**Mathematical Methods and Techniques in Signal Processing - I**
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**Lecture – 34**
**Frequency representation of expanders and decimators.**

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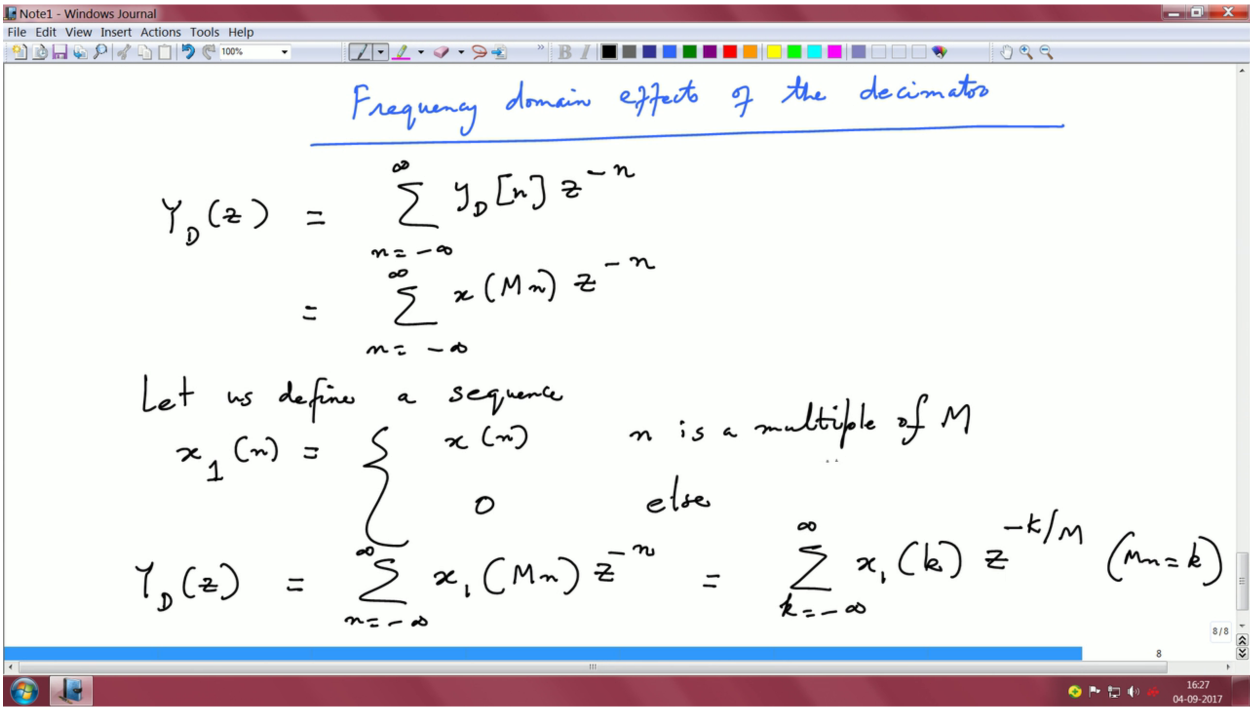


Now we saw the effect of down samplers and up samplers in the time domain. A natural question that arises is, what are these spectral effects, ~~what is~~ what is the frequency domain effect or doing down sampling and up sampling right. So, let us try to investigate into the frequency domain effects of the decimator first. Now, we start with Y suffix ~~d~~ D of z, ~~d~~ D standing for the decimator, it look we take the z transform of the signal which is y ~~d~~ D of n, z power minus n, and we just drop in the equation for, around its the relationship between x of n and y ~~d~~ D of n, and y ~~d~~ D of n is basically x of m times n.

So, this is what we have right. Now this is not enough for us to proceed. So, let us define a sequence x 1 of n equals x of n, when n is a multiple of M and 0 otherwise. Now, I define x 1 of n as x of n when n is a multiple of M and 0 otherwise. So, that I can accommodate, because there are holes right, I mean at these points M n it exists, at other points it does not so, but to take care of those holes we just defined this new sequence x

1 n as follows. Now we can plug this into this equation, the output of the decimator Y, I mean the frequency domain, the transform of this, the output of the decimator Y ~~d~~ D of n is basically Y suffix ~~d~~ D of z, which is summa n eq equals minus infinity to plus infinity x 1 of M times n times z power minus n.

So, this is basically we are fixing those holes at points where they were not multiples of n. And if I let M times n equals k right. The usual trick that we do, I can say this is x 1 of k, where my k equals m times n, z power minus k upon M. Let this happen then this is from k equals minus infinity to plus infinity; x 1 of k times z power minus k upon M, but I am not sort of still complete, because I now have to relate x 1 of n with x of n. I have just written this in this clause x 1 of n is x of n, n is a multiple of M 0 else. So, I have to kind of put this into some compact form, then (Refer Time: 05:09) statement like this right. So, for this we go one step forward, and we say x 1 of n equals some comb sequence times x of n.

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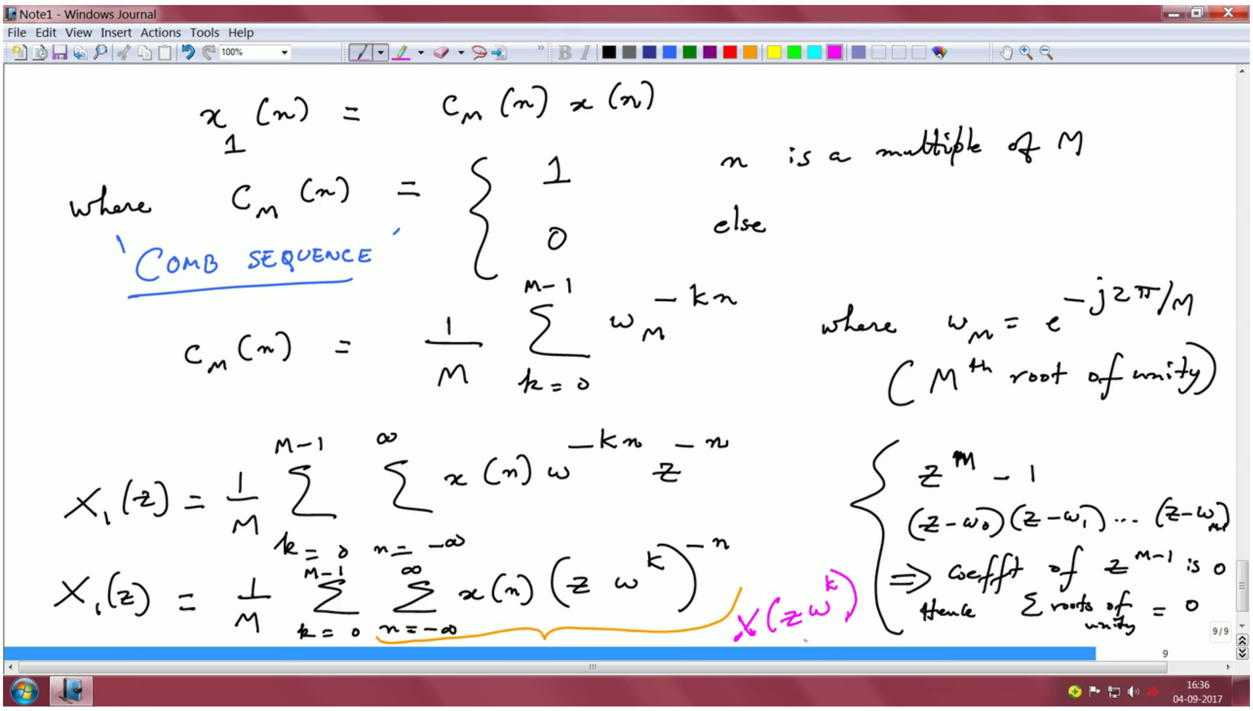


This comb is literally like your comb, the hair comb, where C m of n is 1. If n is a multiple of M and this is 0 otherwise. So, gone a little one step forward right, this is C m of n is 1, when n is a multiple of M 0 else. Still it is like an efelce statement, we think it is not simplifying, but here is an important relationship between. ~~T~~the exact definition of this comb sequence, and this comb sequence can be written in this forum, which links the nth root of unity as follows right. C suffix m of n links the nth roots of unity as follows given by this relationship.

It is not too difficult for you to see, because when n is an integer multiple of capital M right, this is going to be the 0, I mean 0 or M 2 M 3 M so on and so forth right when its in integer multiple of M, this terms becomes 1. So, basically you are adding 1 1 1 1 M such copies and this is basically M, by M upon M this is basically 1 and. If it is not an integer multiple of M we are basically summing the roots of unity and summation of the roots of unity is 0, why basically if you think about this equation z power say M minus 1 right, this is equation that is z minus say omega naught z minus omega 1 dot dot dot z minus omega M minus 1, we have M roots here.

Now, observe the coefficient of the m minus 1 term it is 0, but some of the roots are basically the coefficient of the monomial z power m minus 1 in this product right. So, therefore, this has to be 0. So, therefore, the comb sequence is linked to the roots of unity according to this formula. Now, since this is very clear and interesting to us. So, now, we can say x 1 of z is 1 upon M summa k equals 0 to M minus 1 summa n equals minus infinity to plus infinity x of n omega power minus k n. So, I am just removing this index m in this root it is implied z power minus n. So, just an aside that coefficient of z power M minus 1 is ~~0,~~0 and summation on the roots of unity is equal to is just a proof aside. Now x 1 of z is linked with x of n using this relationship.

So, what I have done is nothing, I am just now I am taking the z transform here. So, this x of n z power minus n is whatever I have in the original z transform right, and this comb sequence is just being replaced by this, this formula nothing special. So, if you went in the efelce statement, saying n is a multiple of M this exists; otherwise it is 0 then you would never have simplified. Since we wanted to fill in the gaps for to account for the 0s when n is not a multiple of M, we had to introduce the comb sequence and link the comb sequence with roots of unity, and that is a trick, and you will see this trick in number theory as well, in various other properties number theory you will see this trick.

So, its a good trick to have this at the back of your mind right. Now we can simplify this further, this is 1 upon M summation k equals 0 to M minus 1 summation n equals minus infinity to plus infinity x of n times z omega power k times. I mean power z omega power k power minus n, is just what in algebra, and then if you sort of just interpret this equation here right, this is basically ~~x~~ X of z omega power k.~~;~~ I~~i~~f x of z corresponds to x of n x of n, I mean the z transform of this is basically x of z omega power k right. Now, we are sort of ready to put this in final form, we say Y ~~d~~ D of z is 1 upon M summa k equals 0 to M minus 1 x of z power 1 upon M omega power k; why, because Y ~~d~~ D of z is ~~x~~ X 1 of z power 1 upon M.

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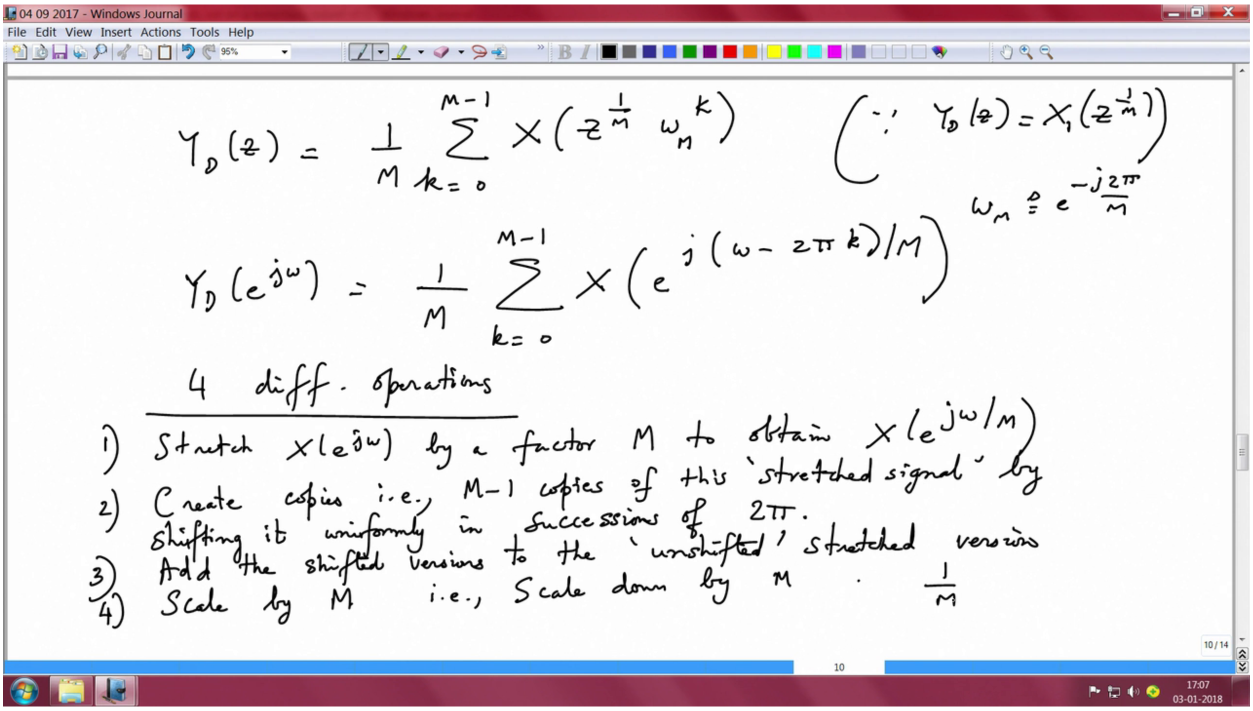


Let us go back here and just look at this equation Y ~~d~~ D of z is x 1 of z power 1 upon M, because see this 1 upon M here right, because of this relationship, we are ready to see how the output of the decimator looks in the frequency domain right. This is the formula, this is the derivation for the frequency domain effect of the down sample, I just have removed subscript capital M from this derivation, but you can it is implied that it is there.

Now, let us work back what it is in the Fourier domain. So, if I look at Y ~~d~~ D of e j omega it is 1 upon M summation k equals 0 to M minus 1 ~~x~~ X of. Now we have to be careful here. So, the nth root of unity omega M again for you to recall is e power minus j 2 pi phi, 2 pi by m right and. So, this is x of e power j times omega minus 2 pi k upon M. So, this is what we get for the response of the decimator in the frequency domain.

Now, what is happening here, just observe this equation here right. So, now, this omega minus 2 pi k for k equals 0 1 2 so on till M minus 1, you can interpret this as basically a translation of your spectrum, and this M basically scales the spectrum right and what is happening is you are basically stretching it, in the frequency domain; whatever is compressing in time, because of slashing of rate is basically stretching in the frequency domain, and basically for every such copy you are evaluating this spectrum and you are summing it up and your amplitude is basically scaled by 1 upon M. So, there are four different operations that are happening.

One; you stretch x of e j omega by a factor M to obtain x of e j omega upon M right, I mean the M is in the exponent, then we create copies that is M minus 1 copies of this stretched signal. So, basically we stretch it, take M minus 1 copies of the stretched signal by shifting it uniformly in successions of 2 pi, then we add the shifted versions to the unshifted stretched version. Last the scale by a factor of M. Well this is scale down by M which is basically you multiply by a factor of 1 upon M ok. So, there are four different operations happening, and you have to realize that the stretched version that is x of e j omega upon M is itself not 2 pi periodic, but after adding these shifted versions, it is 2 pi periodic. So, its good to imagine how this spectrum really looks like.

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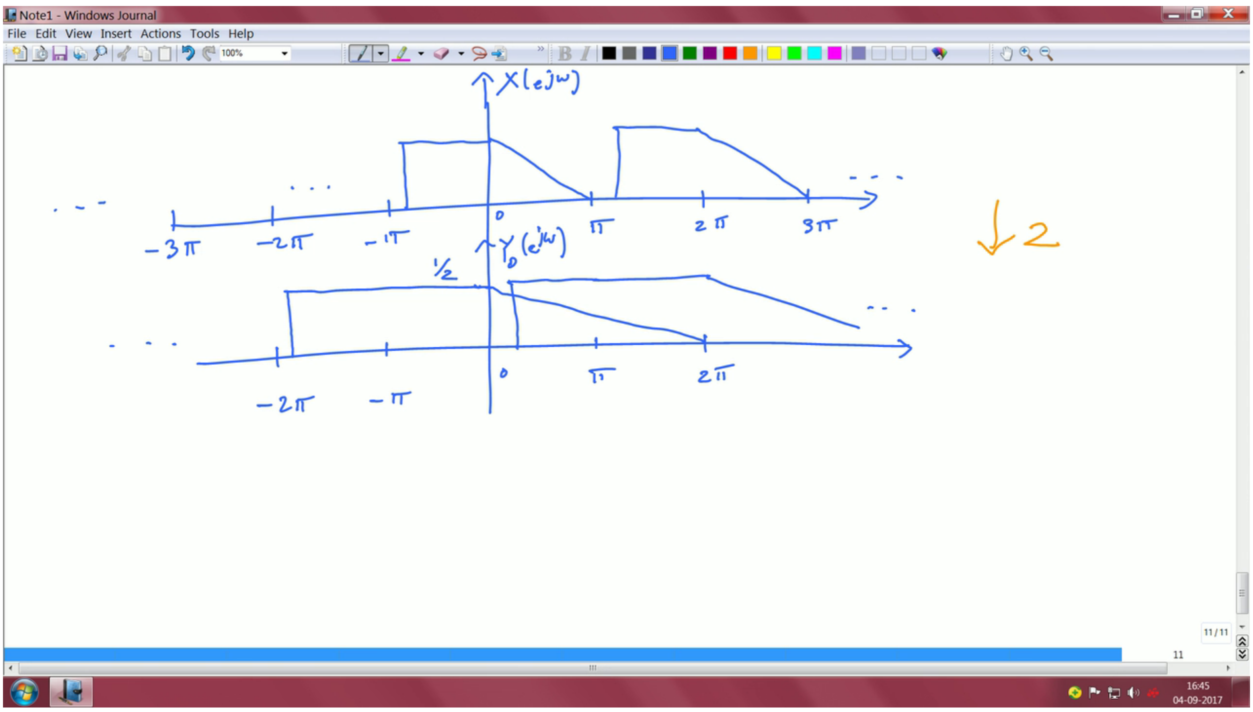

My figure is not perfectly up to scale, but I think you can still follow this up 0 here, just have pi, just have 2 pi, pi here, pi say 3 pi, say minus pi minus 2 pi minus 3 pi dot dot. So, let us say we have our original spectrum ~~x~~ X e j omega and let us say it is periodic so on and so forth. Now if you down sample this by 2 what happens is this base spectrum stretches by 2 and you will have copies of that right. So, basically I start with the this diagram here right; that means, I have to now stretch this thing out from pi it, it moves to 2 pi this point moves somewhere here right, and this is going to be, and then similarly we will have one more copy of this signal like this.

Since it has periodic know you can think about these stretch versions on either side of the origin right. So, this is basically the frequency domain effect of the decimator. So, I have forgotten a scale of, scale factor of half we just have to bring this half as well to say that you have this basically, you are you are doing some translations and basically you are you are folding this. Now similar to the decimator we have the expansion phase as well.

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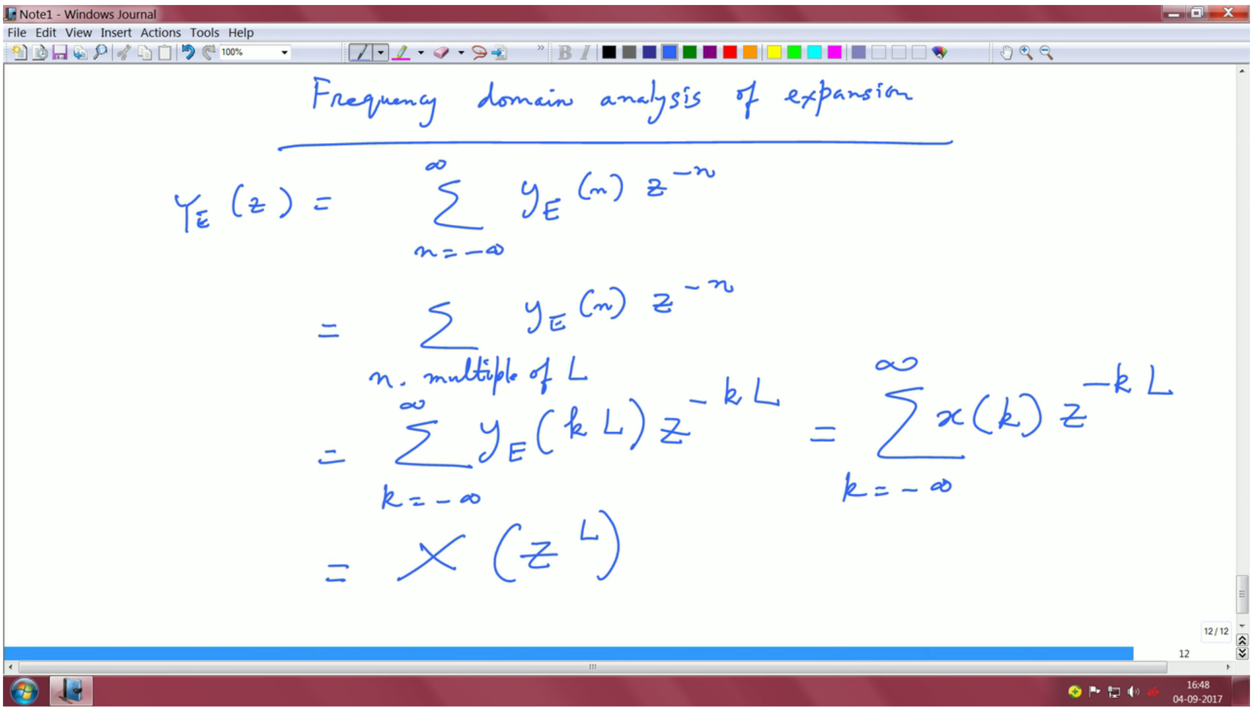


And let us derive the frequency domain analysis of the expansion process. Now Y ~~e~~ E of z is summa n equals minus infinity to plus infinity. Basically I take the z transform of the output of the expander, and this is basically. So, I said this y of n can be written as this summation being valid, when n is a multiple of L. So, I have ~~y~~ Y ~~e~~ E of n expansion of n, of expansion of the expansion process right of x of n. And I introduce a variable k, which is going from minus infinity to plus infinity and n is basically an integer multiple of L. So, this is basically Y ~~e~~ E of k times L ~~z~~ Z power minus k times L, because n is a multiple of L right and.

Therefore I bring that variable k into picture, and now this is very easy for us to simplify, because we do not now have a hole here unlike the decimator, where we had this efelce clause right something for a multiple of n 0; otherwise we here now do not have any such thing here, because its sort of clean, and you can just say this is basically k equals minus infinity to plus infinity Y e of k L is now ~~x of~~ x of k z power minus k L right, because Y ~~e~~ E of n is x of k, k upon n; that is what we had. So, therefore, Y ~~e~~ E of k L is basically x of k. And now this is easy for us and this is ~~x~~ X of ~~z~~ Z power L. So, there we had something z of z power 1 upon M for an M fold decimator and. Here if we observe it is e power L. So, if we are increasing the sampling rate through this expansion process right, you can imagine that the spectral effect is basically shrinking due to just sort of a dual duality right.

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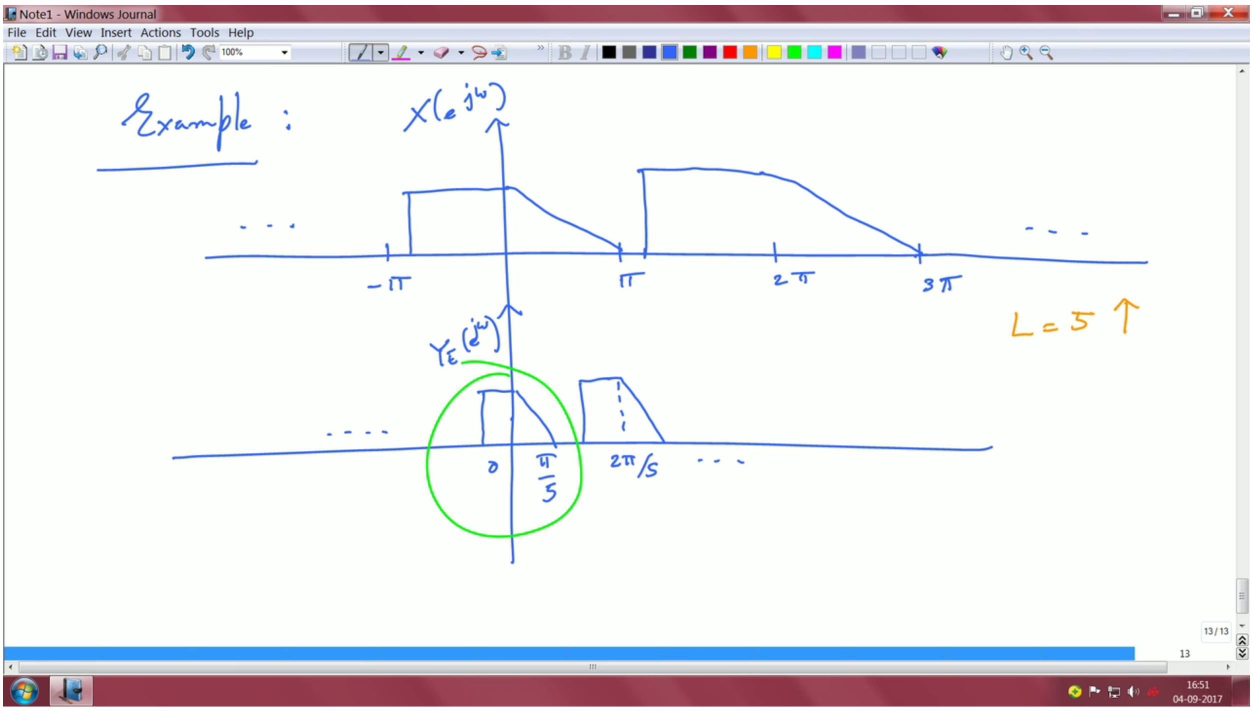


So, let us see through this with an example (Refer Time: 26:47) of pi 2 pi 3 pi dot dot dot. So, let us suppose that we have the same spectrum we started off with, so on and its periodic. Suppose I consider an L equals 5 up sampling right. So, now, this is going to be slashed or its going to be compressed in the frequency domain, and we have, this is a 1 over 5 factor stretch sort of compression in the frequency domain. But since it is a periodic signal right, we can think about having copies 2 pi by 5 so on and so forth; otherwise there are no copies, you just have to have one, you will have just one copy here, unlike the decimator. Since this is unless we are assuming periodicity here, therefore, we can think about that we will have multiple copies right.

So, you just have to focus upon just this. Now the problem if the spectrum is periodic as you can imagine, is the compressed spectrum. If the original spectrum is 2 pi periodic, the compressed spectrum has many images, and these images are basically goes right, and this is this causes imaging effect and if you have to be careful you really have to filter out these images through a filter; that means, you can put a filter which cut off at pi by 5 here. You know where a module is omega is less than pi by 5 just have a sharp cut off at pi by 5, sharp or gradual we will discuss these issues, but you will have to have a filter which cuts off and get just this image one image out and then you do what you want with it. So, this sort of completes this module or this lecture, part of this module in multirate. So, ~~we will~~ we will stop here.