Foundations of Wavelets and Multirate Digital Signal Processing

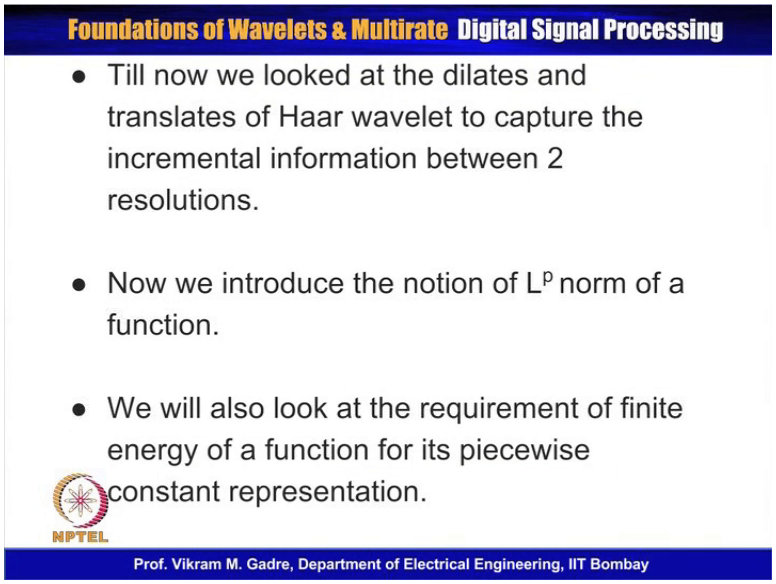
Prof. Vikram M. Gadre
Department of Electrical Engineering
Indian Institute of Technology, Bombay

Lecture - 2

Module - 3

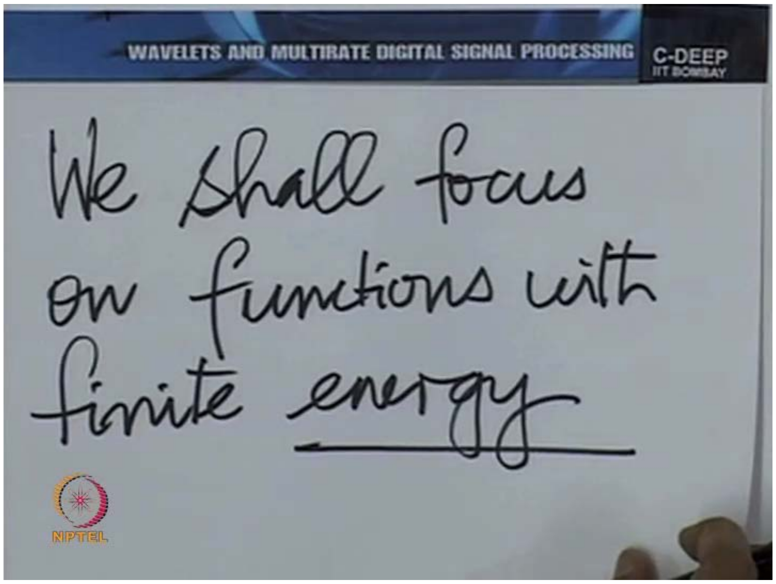
L2 Norm of Function

(Refer Slide Time: 00:16)



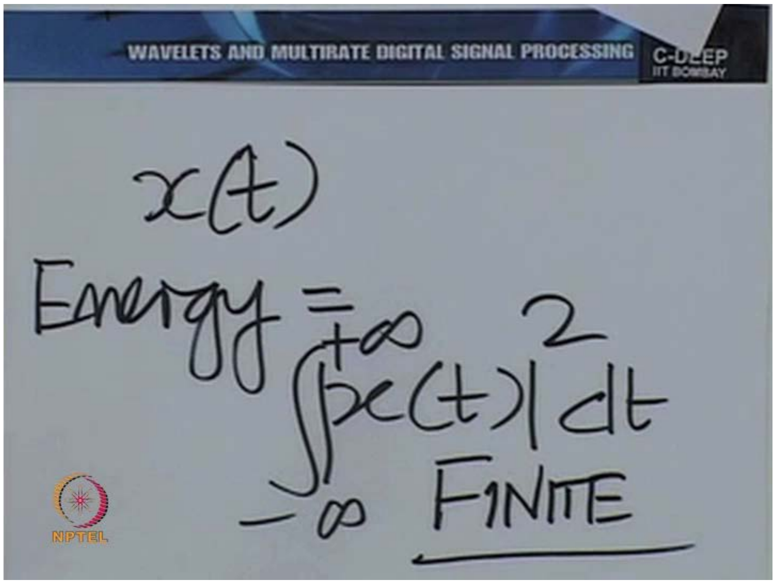
All that we ask for and that's not too unreasonable, is that the function has finite energy. So let us at least put that down, mathematically.

(Refer Slide Time: 00:45)



What we are saying is, we shall focus on functions with finite energy. And what does energy mean. Energy is essentially the integral of the modular square.

(Refer Slide Time: 01:06)

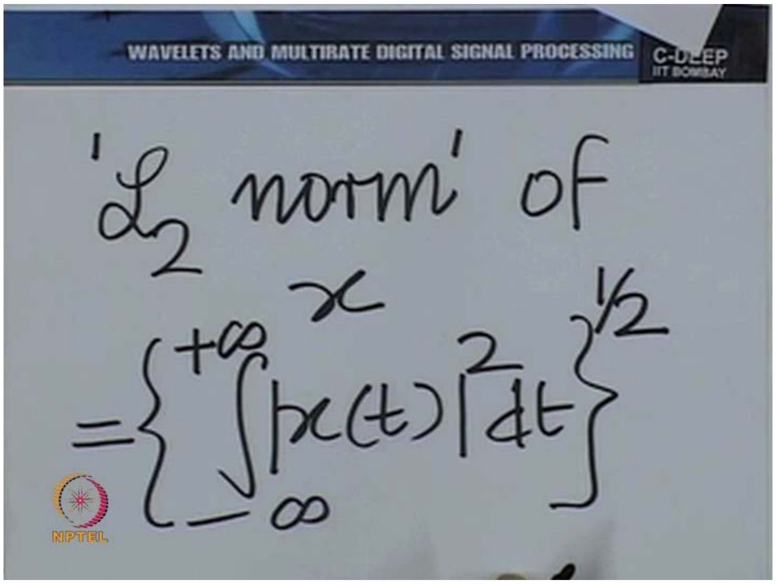


So if I have a function x of T, the energy is XT is the integral mode XT square, over all T. and this needs to be finite, all that we are saying is this.

Incidentally, this quantity has a name in the Mathematical literature, or for that matter even in the literature on wavelets. The energy as we call it in signal processing is called the L2 norm by

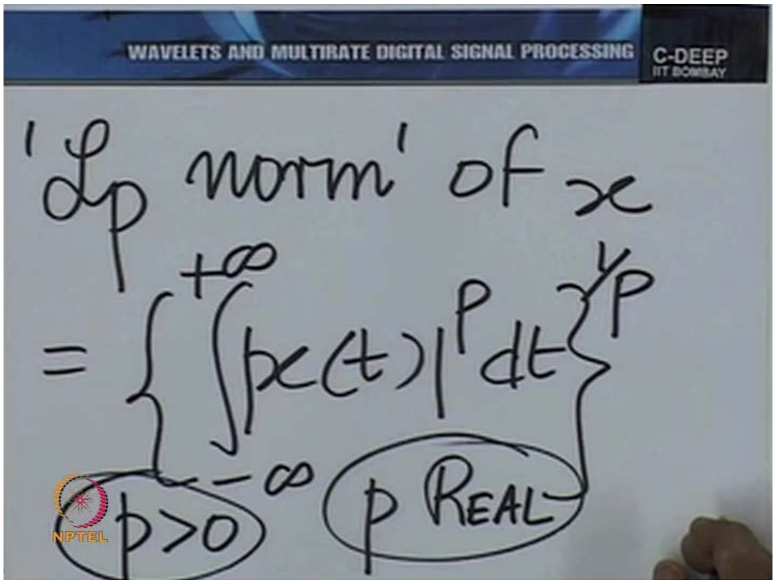
Mathematicians. And you know it helps to introduce terminology little by little from the beginning. Because if one happens to pick up literature on wavelets these terms would be used. So let's introduce that notation slowly.

(Refer Slide Time: 02:11)



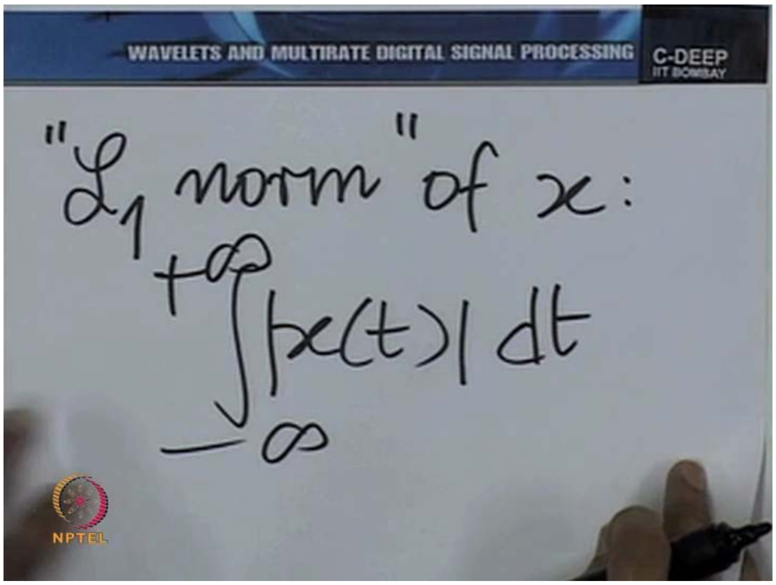
So we say the L2 norm of X, is essentially mode XT squared, DT integrated over all T and to be very precise, this needs to be raised to the power half.

(Refer Slide Time: 02:45)



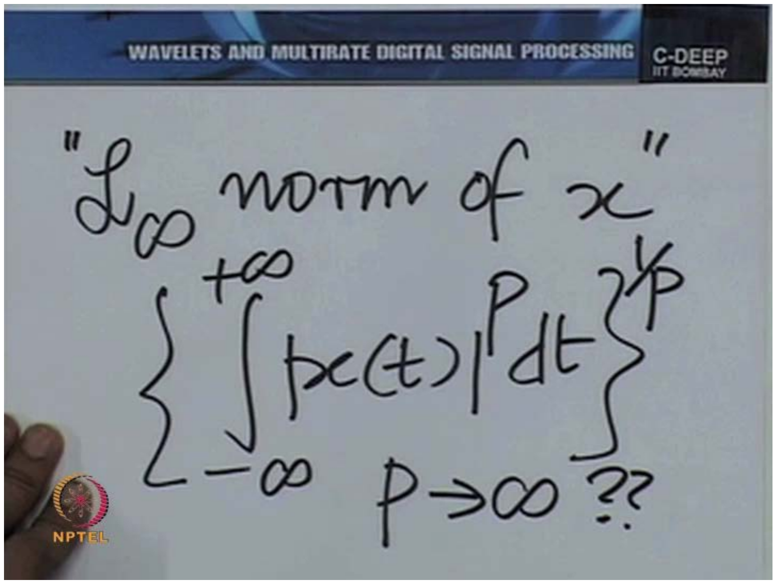
Similarly one can talk of an LP norm. If you could talk about an LP norm of X and that would correspondingly be mode XT to the power P, TT integrated on all time and raised to the power 1 by P. And of course P here is a real number. So for any real (pos...) in fact real and positive. So you could talk about an L1 norm, you could talk about an L2 norm, you could talk about an infinity norm.

(Refer Slide Time: 03:44)



What would an L infinity norm be, let's take an example, and what would an L1 norm be. It would be essentially integral mode XT.

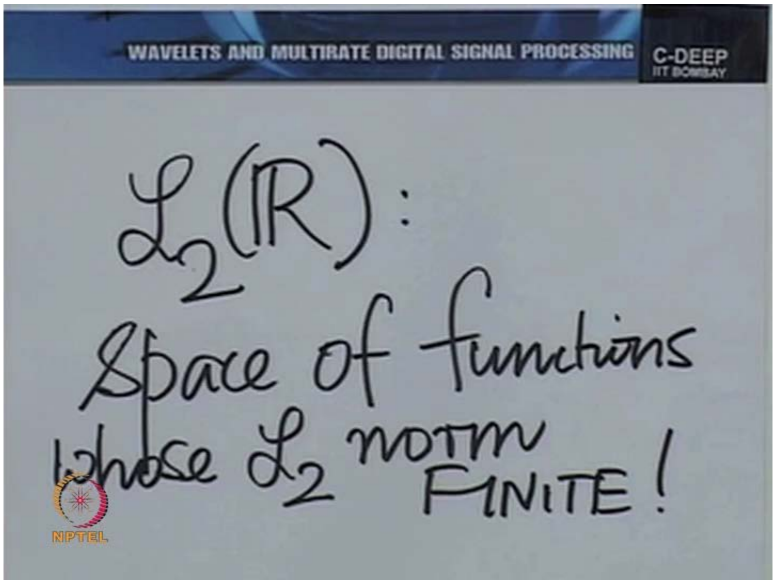
(Refer Slide Time: 04:05)



The l2 norm we already know, what would the L infinity norm be? That's interesting. So you see, in principle, it would be something like this. But what on earth is this? What do we mean by this? You see, as P becomes larger and larger, what are we doing. We are emphasizing those values of XT which are larger, so for a larger value of P, we are emphasizing those values of mode XT which are larger. And as P tends to larger and larger and larger values, as P tends to infinity we are in some sense high lighting that part of XT which is the largest.

So in other words the L infinity norm of X essentially would correspond to the maximum or the supremum, you know the very largest value that XT can attain all over the real axis. So it has a meaning. Even as P tends to infinity. Anyway, this was just to introduce some notations which we are going to find useful. And what we are saying in this language is that we are going to focus on functions which belong, now here, you know we are going to start talking about functions that belongs to a space. We say, you know we say the space L2, what is the space L2?

(Refer Slide Time: 05:50)



L2 over the real axis, it is the space of functions. And it is the space of functions whose L2 norm is finite. Simple. Similarly you could have the space LP. The space LP is the set of all functions whose LP norm is finite. Now the word space is used with an intend. You see space really means that if I take a linear combination of functions in that set it gets back to a function in that set. So if I take any finite linear combination of functions in a space LP, the resultant is also in that space, in that set, LP and that's why we call it a space.

So LP, all the LPs for any particular P are spaces, linear spaces. They are closed under the operation of linear combination. So in other words, we are saying let us focus our attention on the space L2, now what we have said in the Haar analysis that we have talked about a few minutes ago, is that if you take any functions in the space L2, I mean if your adversary picks up any functions in the space L2 and puts before you a value E0, saying please give me an M, so that when I make a piecewise constant approximation on intervals of size T by 2 raised the power of M.

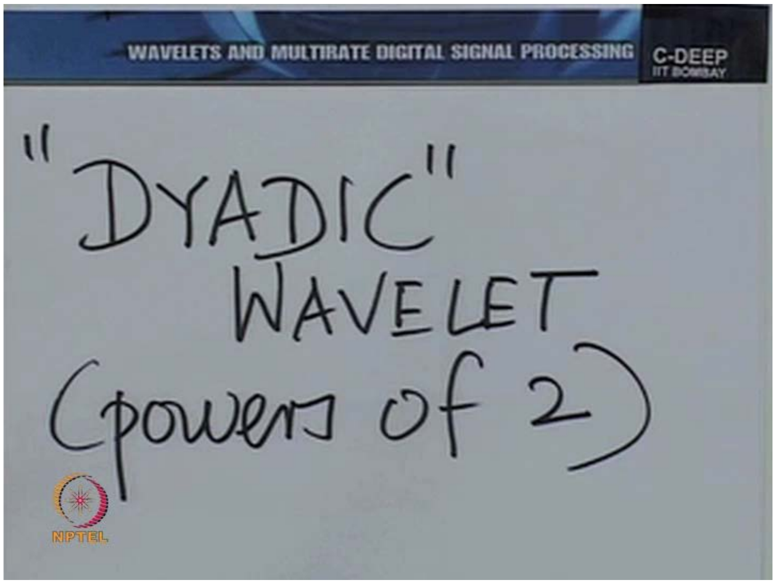
My error, squared error is less than E0, the proponent is able to do so. The proponent is able to come up with an M which gives an answer. And this could be done, no matter how small the E0 is. The proponent will always come out with a suitable M that is the idea of what is called closure. So what we are saying is when we do an analysis using the Haar wavelet, in other words, when we start from a certain piecewise constant approximation on intervals of size let us say 1 for example, and then bring it to the intervals of size half, one fourth, one eighth, one sixteenth, as small as you desire.

You can in principle go as close in the sense of L2 norm, that means if I look at the L2 norm of the error between the functions and its approximation. That L2 norm of the error can be brought down as much as you desire. And in that sense, whatever the Fourier series was doing, after all what does the Fourier series do. It allows you to bring the L2 norms of the error between the function and its Fourier series as small as you desire for the reasonable class of functions. For a wide class of functions, give me the epsilon, give me the E0 and I will give you a certain number of terms that you must include in the Fourier series.

So the adversary says, well here is an E0 for you, the proponent says, OK include so many terms in the Fourier series and you can bring your error down as low as you desire. The same kind of thing is happening here. The proponent, adversary principle. Now this is a deep issue, that one function Psi T is able to take you as close as you desire to the function that you want to approximate. And by the way this is only 1 Psi T which can do it. The whole subject of wavelets allows you to build up, many such psi Ts.

Here we had a good physical, a very simple physical explanation. We started from piecewise constant approximation. We said well, when you want to refine your piecewise constant approximation you could do it by using the Haar wavelet. And this you could do to go from any resolution to the next resolution. Please remember, here we are increasing the resolution or improving the amount of information contained by factors of 2 each time. And that’s why we use the term Dyadic.

(Refer Slide Time: 10:37)



Let me write down that term. Dyadic. So what we introduced in this lecture is the notion of a dyadic wavelet. And dyadic refers to powers of 2. Steps of 2 every time. The Haar wavelet is an example of a dyadic wavelet. And in fact for quite some time in this course, we are going to focus on dyadic wavelets. Dyadic wavelets are the best studied. They are the best and most easily designed, they are the best and most easily implemented. And I dare say the best understood. So for quite some time in this course we shall be focusing on the dyadic wavelet.

The Haar is the beginning. I mentioned in the previous lecture that if one understands the Haar wavelet and if one understands the way in which the Haar multi resolution analysis constructive, many concepts of multi resolution analysis would become clear. What we intend to do now after this, in subsequent lectures, is to bring this out explicitly. So let me give you a brief exposition of what we intend to do in subsequent lectures. And then we shall go down to doing it mathematically step by step.

You see, we brought out the idea of the Haar wavelet explicitly here. What is the Haar wavelet we know, we know what function it is. And we know the dialates and translates functions can capture information in going from one resolution to the next level of resolution in steps of two each time. Now how is this expressed in the language of spaces, after all we talked about the space L2R. L2R is the space of square integrable functions. So how can we express this in terms of approximation of that whole space?

So can we express this in terms of going from one sub space of L2R to the next sub space? And in that case can express this Haar wavelet or the functions constructed by the Haar wavelets and its translates and perhaps also dialates in terms of adding more and more of the sub spaces to go from a coarser subspace, all the way up to L2R on one side and all the way down to a trivial subspace on the other. So we are going to introduce this idea of formalizing the notion for multi resolution analysis. We need to think of what is called a ladder of subspaces. In going from a coarse subspace to finer and finer subspace until you reach L2R at one end and coarser and coarser and coarser subspace until you reach the trivial sub space at the other end.

Further, we are going to see that the Haar wavelet and it’s translates at a particular resolution, at a particular power of 2 so to speak, actually relates to the basis of these subspaces. So we are going to bring out the idea of basis of the subspaces and how the Haar wavelets capture what is called the difference sub space. In fact the altogether compliment to be more formal and precise. Simple but beautiful. And what we do for the Haar will also apply to many other such kinds of wavelets. Let us then carry out this discussion in more detail in the next lecture where we shall formalize whatever we have studied today for the Haar wavelet, by putting down the subspaces that lad us towards L2R at one end and towards the trivial subspace at the other. Thank you.