Stochastic Modeling and the Theory of Queues
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Lecture –55
Introduction to Countable-state DTMC Continued

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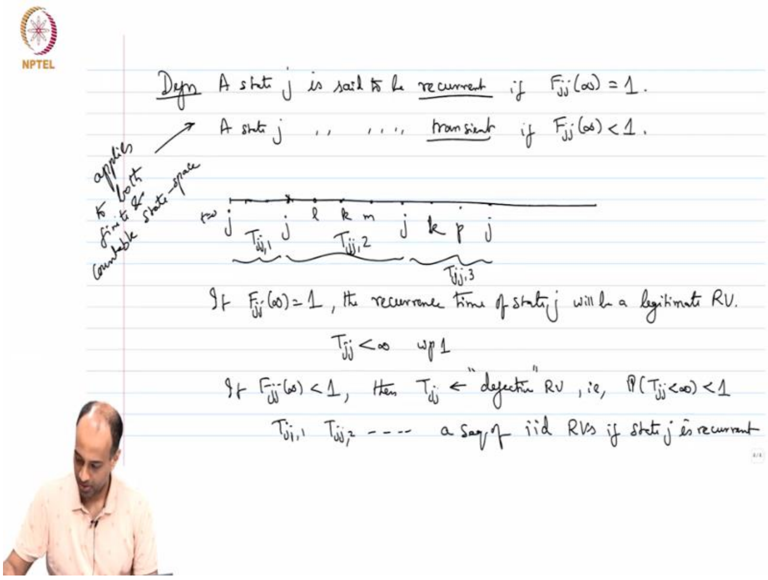

Now if the state j is recurrent then this T jj is a legitimate random variable the time interval between time to return to j starting at j. Now that you have reached j here again you can look at this random variable and this random variable let me call this T jj 2 it is also a random variable is clearly independent of T jj 1 because even that you reach j what you are going to at this point given that you have reach j the previous recurrence time of j is not going to affect the future states the future evolution of the Markov chain this is by the Markov property.

So, clearly the random variable T jj 2 is independent of the random variable T jj 1. Furthermore, these random variables T jj 1 and T jj 2 are identically distributed that is because the Markov chain is homogenous in time. If you are in a state j at any time it does not matter whether the time index is 50 or 5,000 your future evolution and subsequently returning to state j again does not depend on at least the distribution of the recurrence time of state j does not depend on how many time you have been to j before.

So, bottom line is that this if you look at these T jj 1, T jj 2, T jj 3 etcetera T jj 1, T jj 2 dot, dot, dot these form a sequence of i i d random variables if state j is recurrent.

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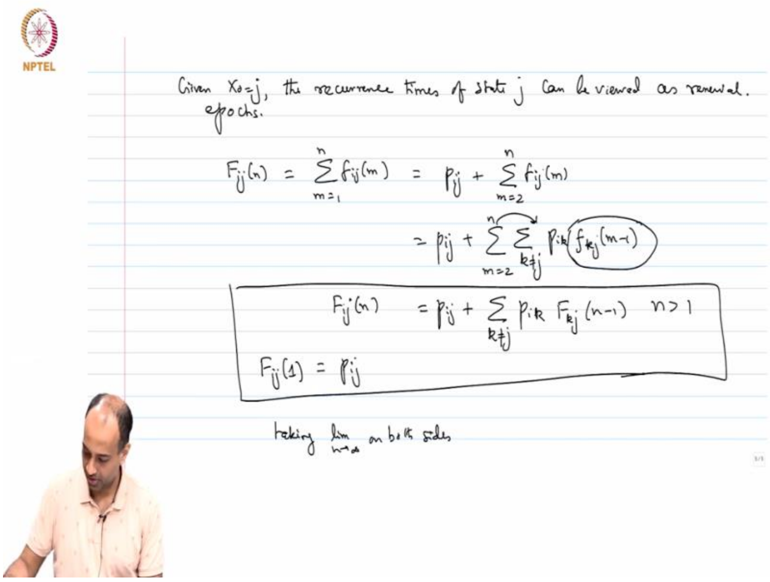

So, in turn what this means is that you can argue that given X 0 equals j recurrence times are of state j can be viewed as renewal epochs. So, what we are saying is that time 0 is started at j then you return to j here and then you return to j here and you return to j here and so on. We just showed that these guys are all iid so these are all renewal epochs. So, times at which the j occurs again a renewal epochs and these intervals are all renewal intervals at iid.

So, these will be a very useful tool because we know a lot about renewal processes as already if you remember from our previous study and we can use results from renewal theory, renewal reward theory and so on to derive some very important and interesting results about these countably infinite states Markov chains. Now let us get back to considering this F ij n. So, this F ij n if you remember is just sum over f ij m, m = 1 to n. So, this is I can pull out P ij which is for m = 1.

And then write sum m = 2 to n f ij of m and f ij of m I already in iteration from earlier. So, I can write this as P ij + sum over m = 2 to n sum over k not equals j if P ik f kj of m – 1 which I can write again as P ij + sum over k not equals j. So, I am just pulling these summation inside this is justified because everything is non-negative so then what do I get? So, if I pull these summation inside I will get sum m = 2 to n of these f kj with what is that equal to?

That will be equal to so P ik comes out and this summation goes in I will get big F kj of n – 1 and this is for n greater than 1 and of course what is F ij of 1 is simply equal to P ij. So, this is the iteration for big F ij of n I have these is the iteration. Now what happens if I take n to infinity?

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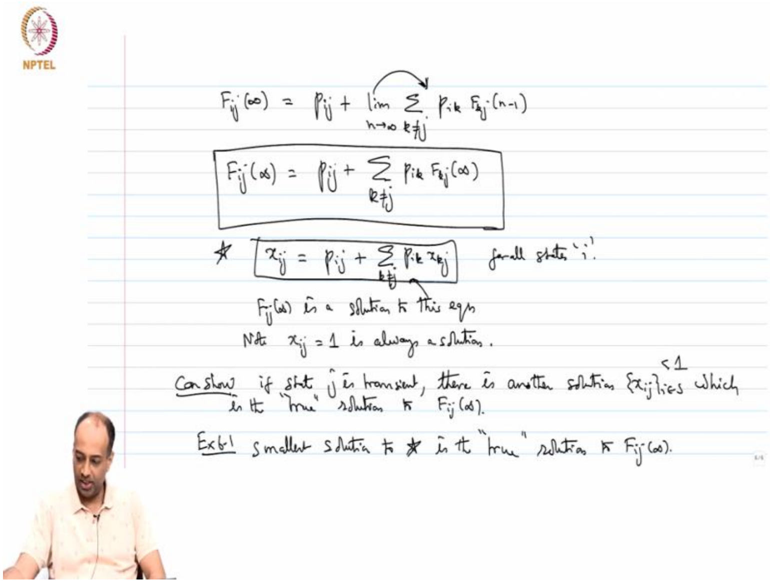

I should have the equations F ij see all these limit exists because they are all monotonic in n. So, F ij infinity = P ij + limit n tending to infinity sum over k not equal to j P ik F kj of n – 1. Now, I am going to push this limit inside the sum this requires a justification this is because the remember now that you are sending a limit inside an infinite sum and this does not always hold it holds in this case.

The justification for this will come from the fact that the terms are non-negative and this is bounded above. So, we can invoke some bounded convergence theorem or something like that and push the limit n. So, let us not worry too much that. So, if we push the limit (()) (08:30) you will get sum over k not equals j P ik F kj of infinity. F ij of infinity = P ij + sum over k not equal to j P ik F kj of infinity.

So, this can be viewed for any j as a set of equations over i and you can solve them. Now so you can just view these as equations on you can write this as X ij = P ij. I am just pretending that F ij infinity is X ij I am just calling it some variables sum over k not equals j P ik X kj and this is for all states i. You can say that F ij infinity which is the probability that eventually hit j having started at i is a solution to this equation.

Now, if you put X kj X ij = 1. Note, X ij equals 1 is always a solution to this above equation. Does it mean that F ij infinity is always equal to 1? No, not necessarily. What we can show is that all the X ij equals one is always a solution to this equation. There may be a solution X ij which is smaller than one also there may be, but it does not have to X ij = 1 is always a solution, but there may also be a smaller solution.

What we can show is that if state j is transient there is another solution X ij i belongs to s which is the true solution to F ij infinity. So, if state j is transient then you may not there is no guarantee that you will hit j from i. In which case F ij infinity will be less than 1 so in that case there will be a solution X ij which is strictly less than 1 which is the true solution to F ij infinity.

And in fact the smallest solution of this is the true solution always. So, it can be shown so there is in fact guided proof in Gallagher that says that the smallest solution is the true solution of let us call this some star is the F ij infinity, but if state j is recurrent then there will be only one solution which is X ij = 1. Now, if you look at this let us particularize this equation for the case when i = j.

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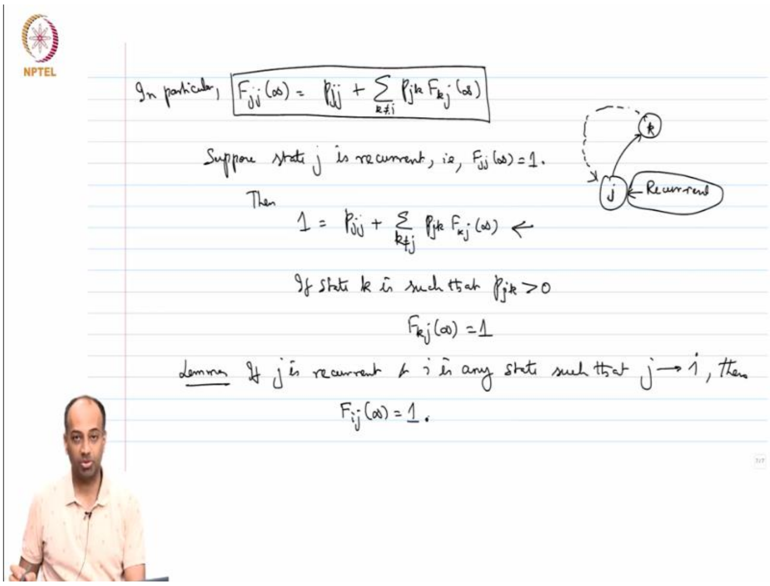

In particular, if I write F jj infinity = P jj + sum over k not equal to j P ik F kj infinity if I look at this equation. Now suppose state j is recurrent ie F jj infinity = 1 then what happens is that you get 1 = P jj + sum over k not equals j P ik F kj of infinity. Now if state k is such that P ik is strictly greater than 0 sorry I should write I think I made a mistake here I should be P jk I am sorry so P jk is strictly greater than 0 that is state j is recurrent.

And there is some other state k such that I can go from j to k with positive probability then if you look at this equation. This equation will imply that since the left hand side is equal to 1 this is only possible that if P kj is strictly positive then F kj of infinity will have to be 1 for all k such that P ij k is strictly positive. What we have essentially shown is that if state j is recurrent if this guy is recurrent if this is a recurrent state and if it is possible to go from j to k P jk is strictly positive.

Then with probability with 1 you are going to eventually come back to j F kj infinity is 1 so starting at k you are assured with probability 1 that you will return to j. In fact, you can extend this logic further this is only a state k which is one hop away from j from a recurrent state j. In fact, you can extend this logic and prove that if j is recurrent and i is any state such that j arrow i meaning that I am able to there is a path of positive probability from j to i.

Then F ij infinity = 1. So, what we have argued above is that if P jk is strictly positive meaning there is a one hop path from j to k then return from k to j is assured. This lemma is generalizing that it is just an iterative you can show that if there is a two hop way to get from j to i or a three hop way to get from j to i you can just inductively show you that return from i to j eventual return is guarantee with probability 1.

This is just an inductive proof using the logic above. You can do the proof yourself inductively. So, this is great. So, if I am considering the recurrent state so the recurrence is given a particular meaning now. If I am at a recurrent state and if there is a state that I can get too from j then I am assured that I will return to j from that state. So, that ties in with the understanding of recurrence that if I start at j I should return to j with probability 1.

So, any state I get to I should be able to get back to j with probability 1 that is what this lemma is saying.

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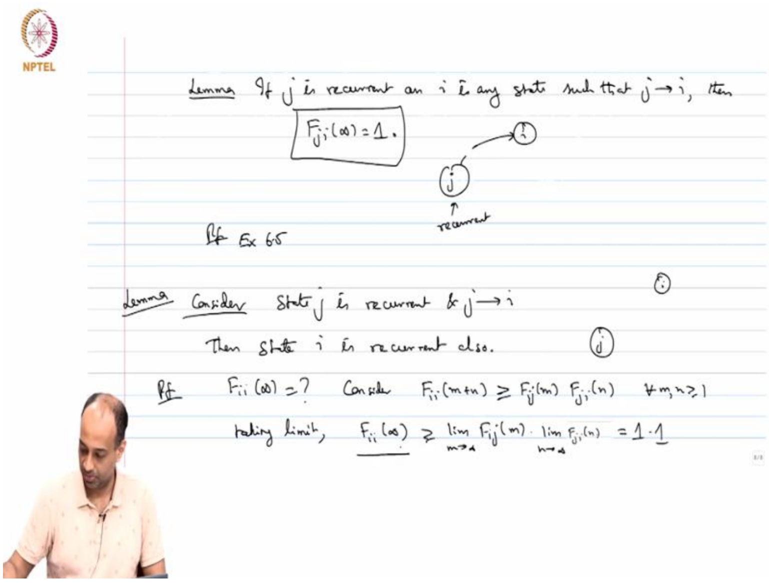

There is another lemma which you can prove which is sort of the opposite of what the above lemma says if j is recurrent and i is any state such that j arrow i then F ji infinity is also equal to 1. So, the situation is the same j is recurrent and i is some other state which I can get to. Let us say I can just get to in one hop or two hop whatever then what we are saying here is F ji infinity = 1.

So, if I start at j I am assured to hit i with probability 1 at some point finite time. Again the proof of this is from using the fact that you are guarantee return j is recurrent so you are assured to return to j whenever you are assured to j you are assured to return to j and these times to subsequent returns are all random variables. So, during one of these returns there is a positive probability that you will eventually get to i that is the logic of the proof.

There is a positive probability alpha that anytime you get to j you may go from j to k, k to l and l to i let us say that is a positive path probability. The probability that let us say this probability is alpha of going from j to l and l to i or something like that. The probability that you keep coming back to j, but you never take this path actually will go down geometrically fast that you can argue some more vigorously.

This is in fact done in exercise 6.5 in Gallagher I will not (()) (21:28) last time on this, but the logic is fairly straightforward. So, we have said two things if this is the picture if j is recurrent and i is the state which you can reach from j if j arrow i then F ij infinity = 1 and F ji infinity is also equal to 1. What that means is that any state which you can get to from a recurrent state you are guaranteed to come back to j that is what the first lemma says.

The second lemma says whenever you are in j you are eventually guaranteed to go to that state i. This is very nice because we can prove another lemma. Consider so a situation was state j is and j arrow i so state i is accessible from state j then we can prove that so you can prove this is lemma then state i is recurrent also. So, as long as j arrow i state j is recurrent and j arrow i then I guarantee that i is also recurrent.

Why is this the case? So, looking at j which is recurrent and you are looking at i. I want to look at F ii infinity = what? If, F ii infinity = 1 I am done. Now you consider F ii of m + n F ii of m + n is surely greater than or equal to F ij of m times F ji of n for all m n greater than or equal to 1. Why is this the case? So, F ii m + n is nothing, but the probability that I start at i and come back to i at some point between (()) (24:17) m + n.

One way of doing this is to go from i to j in m steps within sometime between (()) (24:28) and then sometime between m and m + n m + 1 and m + n go from j to i. So, this equation follows. Now I can just take limit. So, you take limit I get basically m and n tending to infinity F ii infinity is greater than or equal to limit m tending to infinity F ij of m times limit n tending to infinity F ji of n and I know that from the previous lemma I know that both these limits are equal to 1.

So, this is equal to 1 times 1 from the previous lemmas. So, F ii of infinity is greater than or equal to 1 which means that F ii of infinity will be equal to 1 being a probability which means that i is recurrent so which is very nice.

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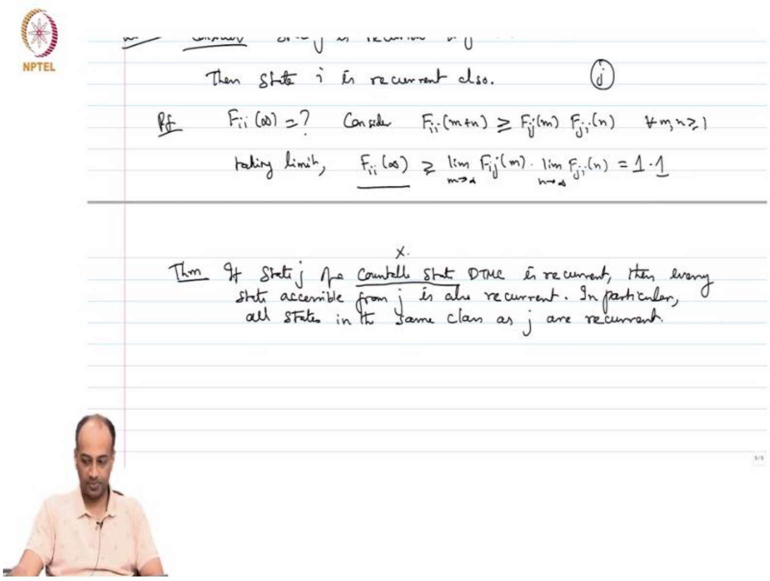

So, we have proven that theorem we have proven the following theorem. If state j of a countable state DTMC is recurrent then every state accessible from j is also recurrent. This means that in particular all states in the same class as j are recurrent. So, if we have a state j any other state you can get to from j is also recurrent. So, we are getting back the old result that we know for finite state Markov chains that all states in a class must be of the same type must be either all recurrent or all transient.

So, this theorem statement is true actually I do not even have to say countable state I can just say I do not have to explicitly say this for finite state also this is true we know this to be true except the definition of recurrence was different over there, but we can show that the current definition holds they are in the finite state case also. So, this is a very nice result. So, in each state in a class must be all class in a state must be all recurrent or transient. We can stop here.