Probability and Random Variables/Processes for Wireless Communication
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Module No. 4
Lecture 22
Special Random Processes - Gaussian Process and White Noise - AWGN Communication Channel

Hello, welcome to another module in this massive open online course on probability and random variable for wireless communication. So in the previous modules, we have been looking at the various properties of random processes. A random process which is wide sense stationary. What are the properties of a wide sense stationary random process and the power spectral density of a random process and transmission of this random process or a wide sense stationary random process through an LTI system, through a linear time invariant system. Now, in today's module, let us start looking at some special kinds of random processes.

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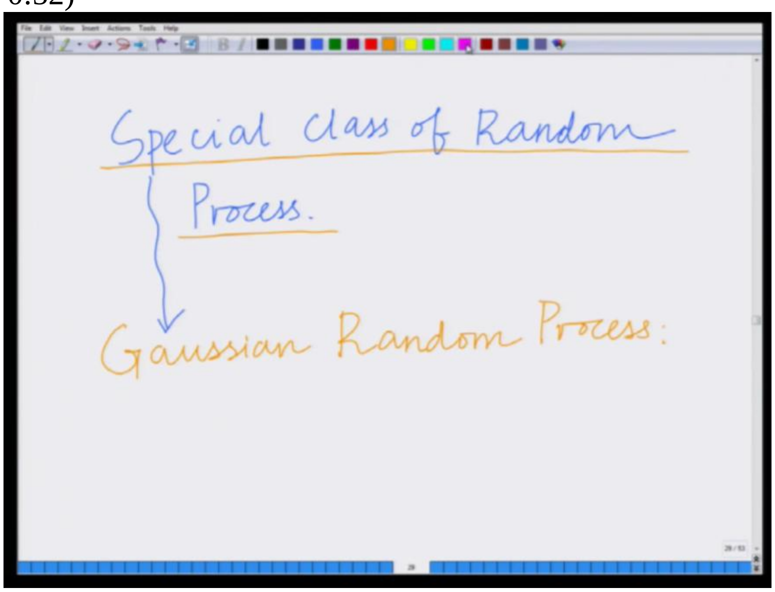

So let us start looking at a special class or a special kind. The 1st such special random process is what we call is a Gaussian random process which is very important to us. What is a Gaussian random process? Let us look at the definition. Let us consider a random process X(t).

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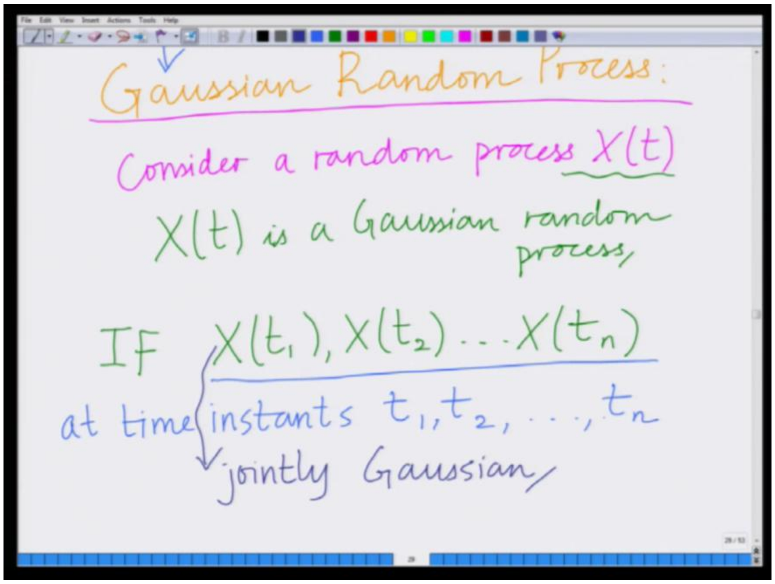

Consider a random process, X(t). Now this random process X(t) is Gaussian random process. This is a Gaussian random process if you consider X(t1), X(t2) so on up to X(tn) . That is if you consider a random process X at t1, t2, …tn. That is X(t1), X(t2), X(tn) at time instants t1, t2, tn. Now these, X(t1), X(t2), X(tn), these are jointly Gaussian. What does it mean to say that these are jointly Gaussian? That is the multivariate density of X(t1), X(t2), X(tn) is Gaussian.

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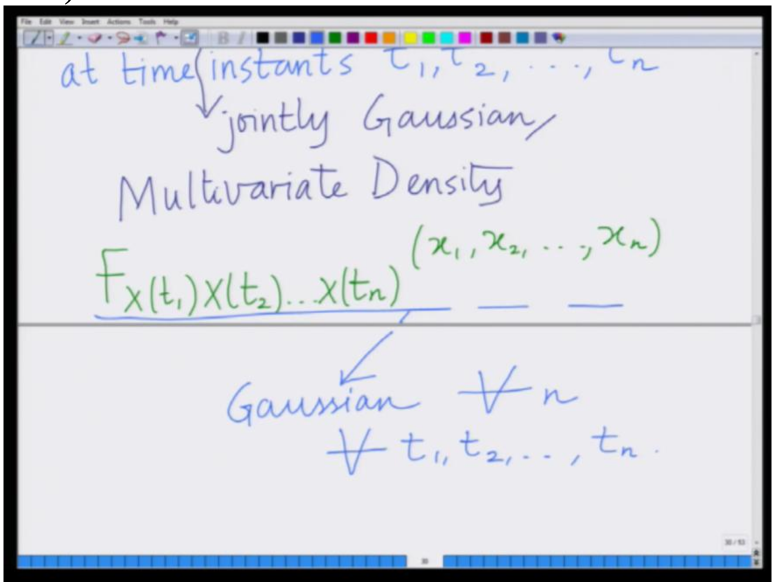

So if you look at the multivariate density, the joint density of X(t1), X(t2), X(tn) that is this probability density function, X(t1) , the joint probability density function X(t1), X(t2), X(tn) which has n random variables, this joint probability density function.

$$F_{X(t1)X(t2)\ldots X(tn)}(x_1, x_2, \ldots, x_n)$$

This multivariate probability density function has to be Gaussian for all possible values of N and for all times t1, t2,…, tn.

We choose n time instants, t1, t2, tn. Right? And we consider X(t1), the random process at these time instants that is X(t1), X(t2), X(tn) and we look at the joint probability density function corresponding to the random process X(t) at these time instants t1, t2, tn, the joint probability density function, this multivariate probability density function has to be Gaussian and then the random process, X(t) is known as a Gaussian random process which holds special importance in communications and specially in wireless communication applications.

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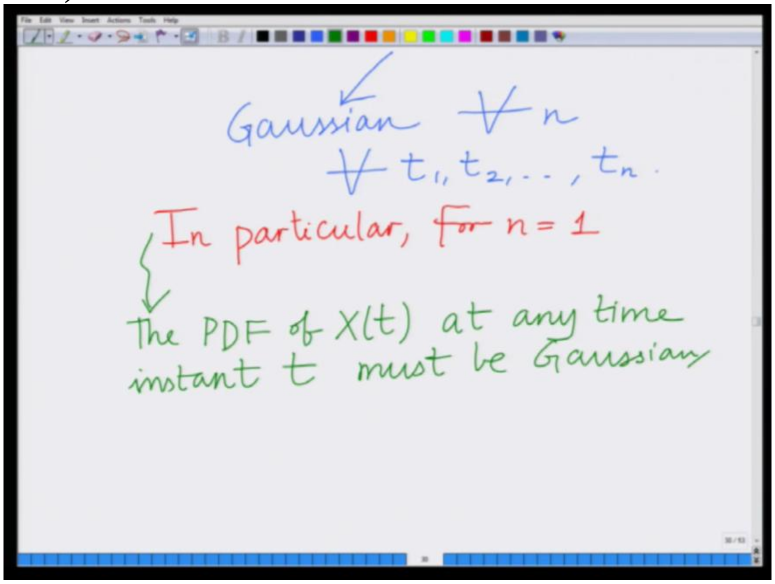

Now, in particular if we set n = 1, this implies, we must have that the distribution of X(t) at any time instant t, the PDF, probability density function of X(t) at any time instant t must be Gaussian. Right ? If the joint probability density function at t1, t2, tn for any n is Gaussian, then if we set n equal to one, it means if we consider X(t) at any time instant t, the probability density function must be Gaussian.

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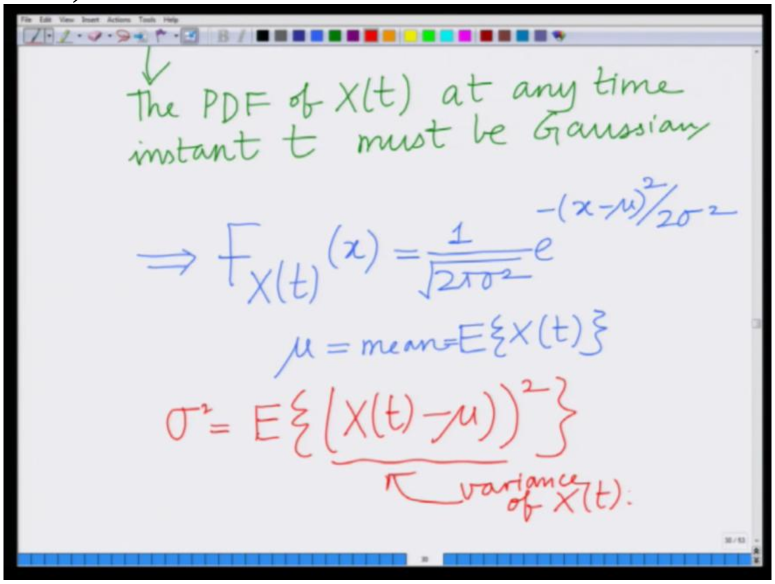

Therefore, this probability density function implies that,

$$F_{X(t)}(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$

Where,

$$\mu = E\{ X(t) \}$$

$$\sigma^2 = E\{ (X(t)- \mu)^2 \}$$

Remember, this is the variance of X(t).

Now of course, for a general random process X(t), both this mean and this variance will be a function of time. So this $F_{X(t)}(x)$ will be a function of time for a general, even for a general Gaussian process.

Therefore, we have to consider a special kind of Gaussian random process which is wide sense stationary. So for a wide sense stationary Gaussian random process we will have…

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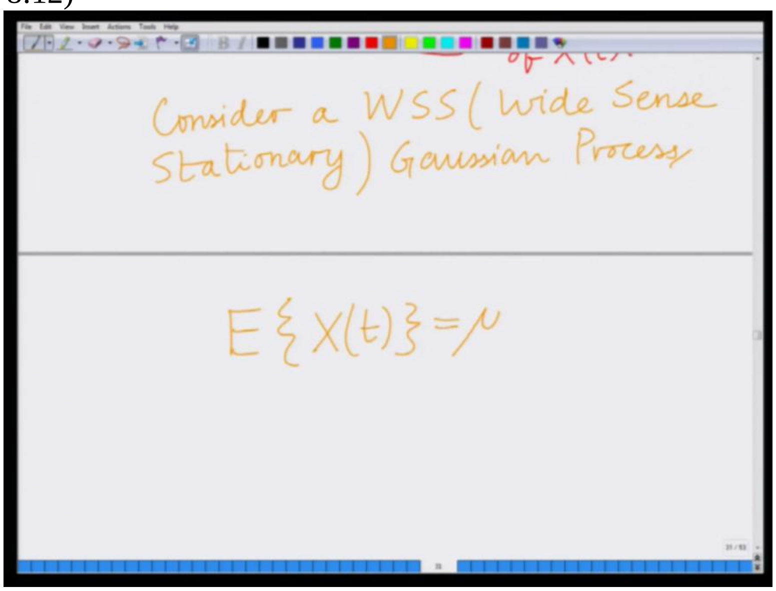

… so now consider a WSS. That is wide sense stationary Gaussian…for a wide sense stationary Gaussian random process we will have E{X(t)} that is stationary in the mean therefore

$$E\{X(t)\} = \mu_X$$

Remember, this is wide sense stationary random process, stationary in the mean.

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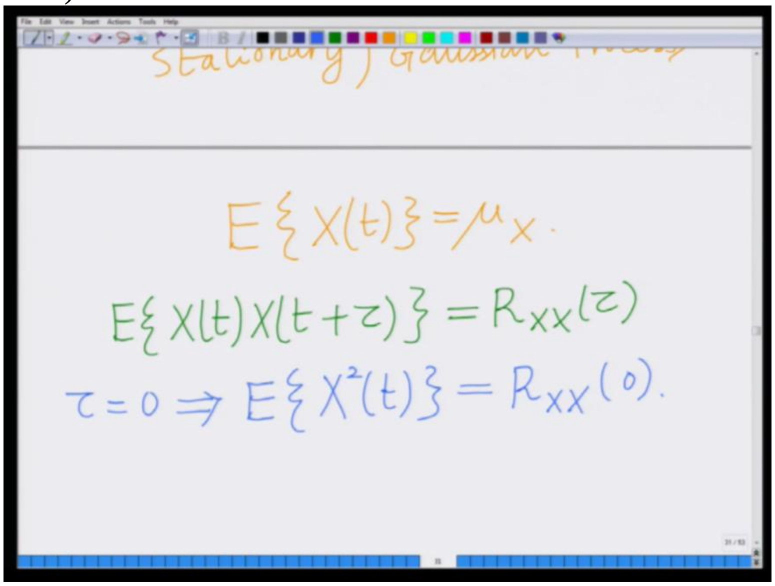

Further,

$$E\{X(t)\,X(t+\tau)\} = R_{XX}(\tau)$$

If $\tau = 0$, then,

$$E\{X^2(t)\} = R_{XX}(0)$$

$E\{X^2(t)\}$ is the average value of t that is the 2nd movement of this equals the average power in X^2. That means, what is their variance?

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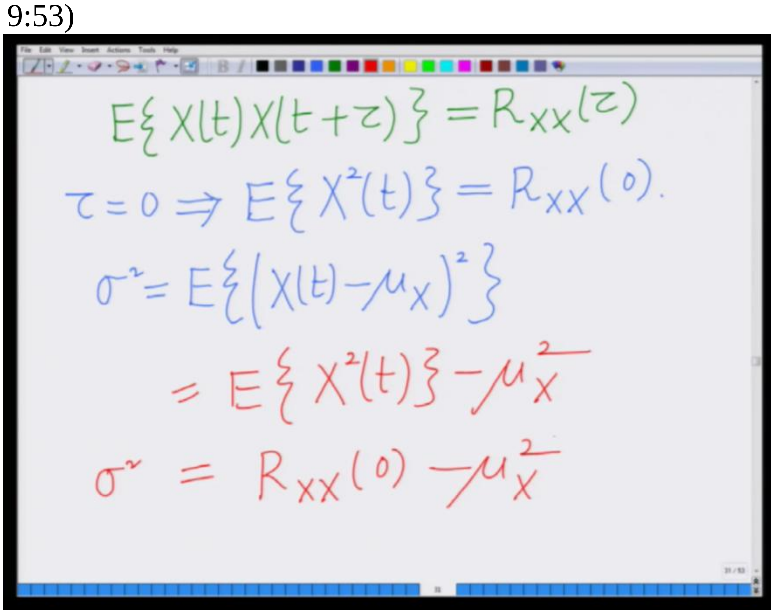

The variance will be –

$$\sigma^2 = E\{ (X(t) - \mu_X)^2 \}$$

$$= E\{ X^2(t)\} - \mu_X^2$$

$$\sigma^2 = R_{XX}(0) - \mu_X^2$$

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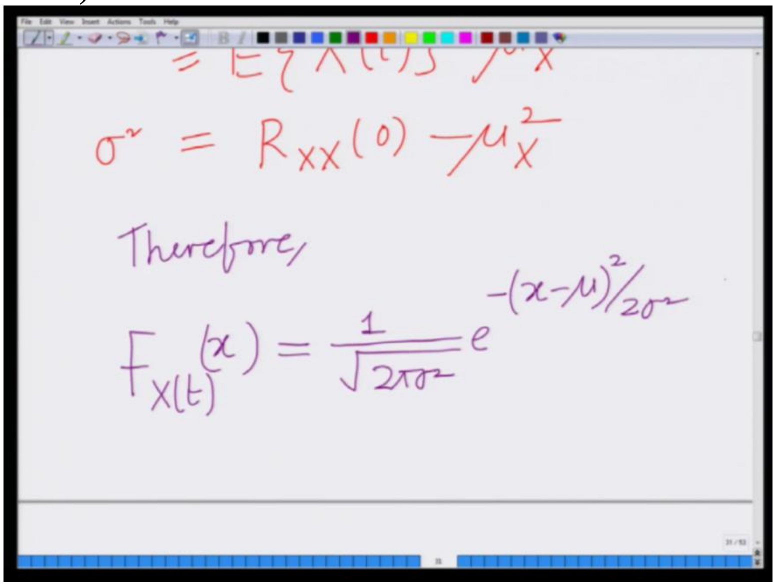

Therefore what we are going to have? The probability density function $F_{X(t)}(x)$ =

$$F_{X(t)}(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$

$$= \frac{1}{\sqrt{2\pi R_{XX}(0) - \mu_X^2}} e^{\frac{-(x-\mu)^2}{2(R_{XX}(0) - \mu_X^2)}}$$

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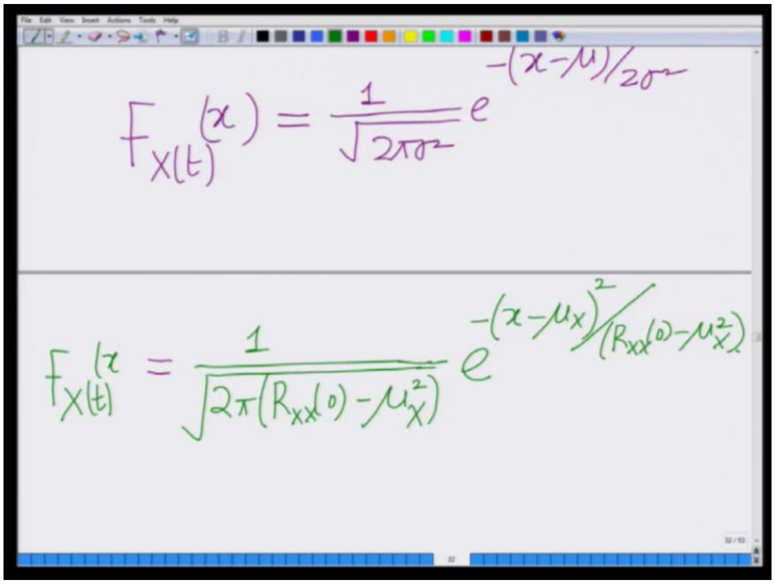

Let us recall what is this? This is the probability density function of the wide sense stationary Gaussian random process.

That is random process X(t) which is both Gaussian and in addition to being Gaussian, it is wide sense stationary. Then this is the probability density function of X(t) for any time instant t.

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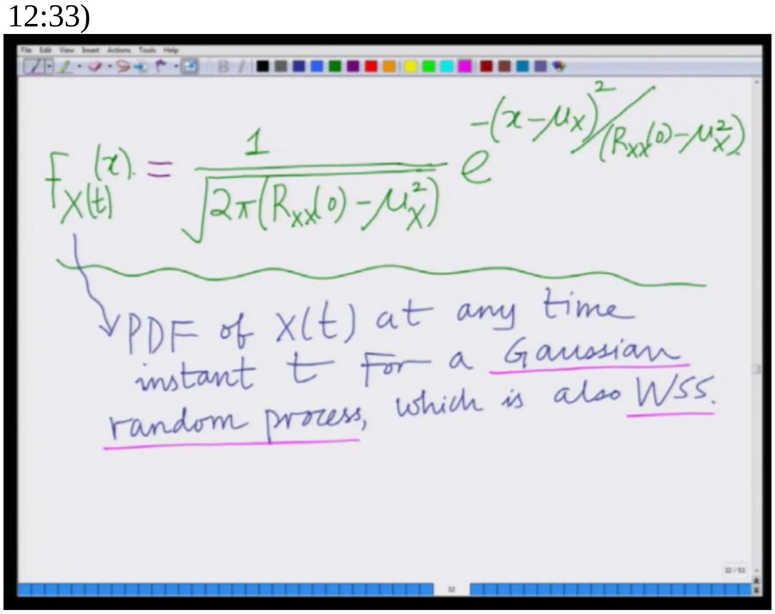

This is a probability density function for a Gaussian that is for a special class of Gaussian process which is also wide sense stationary. So it has to be both, your Gaussian random process and it also has to be wide sense stationary. And then we have this kind of distribution. In fact there will be a factor of 2. Because it is 2 times Sigma square, there will be a factor of 2.

All right? So this shows the probability density function of the wide sense stationary Gaussian random process X(t) at any time instant t. Now let us consider another special kind of a special class of random process which is known as a wide random process with is also frequently termed as white noise. Although in general, any random process can be white, typically it is the noise random process that is the noise in communication systems and wireless communication systems, etc which is white. So this is also typically termed as white noise.

So we have white noise. But it can be, strictly speaking, it should be white random process.

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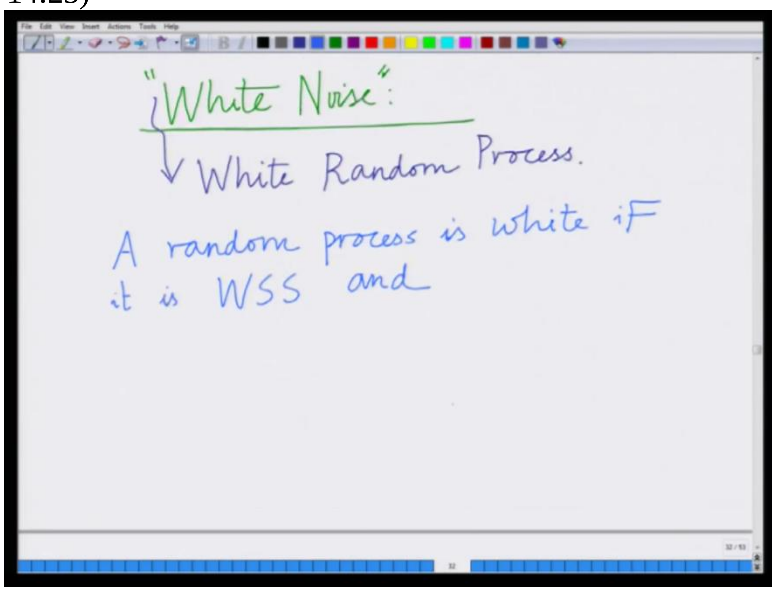

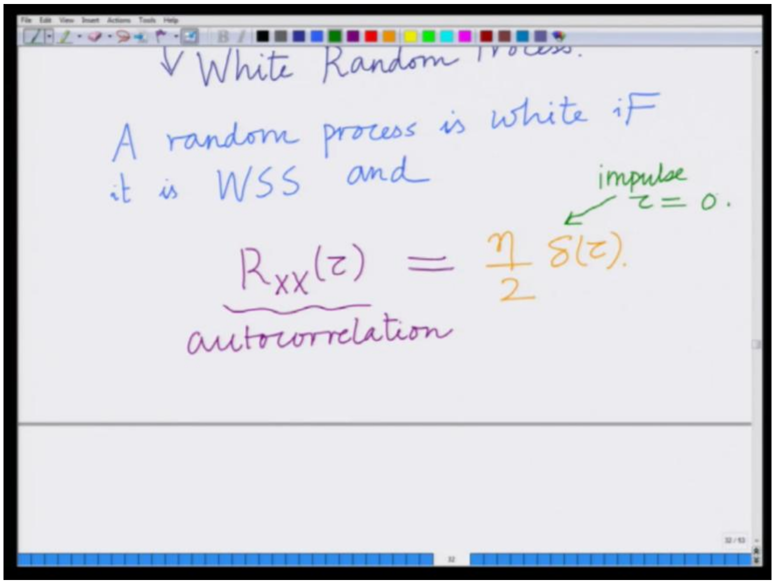

So what is a white noise? So the other thing that we want to talk about is white noise which is also basically any white random process. What is a wide random process? A random process is white if it is wide sense stationary and the autocorrelation function that is $R_{XX}(\tau)$ is such that for a white random process, it is simply equal to –

$$R_{XX}(\tau) = \frac{\eta}{2}\delta(\tau)$$

This is the impulse at τ is equal to 0. So we have our autocorrelation $R_{XX}(\tau)$ in addition for a white random process, when is the random process white or when we call a random process as white noise? If the random process is 1st WSS, that is it is wide sense stationary and moreover, the autocorrelation function $R_{XX}(\tau) = \frac{\eta}{2}\delta(\tau)$. That is which basically an impulse function. That is an impulse at $\tau = 0$.

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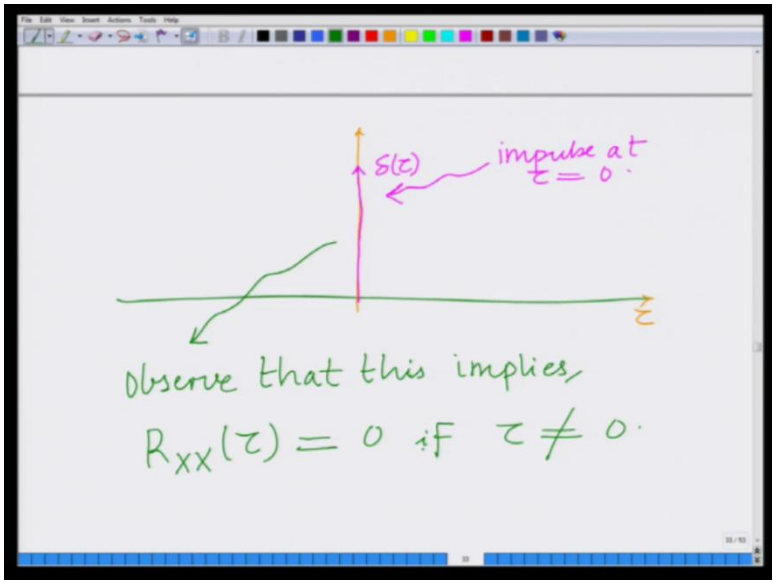

Therefore, I plot it. I plot this autocorrelation function. The autocorrelation function is this impulse which is $\delta(\tau)$. So this is basically your impulse at $\tau = 0$ and this is 0 everywhere else and observe that this implies $R_{XX}(\tau) = 0$ if $\tau \neq 0$. That is $R_{XX}(\tau)$ is an impulse at $\tau = 0$ which implies it is 0 everywhere else. Therefore, $R_{XX}(\tau) = 0$ if $\tau \neq 0$.

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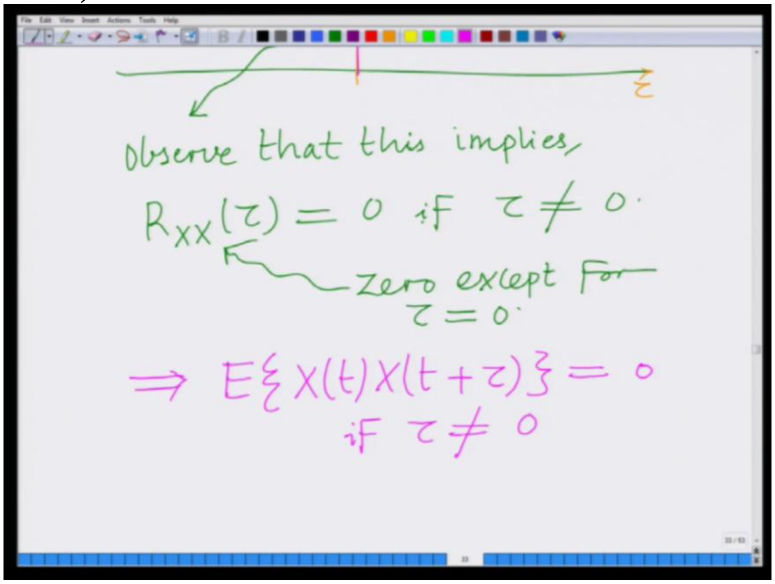

And what does that mean? That means that basically except that is $R_{XX}(\tau) = 0$ that is it is 0 except for $\tau = 0$ which implies, if $\tau \neq 0$, then –

$$E\{X(t)\, X(t+\tau)\} = 0$$

That is for any nonzero shift τ, the correlation between $X(t)$ and $X(t+\tau)$ is 0. So what does this mean?

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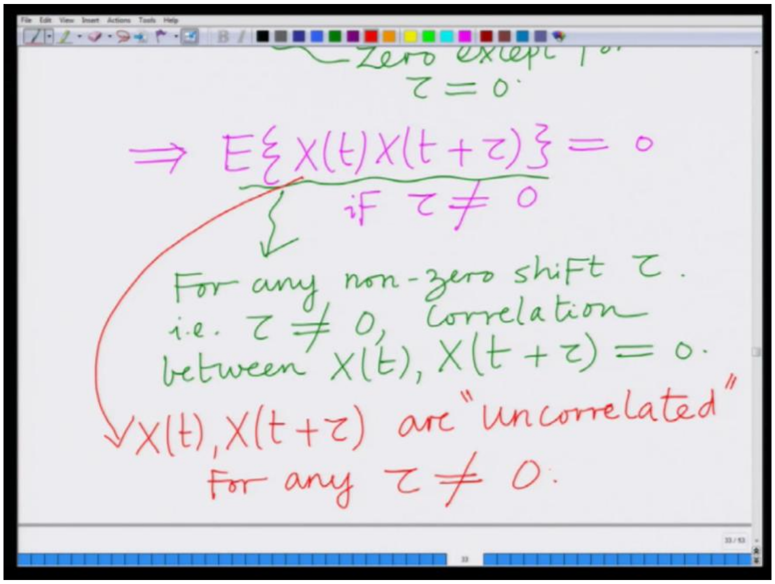

This means that for any nonzero shift τ that is $\tau \neq 0$, the correlation between $X(t)$ and $X(t+\tau)$ is equal to 0. Which also means that $X(t)$ and $X(t+\tau)$ are uncorrelated, which means there is a correlation between X(t) and $X(t+\tau)$ which is 0 for any non-zero shift, τ which means X(t) and X(t plus Tao) are uncorrelated when Tao is not equal to 0. That is X(t1) and X(t2) at 2 different times, t1 and t2 are uncorrelated. Now what is the power spectral density of this white noise?

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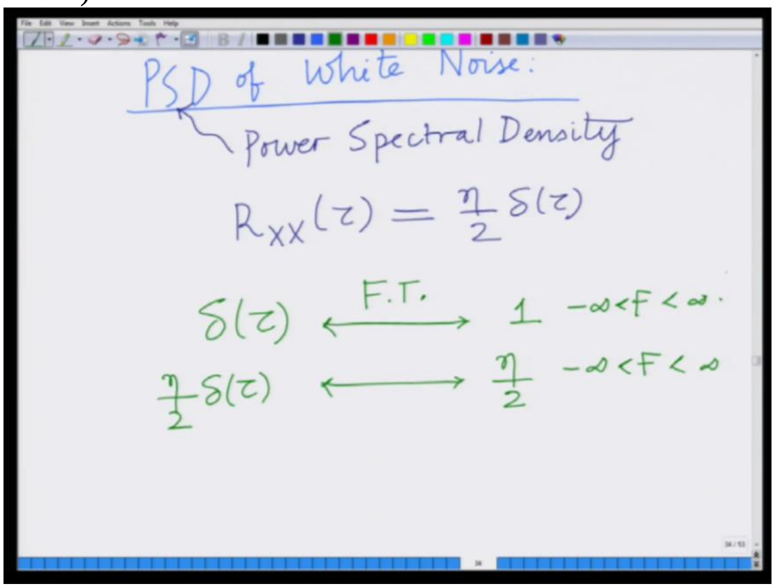

Let us look at the power spectral density of the white noise. PSD is your power spectral density. Remember, power spectral density is nothing but the Fourier transform of autocorrelation. Autocorrelation is $\frac{\eta}{2}\delta(\tau)$ and know if I look at the Fourier transform of $\delta(\tau)$, we all know the Fourier transform.

If I look at FT, the Fourier transform of $\delta(\tau)$, what is that? That is equal to unity over the entire frequency range.

$$\text{FT of } \delta(\tau) \rightarrow 1 \qquad -\infty < f < \infty$$

$$\text{FT of } \frac{\eta}{2}\delta(\tau) \rightarrow \frac{\eta}{2} \qquad -\infty < f < \infty$$

Therefore the power spectral density of white noise which corresponds to the Fourier transform of the autocorrelation is simply $\frac{\eta}{2}$.

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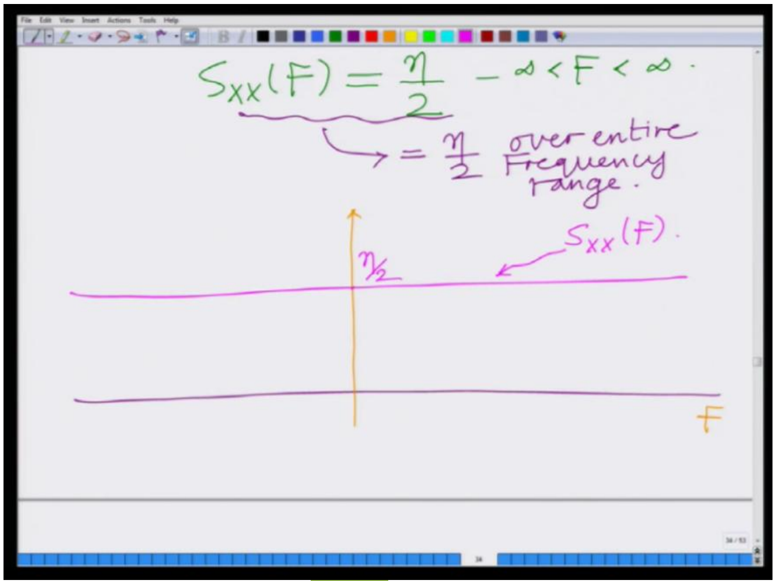

So if I look at the power spectral density $S_{XX}(f)$, what is this? This is nothing but your $R_{XX}(t)$. And $\frac{\eta}{2}$ is your $S_{XX}(f)$. So,

$$S_{XX}(f) = \frac{\eta}{2} \qquad -\infty < f < \infty$$

Remember, this is not $S_{XX}(f)$ for any general random process but for a white noise random process and why is this called white noise because remember what is white noise?

What is white light? White light is something which has uniform power distribution. That is, which is uniformly spread over an entire frequency range that is, it has all the colour components corresponding to all the frequencies. Therefore you look at the spectrum of white light, spectrum of white light has the power over the entire frequency range. Therefore white noise similarly has uniform power distribution over the entire frequency range.

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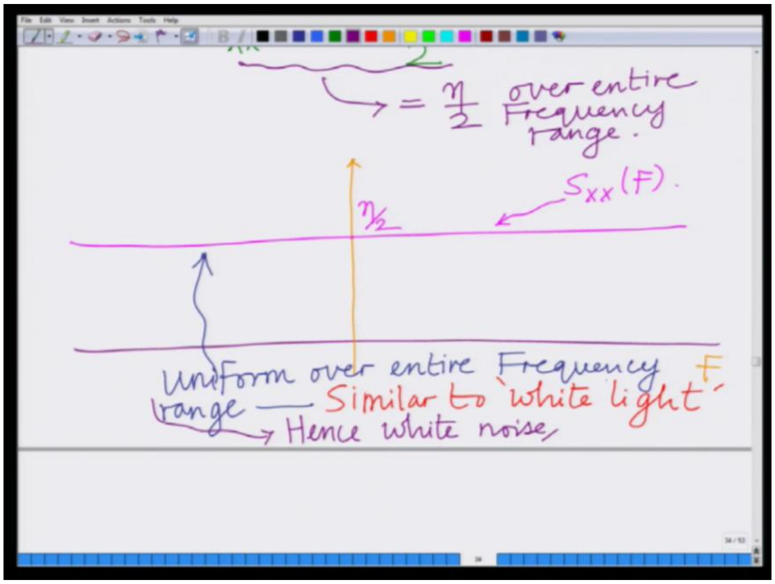

So this is uniform over the entire frequency range. And hence, that is similar to white light. Hence this is white noise. Therefore its power spectral density is uniform over the entire frequency range. So we have looked at 2 special kind of random processes. One is the Gaussian random process and the other is the white random process. Now if a random process or if a noise random process is both Gaussian and white, it is known as a white Gaussian random process. So that is very simple and now we are looking at a very special kind of a random process.

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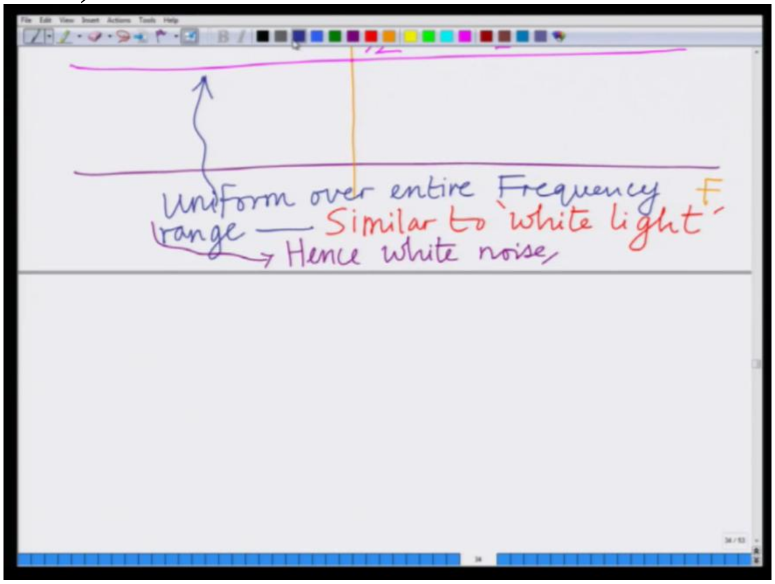

We are looking at white Gaussian noise which basically implies random process which is both Gaussian and white. This is known as a white Gaussian random process or typically this is also white Gaussian noise. So a random process which is both Gaussian and a white random process.

Remember naturally since it is a white random process, we expect it to be a wide sense stationary random process.

So which is both Gaussian and white such a random process is known as white Gaussian noise.
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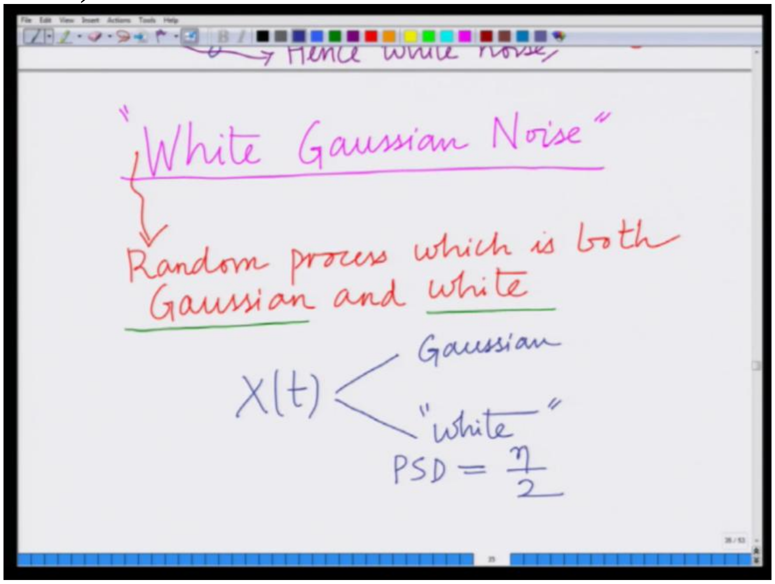

That is any random process X(t) which is both, you have to have your Gaussian property. That means, the joint PDF at time instants t1, t2, tn is Gaussian and it is also white. White meaning, the power spectral density is equal to $\frac{\eta}{2}$. Such a random process which is both white and Gaussian is known as white Gaussian noise.

And this is especially important in communication systems because we consider additive white Gaussian noise. So let us demonstrate a simple application or an example of this in the context of communication systems.

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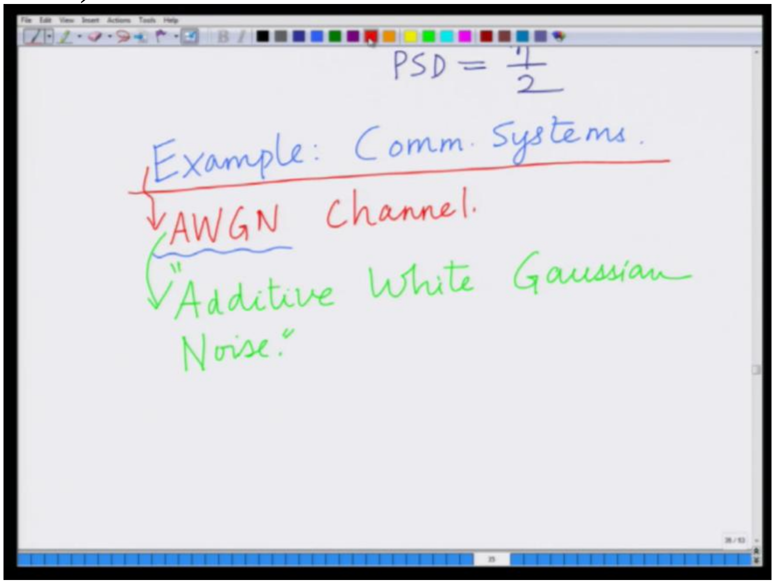

Let us look at a typical communication system. We consider in AWGN. So let us look at an example. Example in the context of communication systems. We consider what is known as an AWGN channel where AWGN stands for additive white Gaussian noise.

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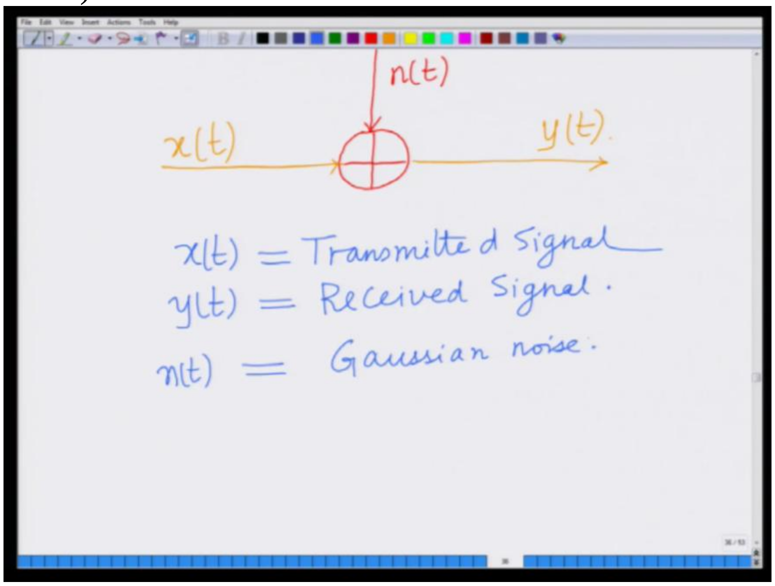

That is basically, if I look at my channel, so let's say I have a transmitted single X(t), so I have a channel or I have my receiver which is adding the noise N(t), I have X(t) which is my transmitted signal and I have my received signal Y(t). So what is this? X(t) is equal to the transmitted signal. What is my Y(t)? Y(t) is equal to my received signal and N(t)is your Gaussian noise.

Further, look at this schematic. We are saying, the noise is adding to the signal, X(t). That is, the received signal is X(t) + N(t). Therefore this is additionally known as Gaussian noise, additive Gaussian noise.

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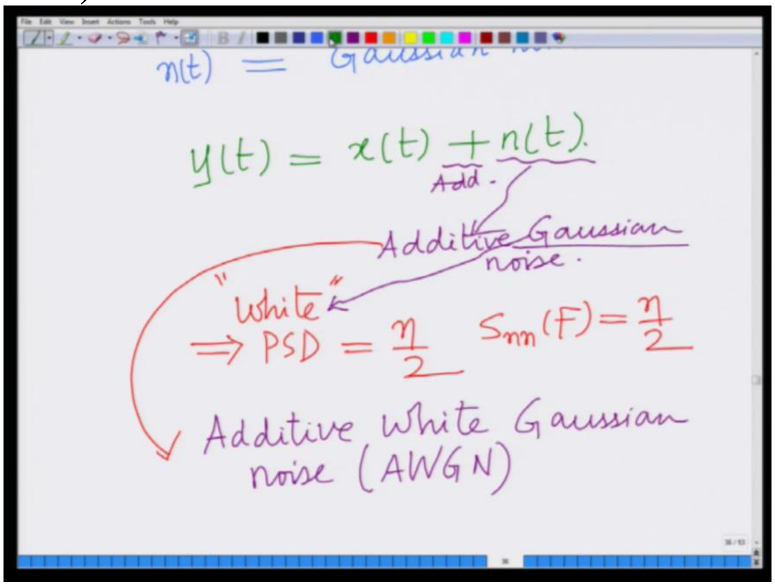

So we have the system models for this is –

$$y(t) = x(t) + n(t)$$

Therefore this is additive Gaussian noise. That is additive in the sense that N(t) adds to the signal X(t). In addition, if this noise is white, that is this Gaussian noise is white, implies its power spectral density equals $\frac{\eta}{2}$. If the power spectral density is $\frac{\eta}{2}$, that is if you look at the power spectral density, $S_{NN}(f)$ that is equal to $\frac{\eta}{2}$ over the entire frequency range.

That is this additive Gaussian noise plus the whiteness becomes your additive white Gaussian noise which is simply represented as this is one of the most common and one of the most popular models for communication channels, this is known as an AWGN. And such a channel which adds AWGN that is such a communication channel which adds additive white Gaussian noise to the signal X(t), that is known as an additive white Gaussian noise channel.

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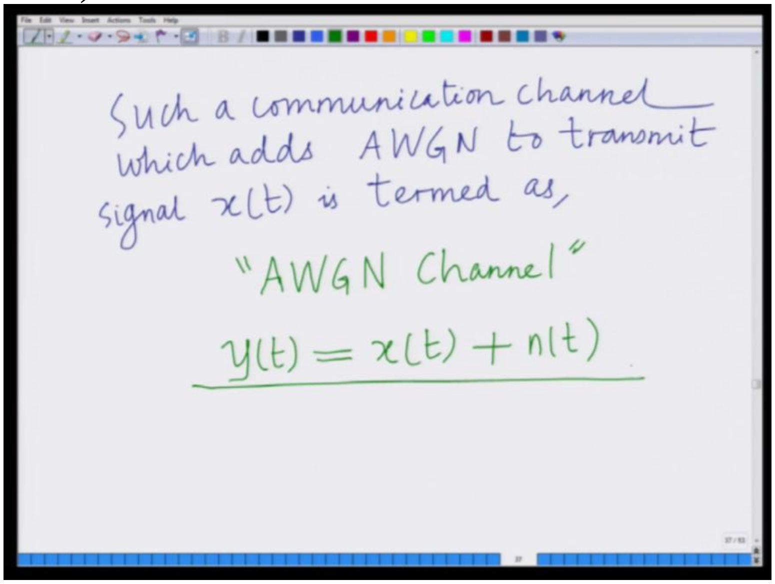

Such a communication channel which adds AWGN to transmit signal X(t) is termed as your basic AWGN and this is the most popular model. That is your Y(t), received signal equals transmitted message signal plus the noise signal.

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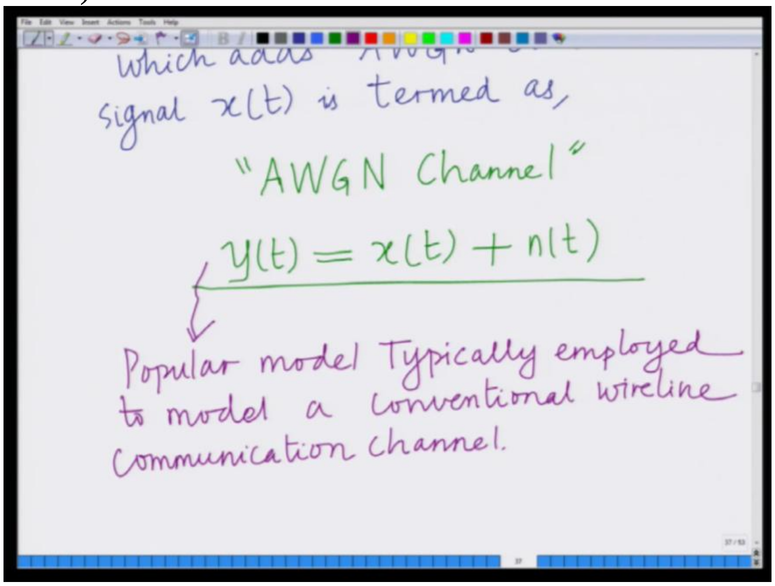

This is the most popular model, a popular model typically employed to model a conventional wireline that is in which there is a wire, a conventional wireline communication system. So what we're saying is this model,

$$y(t) = x(t) + n(t)$$

which is additive white Gaussian noise channel model. It is typically employed to model a conventional wireline communication channel. What is a wireline communication channel? A communication system in which there is a wire for a transmission medium, as a transmission medium between the transistor and receiver.

What is the reason that this is employed only to model wireline channels? The reason is because this lacks the fading. The additional impact of what is known as the fading coefficient which arises in a wireless channel.

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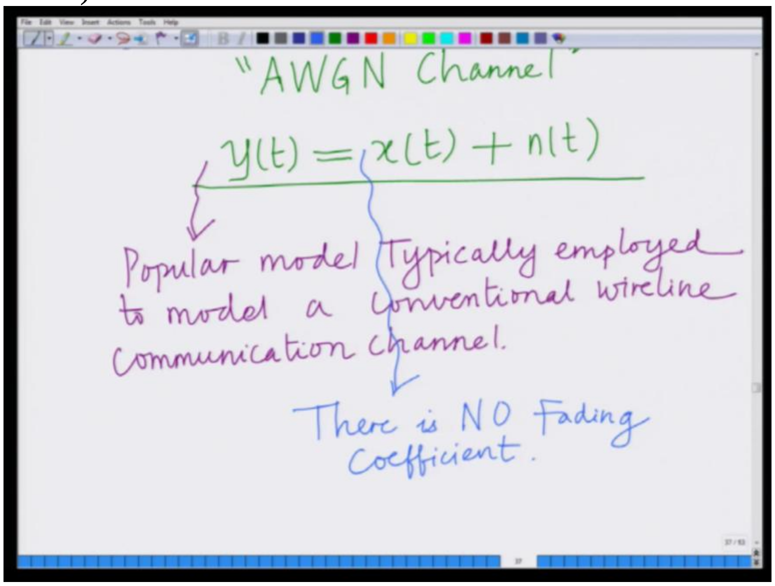

This is only used to model a wireline channel because since, observe here, there is no reason being, there is no fading coefficient. The reason being, there is in an AWGN, there is no fading coefficient. Correct? If additionally, we incorporate a fading coefficient H, this becomes a wireless channel. In addition to the additive white Gaussian noise, if we incorporate the effect of reading coefficient H, this becomes a model of a wireless channel.

Therefore, additive white Gaussian noise that is the noise process at the receiver which is additive that adds to the signal, which is Gaussian in nature and also which is white, that is its power spectral density is uniform $\frac{\eta}{2}$, is a very important and a very special kind of random process which is used extremely commonly which is one of the most standard noise models used in typical communication systems.

That is both digital communication systems as well as wireless communication systems which incorporate fading.

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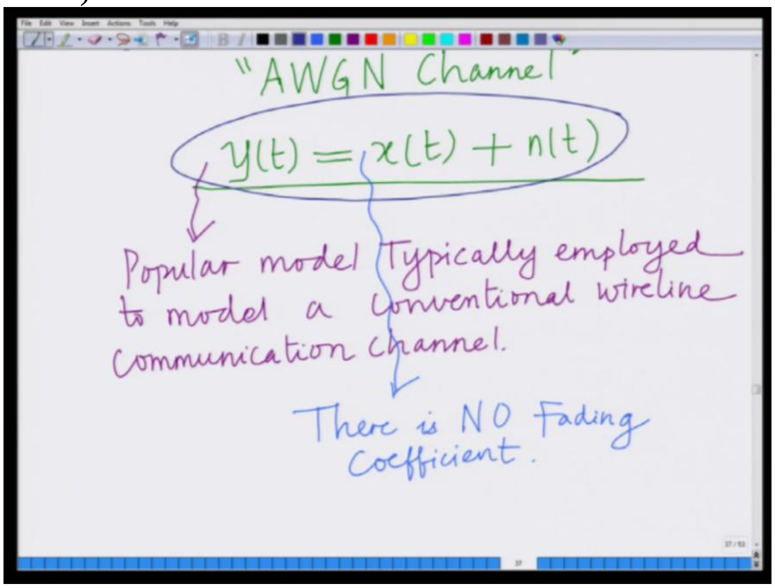

What I am saying is this is the typical model for a conventional digital communication system. That is –

$$y(t) = x(t) + n(t)$$

So what we have seen? In this module, we have seen several aspects. We have seen special kinds of random processes. That is $X(t)$ is a Gaussian random process if the joint statistics are Gaussian that is if you look at the multivariate density of $X(t1)$, $X(t2)$, $X(tn)$ at times since t1, t2, tn for any N and any t1, t2, tn, that has to be Gaussian. Correct?

And a white random process which means its autocorrelation is basically an impulse, an impulse at $\tau = 0$, which means the power spectral density is uniform over the entire frequency range and if the random process typically noise is both Gaussian and white, it is known as white Gaussian noise and additive white Gaussian noise channel is a channel which adds this white Gaussian noise to a single $X(t)$. That is used to model, that is very frequently used to model wireless communication systems and therefore it is very important in the context of communication. So we will stop this module over here. Thank you very much.