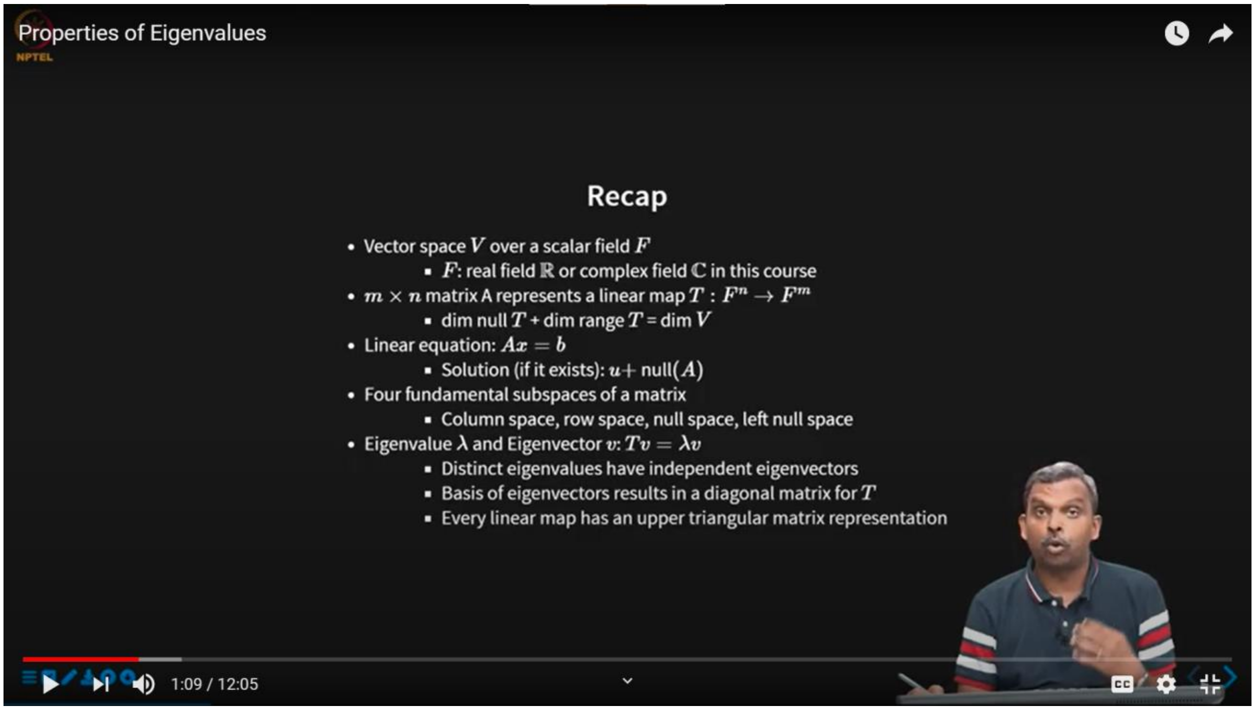

A quick recap. The recap just goes through what we've been learning so far. The vector space V and how a matrix represents a linear map there. We have this fundamental theorem of linear maps and how linear equations can be solved using these notions of null space and all these ideas. And the four fundamental subspaces of a matrix, its importance. And we also looked at eigenvalue and eigenvector. We now know quite a bit about eigenvalues and eigenvectors. The basis of eigenvectors results in a very simple diagonal matrix representation for the linear map. And we also saw that every linear map has an upper triangular matrix representation. All of these things are important and we'll use some of them and expand on them to study more properties of eigenvalues, eigenvectors in connection with the matrices, okay?

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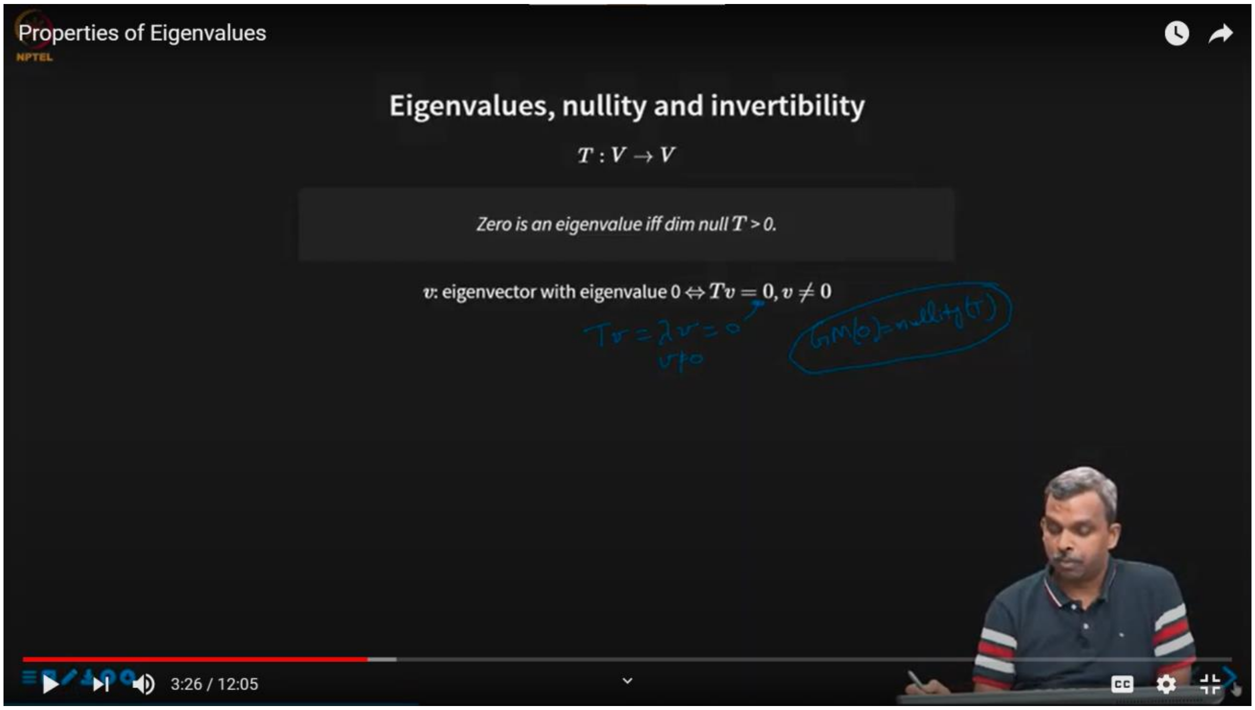

So let's see that. First thing we'll see is the connection between the null space, nullity and then you know, invertibility and eigenvalues. So if you have a linear map from, you know, linear operator from V to V, of course, there is a very nice connection between the eigenvalue 0 and null space vectors, okay? So it is very easy to connect. It's, if you think about it, you can see 0 being an eigenvalue, right? If 0 is an eigenvalue, then the eigenvector corresponding to 0 has to be in the null space, right? So v is the eigenvector with eigenvalue 0. Then what happens, right? What happens is: you have $Tv = \lambda v$ and if λ is zero, Tv becomes zero, right? And v is non-zero, right? So every linearly independent eigenvector with eigenvalue zero gives you a linearly, you know, gives you a set of linearly independent vectors for the null space, okay? So that is like an if and only if, right? It goes both ways. So how many linearly independent eigenvectors will you have for an eigenvalue 0? It depends on the, it is exactly equal to the dimension of the null space of T, okay? So nullity is also the geometric multiplicity of the eigenvalue 0, okay? So that is another way of thinking about it. GM of 0 is nullity of T, okay? So this is another relationship you can think about. The geometric multiplicity of the zero eigenvalue is the number of linearly independent vectors in the null space, right? So this is a nice relationship to remember. So eigenvalue 0 corresponds to null space eigenvectors. So that's a good simple relationship. Now this gives you a very obvious connection to invertibility, okay? So only if every eigenvalue is non-zero, okay? So there cannot be an eigenvalue 0 for an invertible operator T, okay? So this is a good check. Both ways it goes. If T is invertible, no eigenvalue is zero. If no eigenvalue is zero, T is invertible. It is quite a very direct corollary of the above. It's an operator, V to V, so all of them are equivalent, right? Injective, surjective, invertibility. And null space dimension being 0 is the condition for invertibility. You can go back to the previous thing and come up with quite an easy conclusion.

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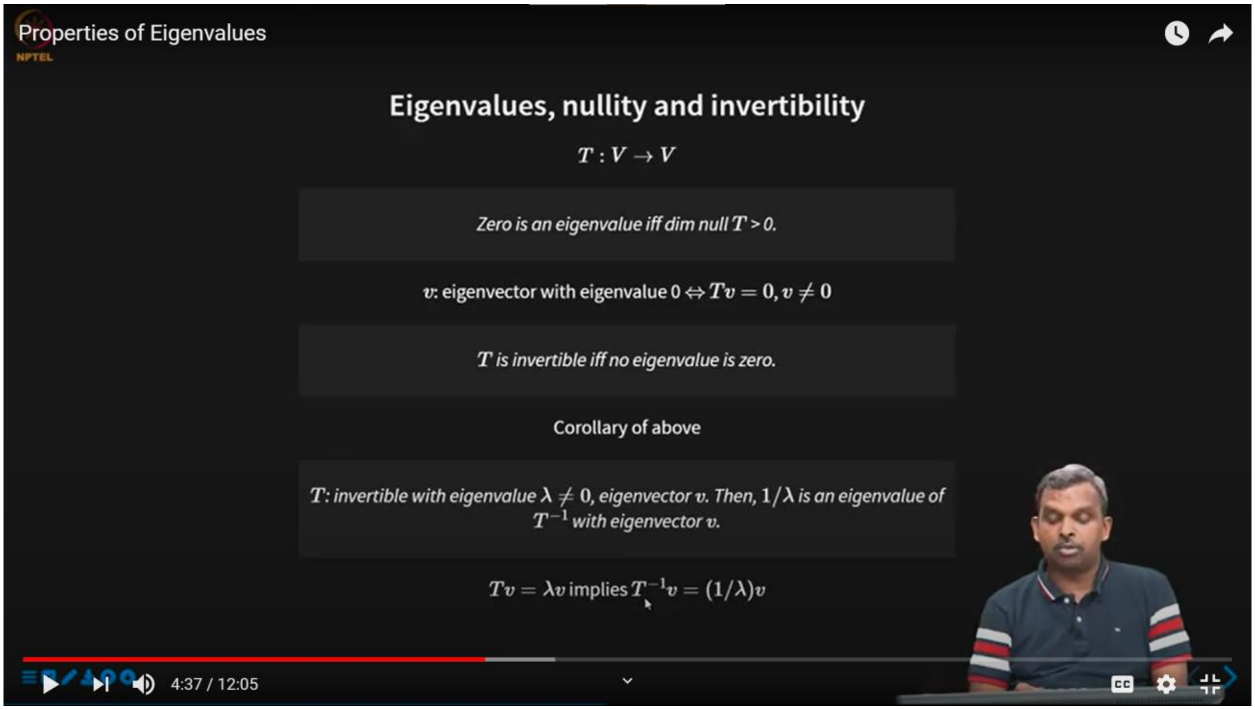

Now when T is invertible, is there a connection between the eigenvalues of T and T^{-1}, okay? It turns out there is a very direct connection, okay? So T is invertible and λ non-zero is an eigenvalue with eigenvector v. Then T^{-1} has eigenvalue $1/\lambda$ with the same eigenvector v, okay? So the proof is sort of trivial. $Tv = \lambda v$. So you can operate with T^{-1} if you like on both sides. And then you will write $T^{-1}v = \frac{1}{\lambda}v$. So you see v becomes an eigenvector for T^{-1} and with the eigenvalue $\frac{1}{\lambda}$, okay? So if you know the eigenvalues, eigenvectors for T, you know the eigenvalues and eigenvectors for T^{-1} and they have that relationship of reciprocal to each other, okay? So even multiplicities, everything will carry over. So T and T^{-1} are very, very strongly connected with respect to eigenvalues and eigenvectors also, okay? So that's an important thing to know.

Okay. What other properties are there? Let's look at eigenvalues and transpose. We saw inverse has this connection, what about transpose? It turns out they also share the same set of eigenvalues, okay? So if λ is an eigenvalue of A, then λ is an eigenvalue of A^T also. The proof again is quite simple. If λ is an eigenvalue, then $(A - \lambda I)$ is non-invertible, okay? But we know rank of $(A - \lambda I)$ is the same rank of $(A^T - \lambda I)$. How do I know that? So you know row and column space get turned around. But their dimensions remain the same whether you do A or A^T. So you do $(A - \lambda I)$, the dimension of its row space or column space will be the same as the dimension of row space or column space of $(A^T - \lambda I)$. So the rank is the same which means $(A^T - \lambda I)$ is also non-invertible. So if $(A - \lambda I)$ is non-invertible, $(A^T - \lambda I)$ is not invertible. So λ becomes an eigenvalue. But interestingly, the eigenvectors are now going to be different, okay? So that's not very easy to figure out, right? So given an eigenvector for A with eigenvalue λ, one can't very immediately go to an eigenvector for A^T. The eigenvalue is the same but the eigenvector is going to be very different, okay? So there they will, the subspaces are different. So the way the eigenvectors will map will be very different. They will, you will get some equivalent eigenvectors but they can be very different from the eigenvectors for A and A^T, okay? So because they multiply different sets of rows or columns, okay? So this is the connection for eigenvalues and transpose.

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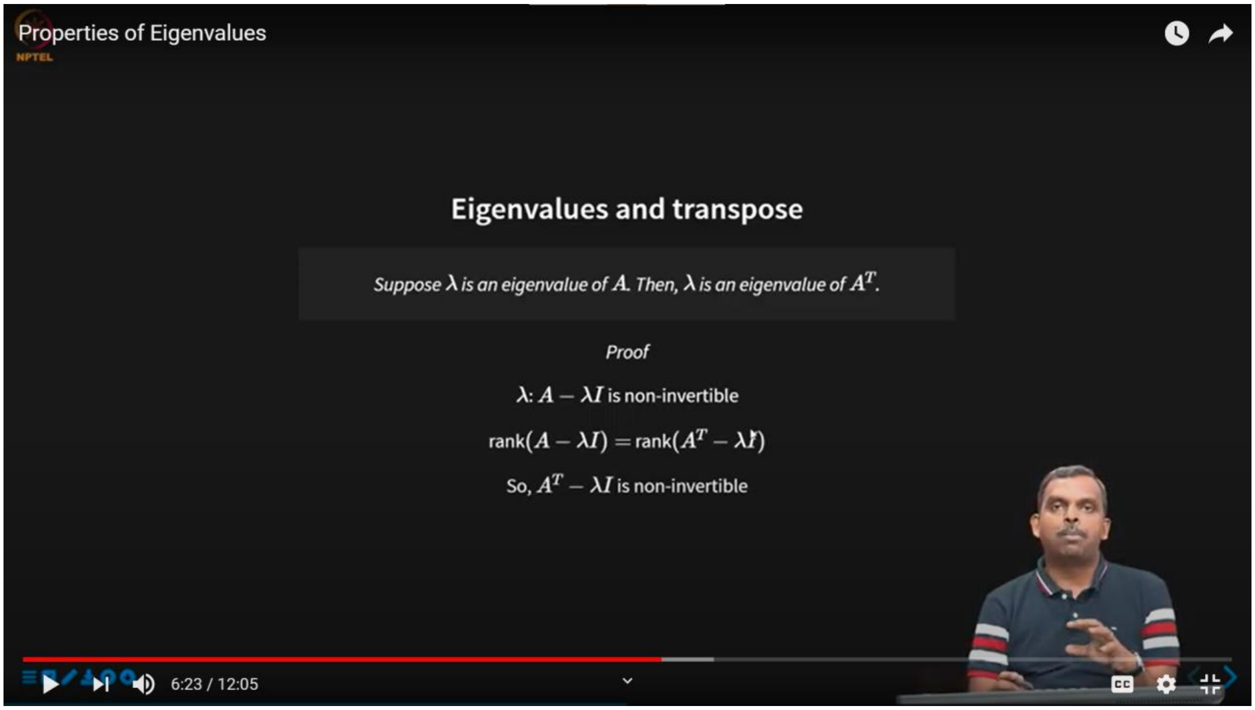

Determinant, okay? So that's another property that we saw for an operator, right? Determinant of the operator is a very simple form, you can compute it using many interesting methods. You can do row reduction and compute etc. It has so many wonderful properties. Here is another nice property for determinant. Determinant is equal to the product of all the eigenvalues, okay? Once again the proof is very easy. You find a basis in which the linear map is going to be upper triangular and determinant is the product of the diagonal elements. And those diagonal elements are exactly the eigenvalues and you're done, okay? So proof is very easy. But this is a powerful property as well. So determinant is equal to the product of eigenvalues. So you can again see if any of the eigenvalues are zero, determinant is going to be zero, the the map is non-invertible, okay? So if the, if none of the eigenvalues are zero, determinant is non-zero and the linear map is non-invertible. All of these are connected in some very nice way, okay? So this is a simple connection and property for connecting eigenvalues and determinants. So this is quite useful sometimes in small problems or any other problem also if you want to compute eigenvalues sometimes the, you know, problem can be a little bit more complicated, you can look at determinant and get an idea for the product of the eigenvalues and that might give you some idea about how, you know, eigenvalues can behave. So this kind of property is very useful for that matter.

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Another interesting property of an operator which we have not defined so far and we have not spent too much time on. Later on maybe we will see it. But for now this is a very important property. It has a lot of connections to many other applications also, okay? So it is called trace, okay? The trace, I will define it in one way and justify why the definition is correct. I will do it like that. So an operator T has something called trace associated with it. And how do you compute the trace? I am going to say the following thing to compute the trace. You fix some basis, and find the matrix representing this operator T and then you sum up the diagonal elements of that matrix, okay? Is that clear? So you fix some matrix, fix some basis for the vector space and find the matrix corresponding to the linear map and add up the diagonal elements, you will get something in the field, scalar field. And that is called the trace of the operator T. Now notice I am saying operator, trace for the operator. But I am working with a particular matrix. What if there is some other matrix? It turns out whatever basis you pick, whatever matrix, as long as it represents the same linear map in any basis, the trace will be the same, okay? You can't change by changing the basis. You can pick any basis you like, you'll get the same trace, okay? So for that... Why is this definition valid? It's because the sum of diagonal elements of SAS^{-1} for any invertible S, and the matrix A are equal, okay? So once I show this, a similarity transform, a change of basis is not going to affect the trace, okay? So trace can be thought of as a property of the, or the number corresponding to the linear map rather than the matrix, okay?

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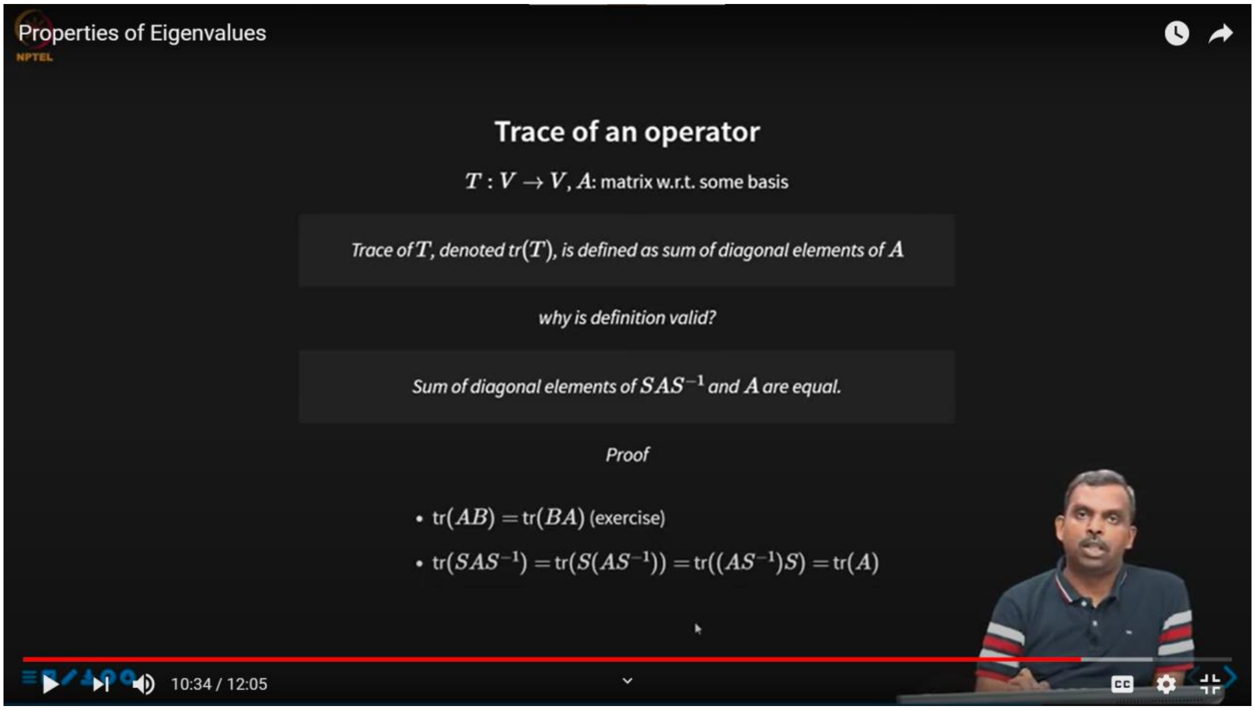

So how do you prove this? The proof is not very hard. There are two properties. The first property is $trace(AB) = trace(BA)$, okay? Remember AB and BA have to be, you know, square matrices. But, you know, A and B need not even be square for this property to hold, okay? So, but AB needs to be square, BA needs to be square. Then $trace(AB) = trace(BA)$. This is a very simple exercise. Just a question of finding out the diagonal elements, summing them up and seeing that whether you do AB or BA, the diagonal elements will sum up to the same value. It's not very hard to prove, okay? One can prove this. Once you show this, showing the trace of SAS^{-1} and A being equal is quite easy, okay? So you read SAS^{-1} as $S(AS^{-1})$, and this is your A B. And that's same as trace of BA. And when you do BA, you get $S^{-1}S$ and that will cancel and you get trace of A, okay? So it's a very simple proof. So you see that trace is a very well defined property of the operator. Seems to be an interesting property. Does not change when you change a basis, okay?

So is there a connection between trace and eigenvalues? Yes there is. It’s a very nice and very interesting connection. Trace is equal to sum of all the eigenvalues, okay? When I say sum of eigenvalues, remember, the multiplicity matters. If it occurs multiple times, you have to add it multiple times. Same thing with product also. If the eigenvalue appears multiple times, you have to multiply it all the times, okay? So keep that in mind. Again the proof is very simple. You just simply find the upper triangular matrix representation for T and the trace is the sum of diagonal entries. And the diagonal entries are the eigenvalues and you're done, okay? So it's a very simple proof once again. And you have this wonderful property that trace is equal to sum of the eigenvalues. So now this can also be very useful in practice. Quite often the matrix is given to you. Maybe it's big, maybe it has some structure to it. But you can find the trace. Trace is simply the sum of the diagonal elements. And you know that the eigenvalues all add up to that number and that can give you some good input to quickly find out the eigenvalues themselves, okay? So this is also something important.

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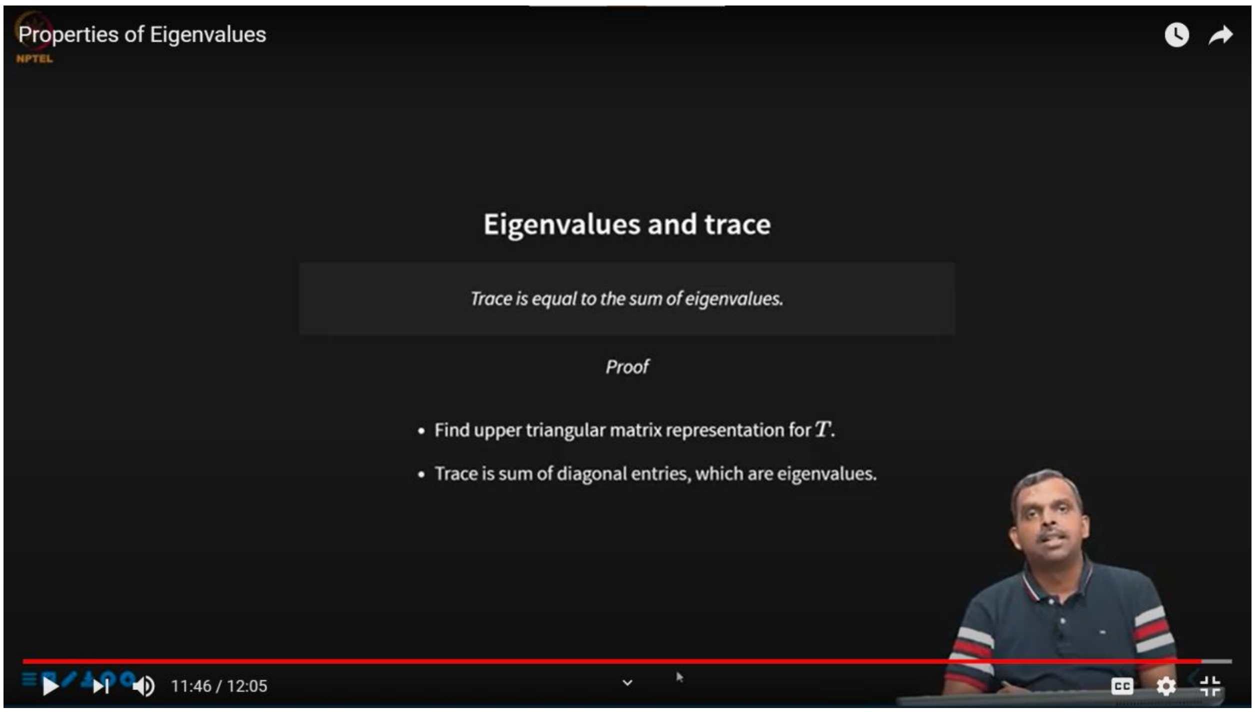

So in this short lecture to summarize we saw a few nice and simple properties of eigenvalues, eigenvectors. And you can see that they tell you more and more about the linear operator, okay? So they are very nicely tied up with a lot of foundational, fundamental properties, okay? So we'll stop here as far as this lecture is concerned.