Probability and Random Variables/Processes for Wireless Communication
Professor Aditya K. Jagannatham
Department of Electrical Engineering
Indian Institute of Technology Kanpur
Module 4
Lecture No 23
Gaussian Process Through LTI System – Example:WGN Through RC Low Pass Filter

Hello, welcome to another module in this massive open online course on probability and random variables for Wireless Communication. So in the previous modules, we have been looking at wide sense stationary random process and several interesting properties of wide sense stationary random process. So we have seen if X(t) is a wide sense stationary random process so let us consider X(t)…

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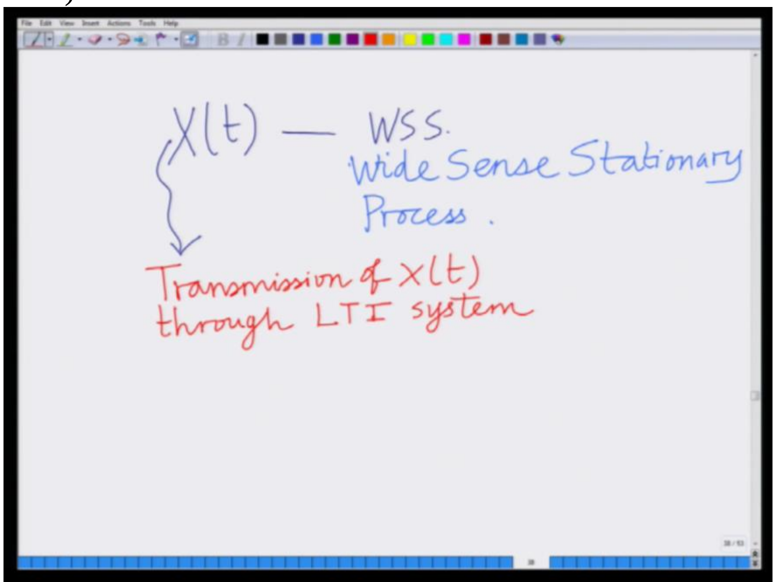

…which is a WSS that is wide sense stationary random process. If X(t) is your wide sense stationary random process, then we have seen, what happens in X(t) is transmitted that is transmission of X(t) through a linear or basically through a linear time invariant system.

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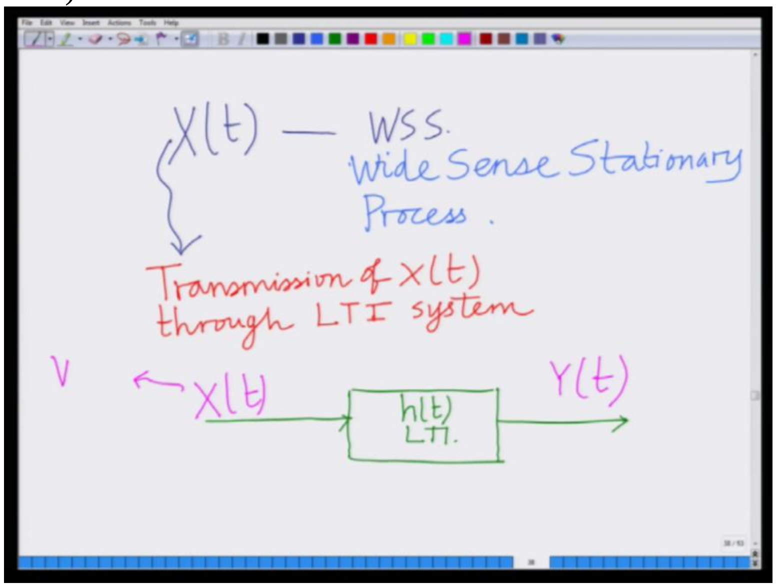

So let's say we have a linear time invariant system that is represented by this impulse response, h(t). So I have an input X(t) which is the wide sense stationary random process. Output, Y(t).

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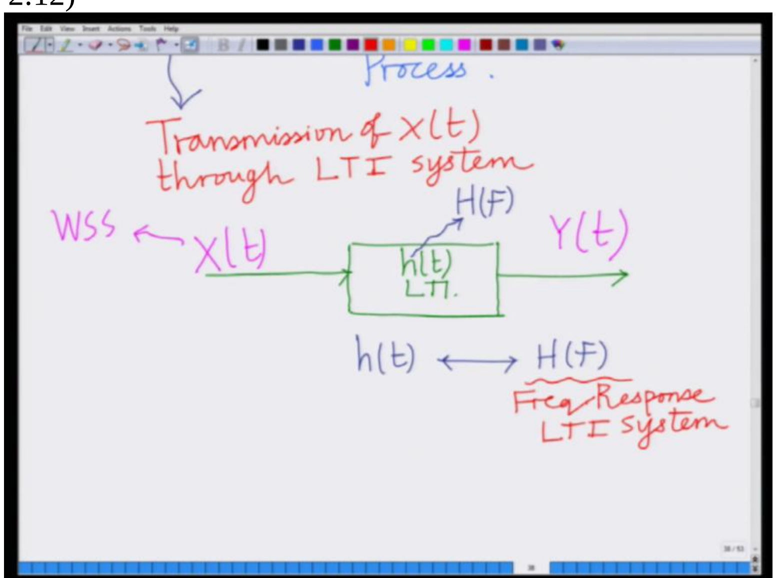

This h(t) is an LTI system. Also, we said that the Fourier transform of h(t) is H(f). This is also known as the frequency response of the LTI system. And when this wide sense stationary random process X(t) is transmitted or is passed through this LTI system with impulse response h(t), we said that the output random process Y(t) is also wide sense stationary. The interesting fact is when a wide sense stationary random process is an input to an LTI system, the resulting output is a random process and more interestingly, the resulting output is also a wide sense stationary random process.

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So this Y(t) over here is also a WSS, that is your wide sense stationary random process. Further we had a very interesting relation for the power spectral density. That is if the input power spectral density is $S_{XX}(f)$, the output power spectral density of Y is $S_{YY}(f)$, then we had –

$$S_{YY}(f) = S_{XX}(f).\ |H(f)|^2$$

This is a very interesting result we have shown which relates the input power spectral density to the output power spectral density. That is the output power spectral density $S_{YY}(f)$ is the input power spectral density, $S_{XX}(f)$ times $|H(f)|^2$ where H(f) is the frequency response of the system. Further we have also looked at another interesting aspect. That is, we looked at a special kind of a random process that is a Gaussian random process.

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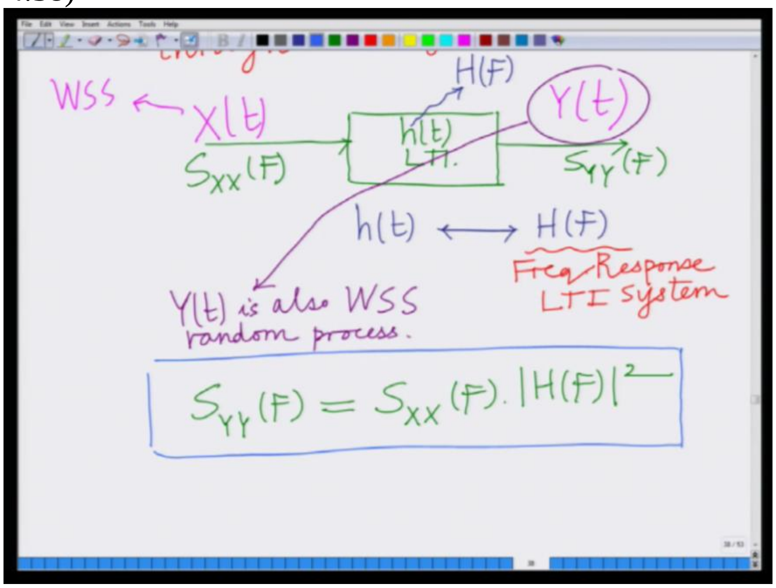

So we have also looked at what is known as a Gaussian random process. And what is a Gaussian random process? X(t) is Gaussian if -

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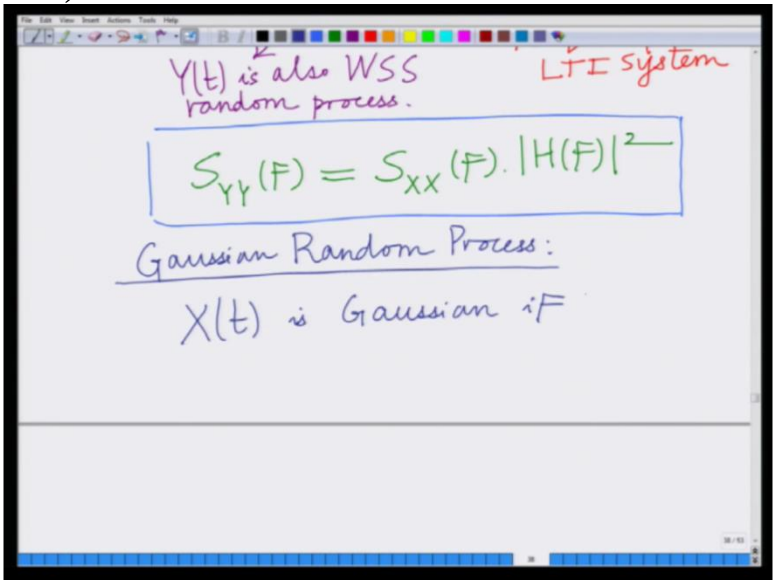

… If we look at the joint distribution, $F_{X(t1)X(t2)\ldots X(tn)}(x_1,x_2,\ldots,x_n)$ of X of T1, X of T2, that is random variables $x_1,x_2,\ldots,x_n$ that is these N random variables if we look at this joint distribution at N time instant. If this PDF is jointly Gaussian. Correct? If the joint PDF of the random process X(t) at instant $t_1,t_2,\ldots,t_n$. That is joint PDF corresponding to X(t1), X(t2), X(tn). If this is jointly Gaussian for all possible values of N and also at all times, $t_1,t_2,\ldots,t_n$. That is, if this is jointly Gaussian for all N.

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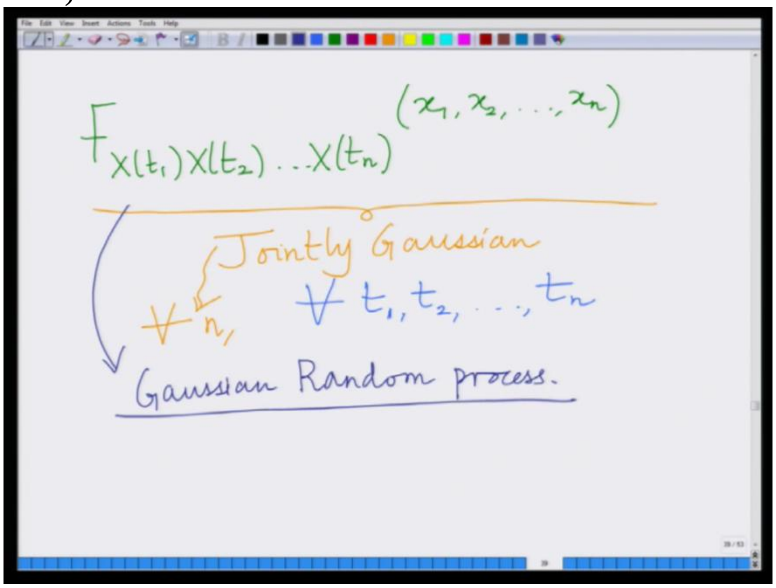

For all number of time instants N and for all X(t1), X(t2),, or for all time instants, $t_1, t_2, \ldots, t_n$, then this is a Gaussian. Such a random process $X(t)$ is known as a Gaussian, such a random process $X(t)$ whose joint statistics at any set of time instants, $t_1, t_2, \ldots, t_n$ is Gaussian. And in the context of Gaussian random variables, we had also seen a very interesting property that is a linear transformation or a linear combination of Gaussian random variables leads to another Gaussian random variable. So, we had seen that for a Gaussian random variable what have we seen? We have seen previously…

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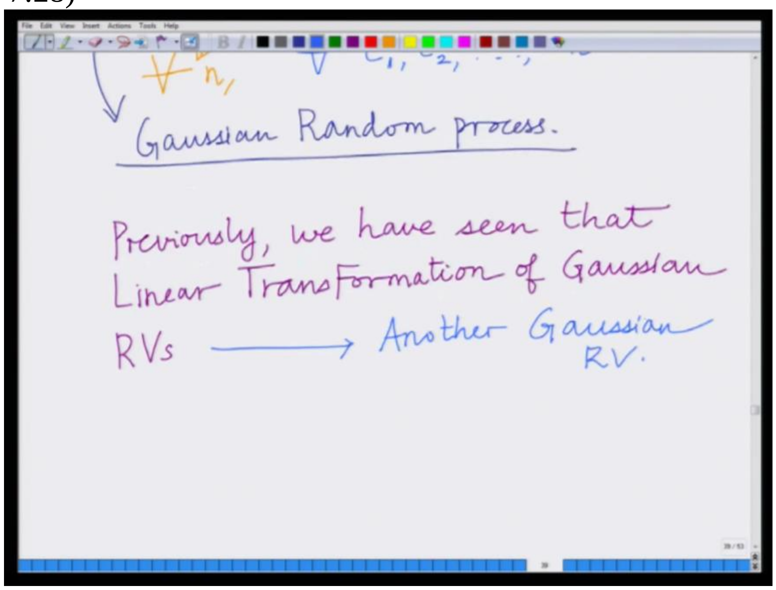

… Previously we have seen that a linear transformation of Gaussian RVs, a linear translation of Gaussian random variables leads to another Gaussian random variable. This we had seen not in the context of a random process but we had seen this in the context of random variables.

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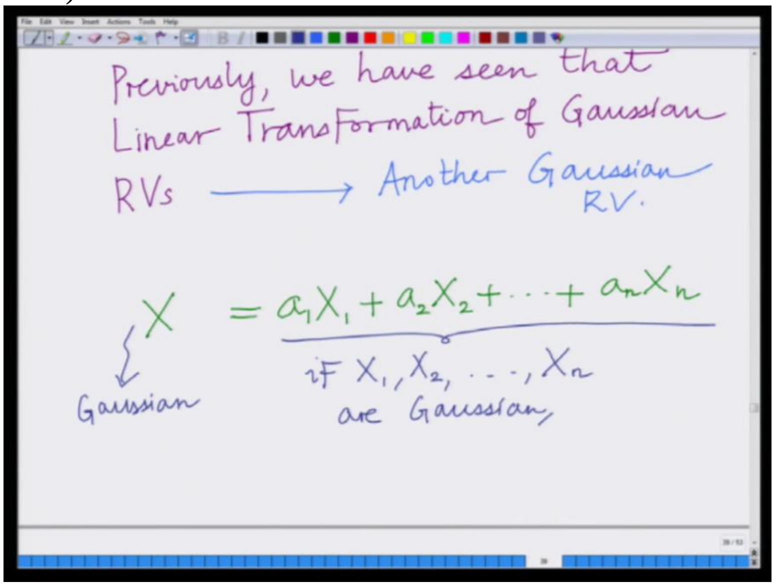

That is, we said that if X1, X2, Xn are Gaussian random variables and the random variable X is –

$$X = a_1X_1 + a_2X_2 + \dots + a_nX_n$$

is Gaussian if X1, X2, so on up to Xn are Gaussian. That is if these random variables X1, X2, up to Xn if these are Gaussian, then the linear combination, that is $a_1X_1 + a_2X_2 + \dots + a_nX_n$ is another Gaussian random variable that is X. Now, what is the applicability of this? What is the interesting implication of this in the context of random process? It has a very interesting implication in the context of random process.

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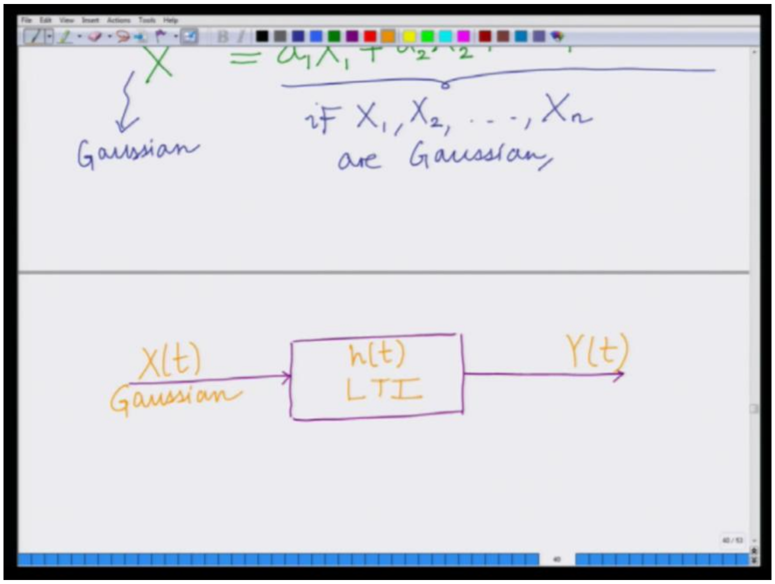

If we consider a linear system, what is the implication of this in the context of random processes and specially a Gaussian random process? What we have? Let's say we have a linear system again characterized by impulse response H(t) and the moment we say it has an impulse response, it means that if it is characterized by an impulse response, it means that it is an LTI system. Then, if we input a Gaussian random process to it and we look at the output random process Y(t) that is to an LTI system, linear time invariant system, if we input a Gaussian random process, then the output, Y(t) naturally is a Gaussian random process. Because the linear time invariant system is a linear system. So we are transmitting or we are passing this Gaussian random process that is linearly transforming this input Gaussian random process. Therefore, the output random process is also Gaussian which is a linear transformation of the input random process.

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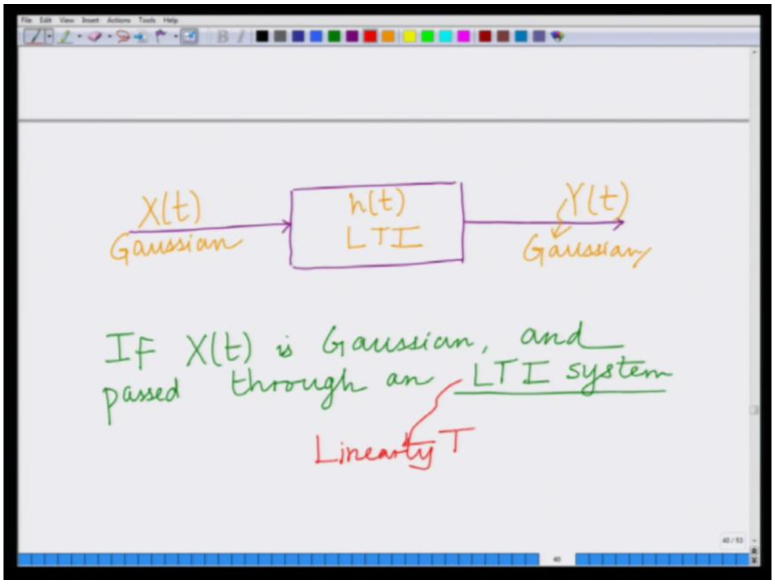

Therefore the interesting result is if you pass a Gaussian random process through an LTI system, the output is Gaussian, another Gaussian wide sense stationary random process. Both wide sense Stationarity and Gaussianity are preserved when you transmit a Gaussian random process through an LTI system. So basically this result states, if X(t) is Gaussian and passed through an LTI system, LTI system is basically one which linearly transforms, LTI system is nothing but a linear transformation. So this linearly transforms the input.

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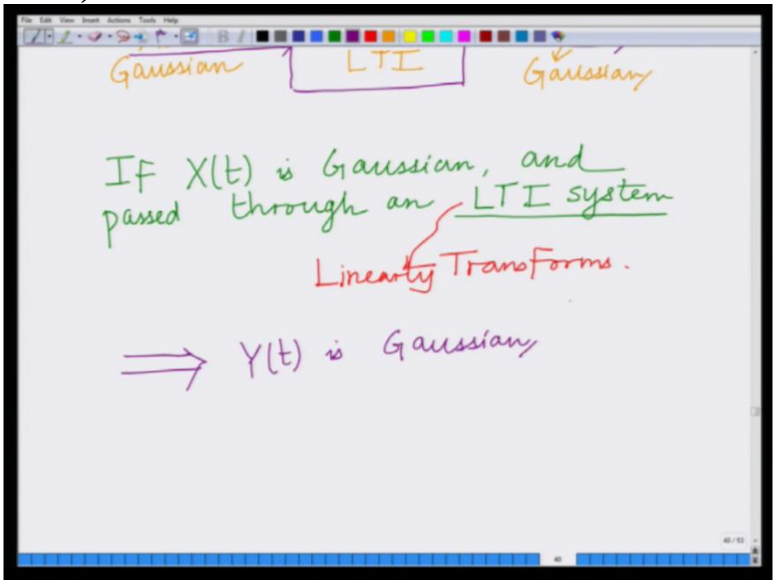

Therefore, we can say that the random process Y(t) is Gaussian. We had already seen that if the input is wide sense stationary, the output is wide sense stationary. Now we are saying, if the input is Gaussian, then the output is also Gaussian because Gaussianity of the random variables or the Gaussianity is preserved under linear transformation. Therefore, if the input to an LTI system is a Gaussian random process, then the output to the LTI system is also a Gaussian random process. And this is a very interesting result. And we have also seen other special kind of random process which is white noise.

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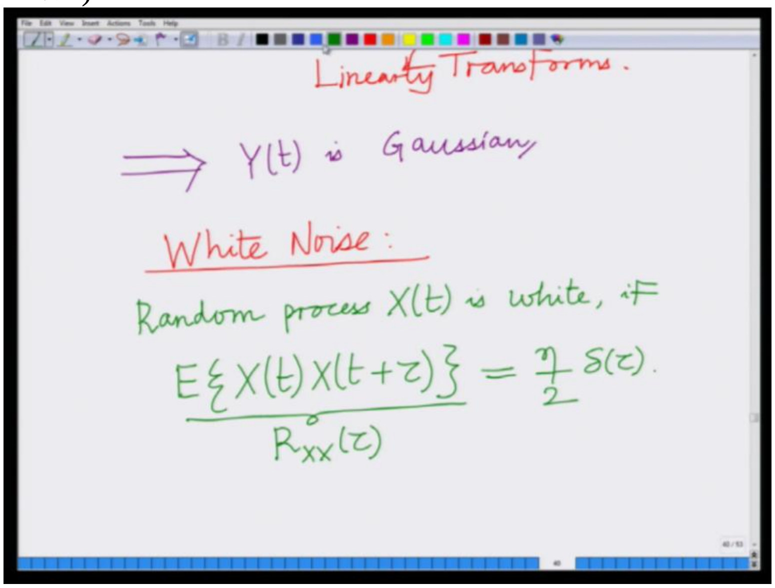

So we had seen another special kind of random process which is a very special kind of a random process which is white noise. And what is white noise? White noise is a random process, X(t) is white if it is wide sense stationary and the autocorrelation, E{X(t) X(t+ τ) }, remember this is your autocorrelation function, $R_{XX}(\tau)$, this has to be equal to –

$$E\{X(t)\ X(t+\tau)\} = R_{XX}(\tau) = \frac{\eta}{2}\delta(\tau)$$

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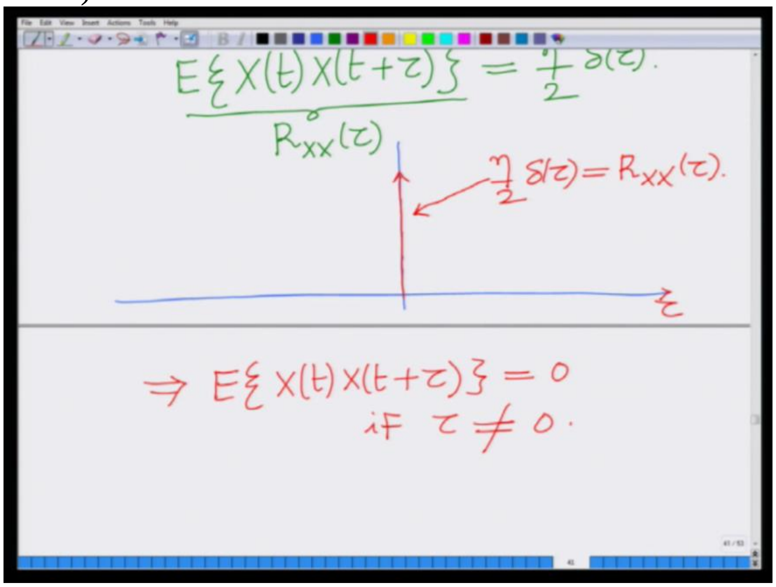

That is, if you look at the autocorrelation function, then the autocorrelation function of white noise is basically your impulse. That is the autocorrelation of white noise is basically, this is the impulse $\frac{\eta}{2}\delta(\tau)$ as a function of τ and this is your autocorrelation $R_{XX}(\tau)$. And this also means -

$$E\{X(t)\ X(t+\tau)\} = 0 \quad \text{if } \tau \neq 0$$

That is if you look at the covariance or the correlation of the random process X(t) at two-time instants t and some nonzero timeshift t+ τ, these two, X(t) and X(t+ τ) are uncorrelated. And also, if we look at the power spectral density in this white noise, it has a very interesting property.

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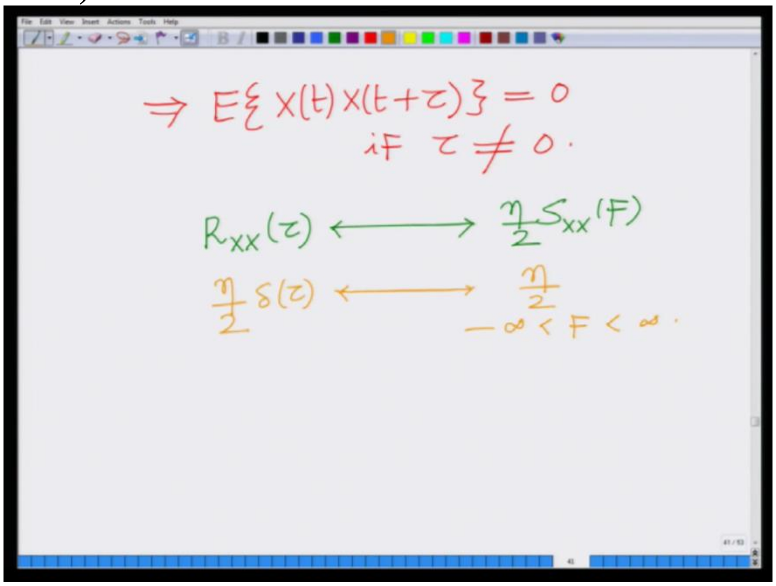

That is we have, if you look, what is the power spectral density? That is, we have the autocorrelation. We take the Fourier transform of this. And that gives us the power spectral density. And now we have for a white noise, the autocorrelation is the impulse. Therefore, the power spectral density is simply $\frac{\eta}{2}$ for $-\infty < f < \infty$. And we had said, we derived this as follows. For a simple impulse, the power spectral density is uniform 1 over the entire frequency band. Therefore, for $\frac{\eta}{2}\delta(\tau)$, the power spectral density is a simply $\frac{\eta}{2}$, over the entire frequency band.

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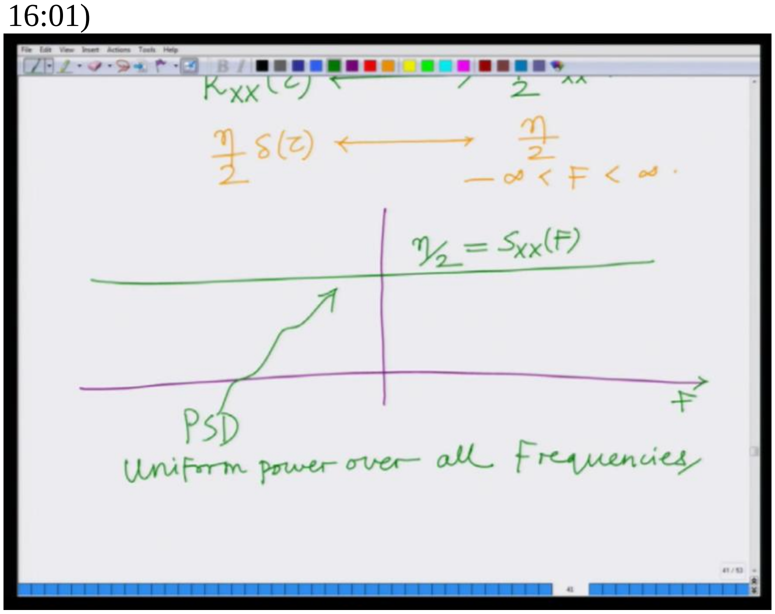

So if I look at this power spectral density, it has a very interesting property. This power spectral density looks simply, it looks like this. It is flat over the entire frequency band. That is if this is your frequency band, this is your PSD which has uniform power over all frequencies. That is it has uniform power over all frequencies which is similar to white light which has all the components or uniform power over all the components of the frequency spectrum. Therefore, this is known as white noise which has a white power spectral density. This is the characteristic feature of white noise.

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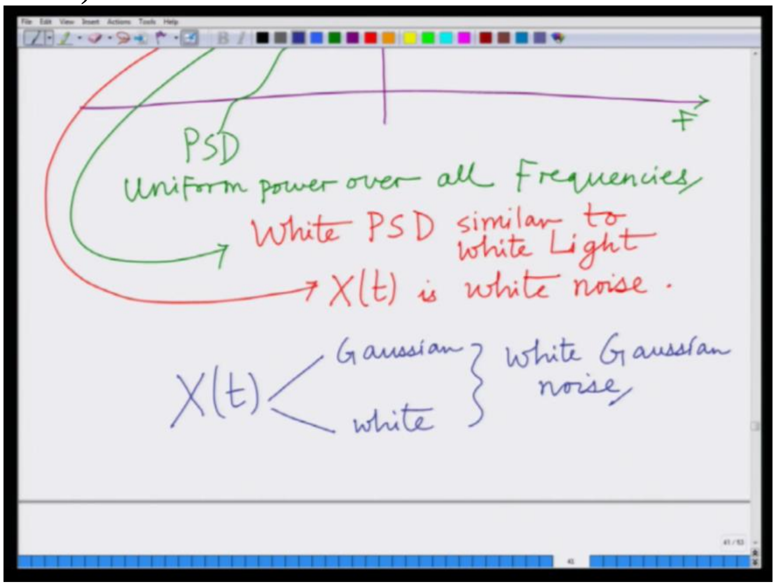

So this is the characteristic feature of white noise. It has a white power spectral density which is similar to white light or it has a white spectrum. Therefore such a noise X(t) is known as, it is a white random process which is also frequently associated with noise therefore it is known as white noise. Now X(t) additionally something very interesting which occurs very frequently in communication systems, if X(t) is both Gaussian and white, we said this is your white Gaussian noise which is very important in the context of communication system. That is if the random process X(t) is both white and Gaussian, then we say this is white Gaussian noise which is frequently, the assumption that is used at the receiver in any communication system, digital communication system, also typically in an analog communication system and a wireless communication system, we use the assumption of additive white Gaussian noise.

And that is what has been explained in the previous example. That is, the noise is additive. It adds to the signal and it is white, Gaussian in nature. Such noise is known as additive white Gaussian noise. And the channel which has this additive white Gaussian noise feature is known as an AWGN channel which is frequently employed to model a digital communication system or a wireline communication system.

That is what we had seen in the previous module. Let us now look at another interesting example, let us now look at an interesting example relating to the processing at the receiver. Let us observe what happens when we employ, when we process the white noise at the receiver. Or at the receiver, we process or we pass a white noise through a filter, we filter this white noise at the receiver. Or in other words, this can also be said as, when we filter the signal plus noise, what happens to the noise at the output of this filter?

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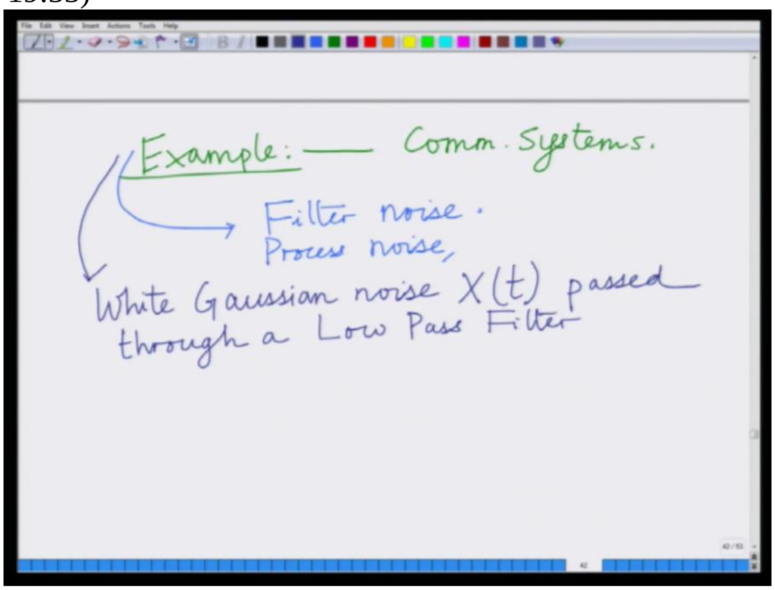

So let us look at a typical example from a communication. So let us look at our typical example in the context of a communication system. That is what happens when we process white noise or let's say what happens when a white Gaussian noise which is passed through a filter or white Gaussian noise X(t) passed through a filter. Let's say, passed through a lowpass filter.

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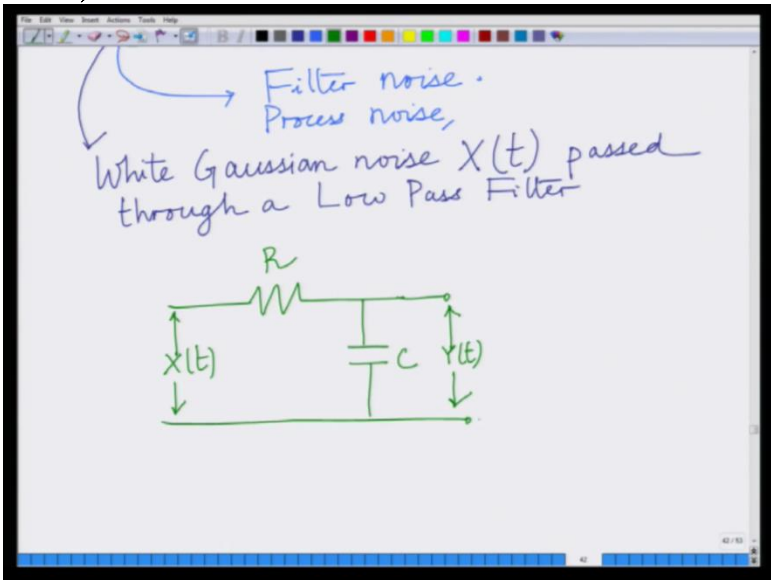

Let us consider a simple lowpass filter. Let us consider this RC lowpass filter. So the input to this lowpass filter is your random process X(t), the output is your random process Y(t). So we are saying let us consider what happens to a noise process at the receiver in a communication system when you pass it through a filter. We are assuming that this noise process X(t), which is a white Gaussian random process. Right? This is a white Gaussian random process.

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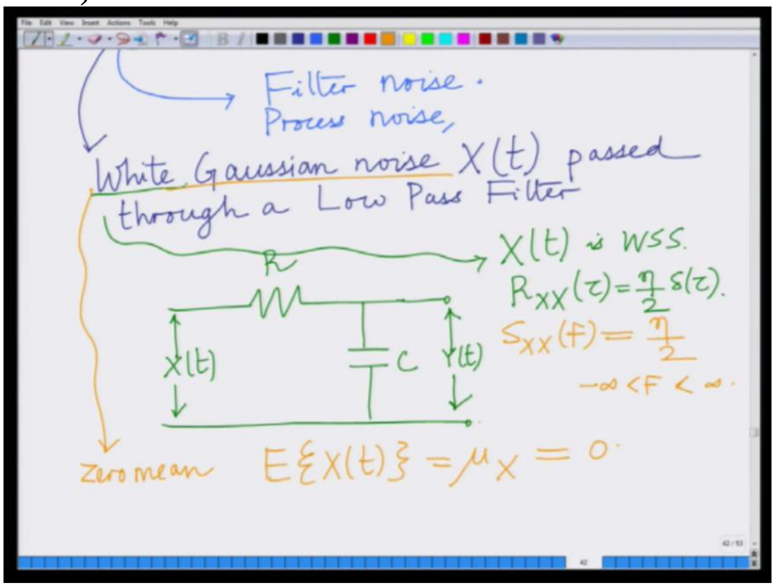

Therefore, we are also assuming that this is basically a wide sense stationary. So X(t) is wide sense stationary. Further this is white. So we are given that the power spectral density is –

$$R_{XX}(\tau) = \frac{\eta}{2}\delta(\tau)$$

$$S_{XX}(f) = \frac{\eta}{2} \text{ for } -\infty < f < \infty.$$

Further, let's also as you that this noise is 0 mean. So we are assuming, white Gaussian noise, let us also assume that this noise is 0 mean Gaussian. That implies –

$$E\{X(t)\} = \mu_X = 0$$

which means $X(t)$ is not only Gaussian but it is 0 mean. That is, $E\{X(t)\} = \mu_X = 0$ at every time instant t.

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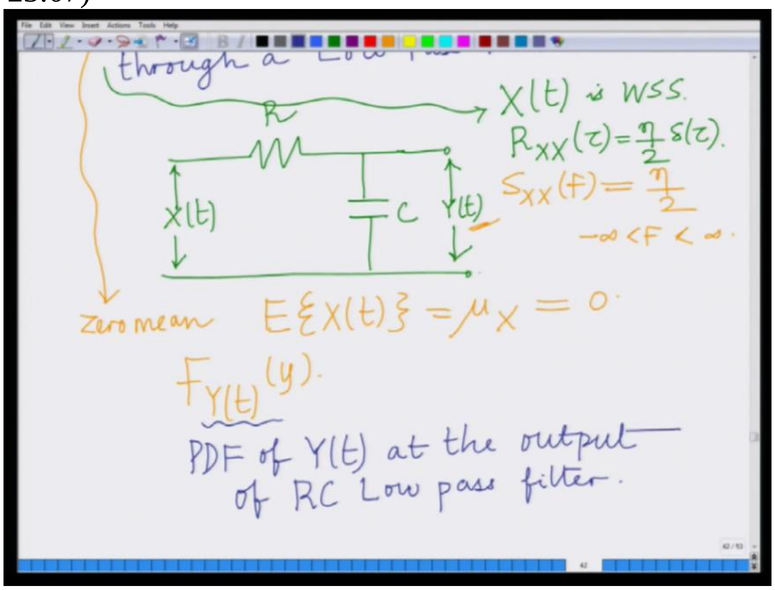

Now what we want to find in this system. So this $X(t)$ is white Gaussian noise, 0 mean white Gaussian noise with power spectral density, $\frac{\eta}{2}$, which we are passing through this RC filter and this $Y(t)$ is the output, what we want to find is we want to find the probability density function $F_{Y(t)}(y)$. So we want to find the probability density function, the PDF of $Y(t)$ at the output, that is the noise output of your RC lowpass.

That is, in communication system, we employ processing. So let's say, we are processing the signal by passing it through a low pass filter. And what happens to the noise at the output? What is the noise? That is, the input is white Gaussian, 0 mean white Gaussian noise. What is the distribution of the noise at the output? Okay?

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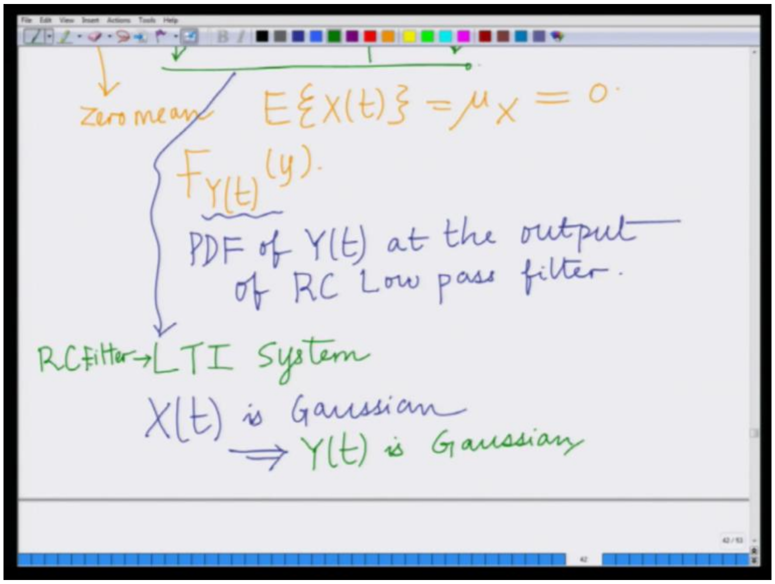

So 1st thing that we have seen? We can say is basically the RC, this low pass filter, this is a linear system, this is a linear time invariant system. So your RC filter is basically, is a classic example of an LTI system. So therefore, as we had seen earlier in this module since we are taking X(t) which is Gaussian, and passing it through an LTI system, so X(t) is Gaussian which is passed through an LTI system, which means that the output random process Y(t) is also Gaussian. The input to the RC filter is a Gaussian random process and it is an LTI system. Therefore the output random process, Y(t) is also basically it is also a Gaussian random process.

Now, let us find the power of this output random process.

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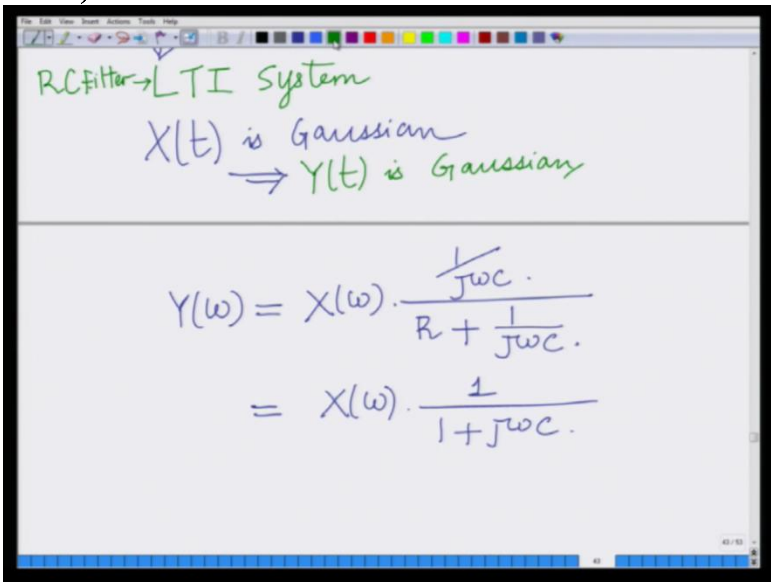

Now let us 1st find the transfer function of the system. Now most of you must be familiar with phaser analysis. So if I look at the impedance, the complex impedance of this capacitor, the complex impedance is $\frac{1}{j\omega C}$ as a function of the circular frequency ω.

Therefore,

$$Y(\omega) = X(\omega). \frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}}$$

$$= X(\omega). \frac{1}{1 + j\omega RC}$$

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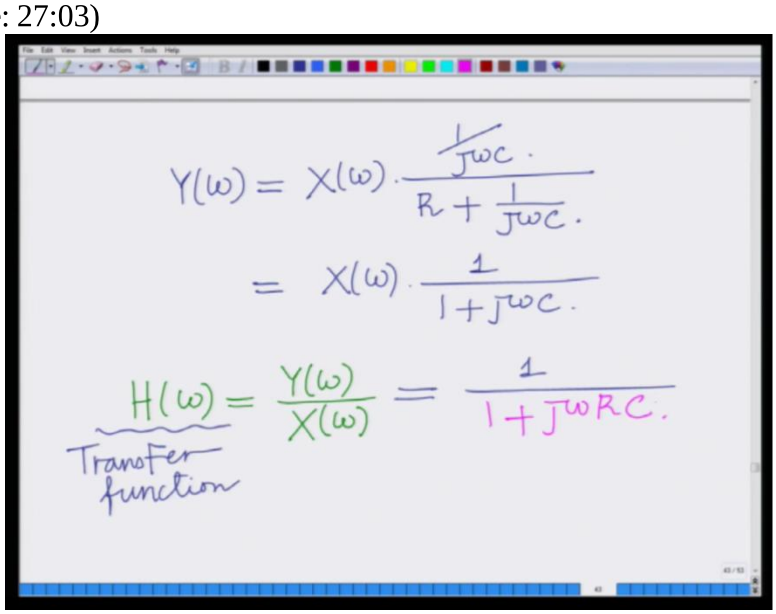

Now therefore, if a look at the transfer function. The transfer function is nothing but the ratio of the output frequency response Y(ω) to the input frequency response X(ω). What is this? This is H(ω), which is your transfer function,

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{1}{1 + j\omega RC}$$

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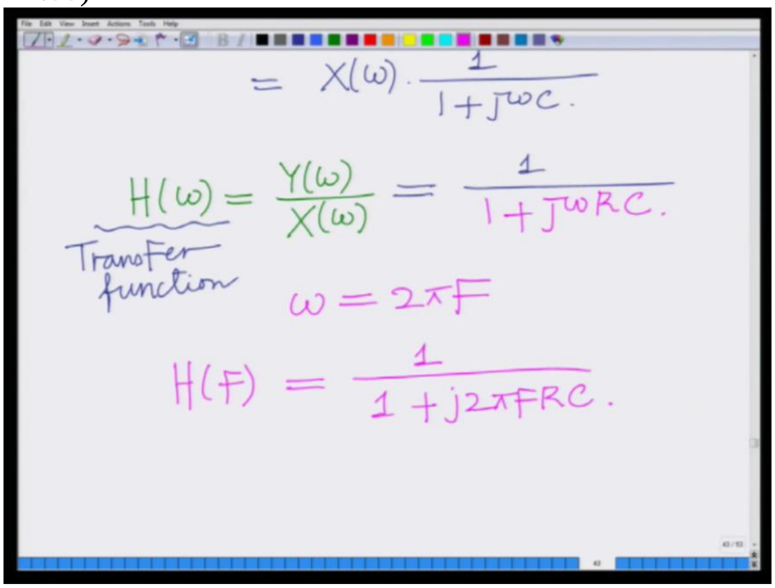

Now if I substitute ω = 2πf, that is the circular frequency ω = 2πf, then I have the transfer function in terms of f equals –

$$H(\omega) = \frac{1}{1 + j2\pi fRC}$$

R is the resistance of the circuit, C is the capacitance.

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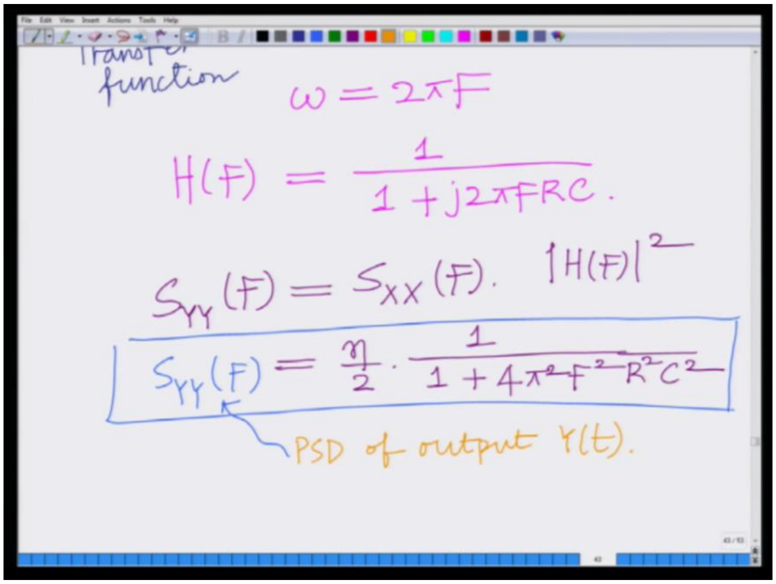

And now therefore if we look at the output power spectral density. For this LTI system, the output process Y(t) is wide sense stationary and the output power spectral density is –

$$S_{YY}(f) = S_{XX}(f)\ .\ |H(f)|^2$$

Now,

$$S_{XX}(f)= \frac{\eta}{2}$$

And therefore,

$$S_{YY}(f) = \frac{\eta}{2}\ .\ \frac{1}{1+4\pi^2 f^2 R^2 C^2}$$

This is the power spectral density of your output random process Y(t).

So we have inferred 2 things. One is, the input random process X(t) is Gaussian. Therefore the output random process, Y(t) is also a Gaussian. Further, the power spectral density of the output $S_{YY}(f) = \frac{\eta}{2}\ .\ \frac{1}{1+4\pi^2 f^2 R^2 C^2}$.

Now we can find the power of this output random process. That is simply the integral of the power spectral density from $(-\infty , \infty)$.

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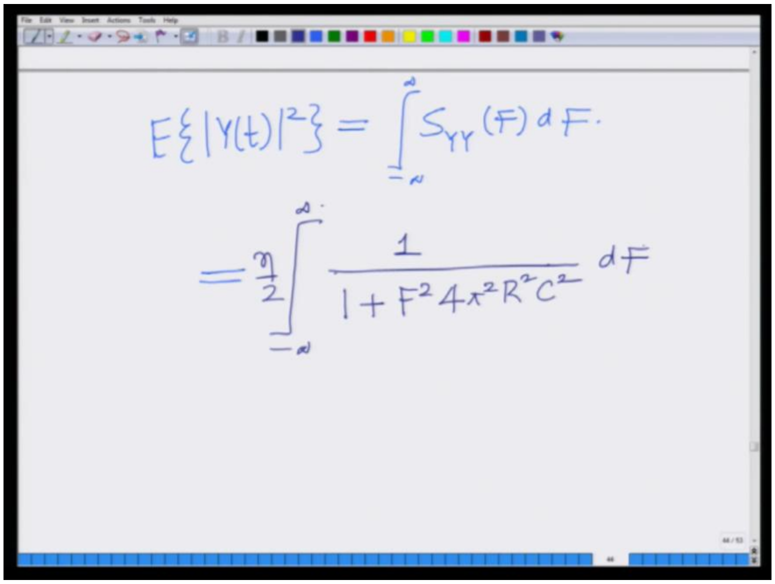

Remember there are several concepts that are coming together in this example. That is –

$$E\{\,|Y(t)|^2\,\} = \int_{-\infty}^{\infty} S_{YY}(f)\,df$$

$$= \frac{\eta}{2} \cdot \int_{-\infty}^{\infty} \frac{1}{1+4\pi^2 f^2 R^2 C^2}\, df$$

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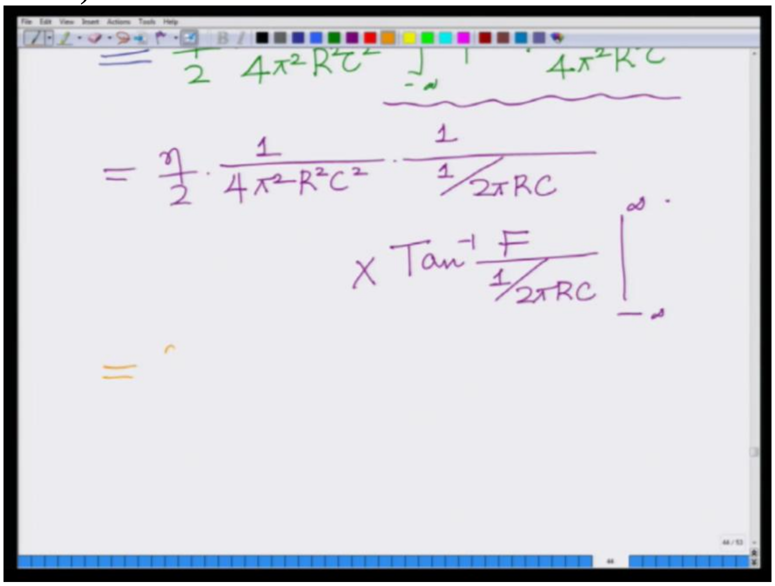

$$= \frac{\eta}{2} \cdot \frac{1}{4\pi^2 R^2 C^2} \cdot \frac{1}{\frac{1}{2\pi RC}} . \tan^{-1} \frac{f}{\frac{1}{2\pi RC}} \Big|_{-\infty}^{\infty}$$

And now if I simplify this further, what we have is…

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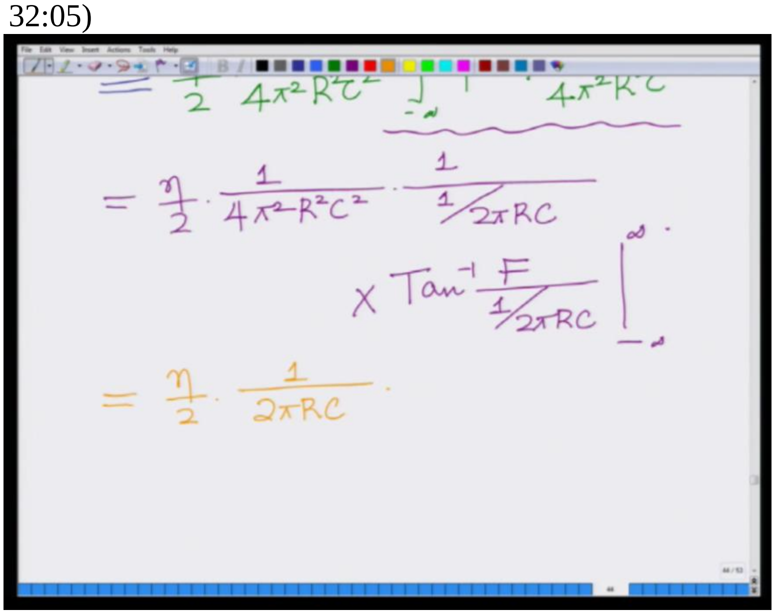

$$= \frac{\eta}{2} \cdot \frac{1}{2\pi RC} . \pi$$

$$= \frac{\eta}{4RC}$$

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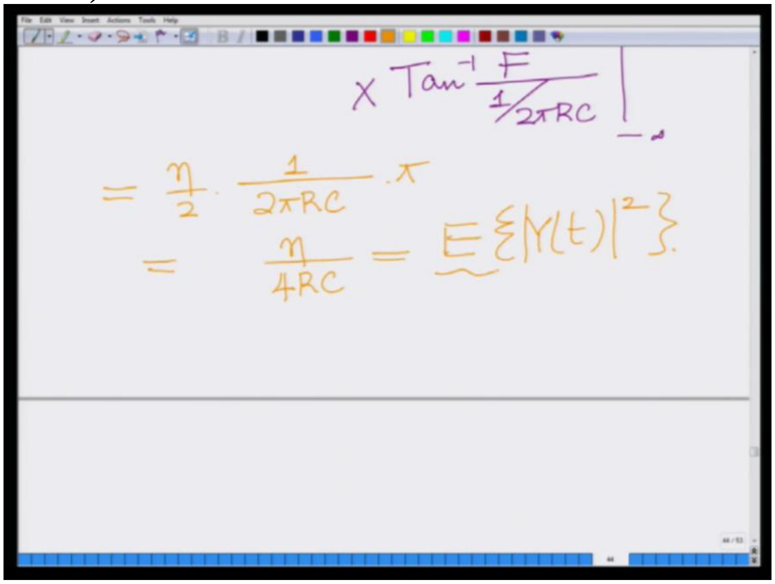

And remember, what is this quantity? This is your $E\{|Y(t)|^2\}$. Or this is basically the power in the Gaussian random process. This is integral of the power spectral density of Y over the entire frequency range.

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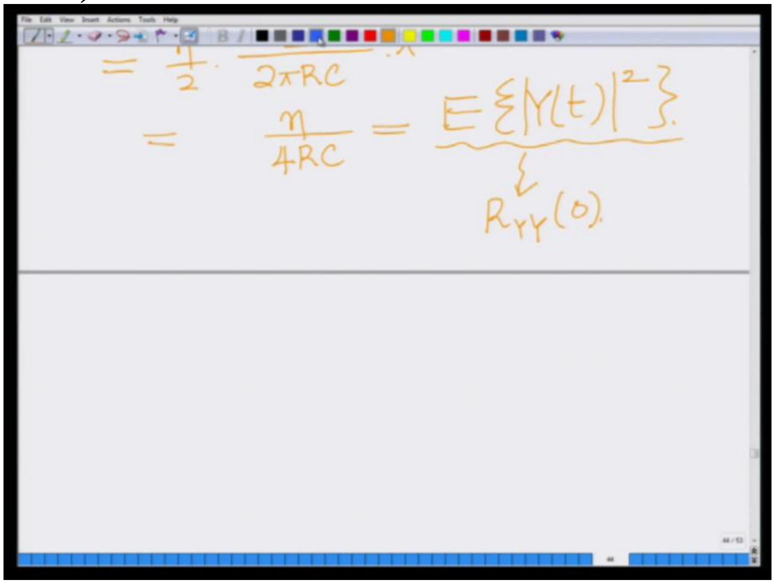

This is also $R_{YY}(0)$. That is autocorrelation function of Y evaluated at 0. So we have found the power in the Gaussian random output process Y.

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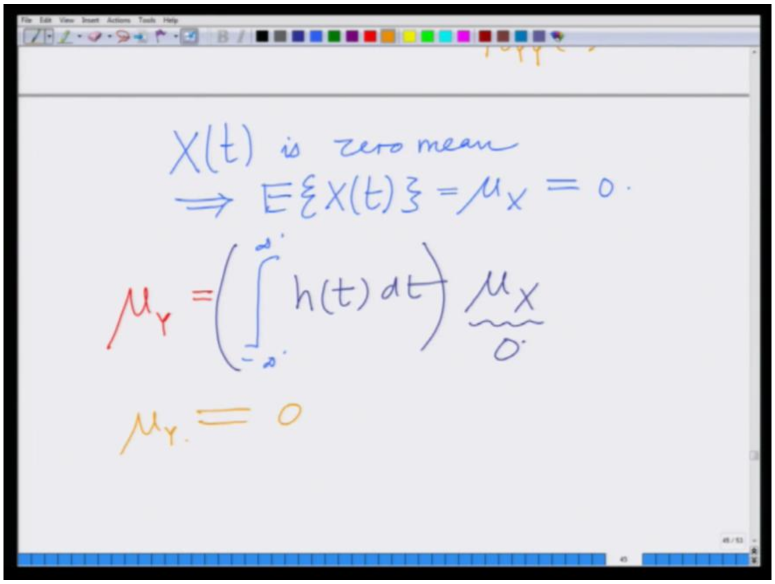

Now also we are given that X(t) is 0 mean. Now that implies,

$$E\{X(t)\} = \mu_X = 0$$

$$\mu_y = \mu_X \cdot \left(\int_{-\infty}^{\infty} h(t)\, dt\right)$$

$$\mu_y = 0$$

The mean of the input random process X(t) is 0. Therefore the mean of the output random process Y(t) is also 0. Which means the power of the random process is same as the variance. So the mean is 0 which means…

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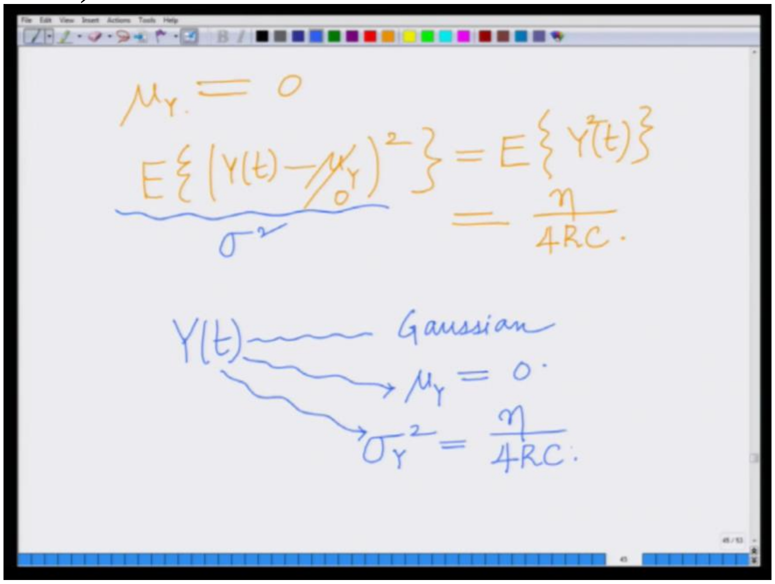

… expected, the variance is –

$$E\{ (Y(t)- \mu_Y)^2 \} = E\{ Y^2(t)\} = \frac{\eta}{4RC}$$

This is basically the variance of your Gaussian random variable. So we have $Y(t)$ is a Gaussian random process. So at each time instant T, $Y(t)$ is basically Gaussian, remember the joint statistics are Gaussian. So at each time instant, it is Gaussian. The mean $\mu_Y = 0$. The variance, if you were to call it σ_Y^2 , that is equal to –

$$\sigma_Y^2 = \frac{\eta}{4RC}$$

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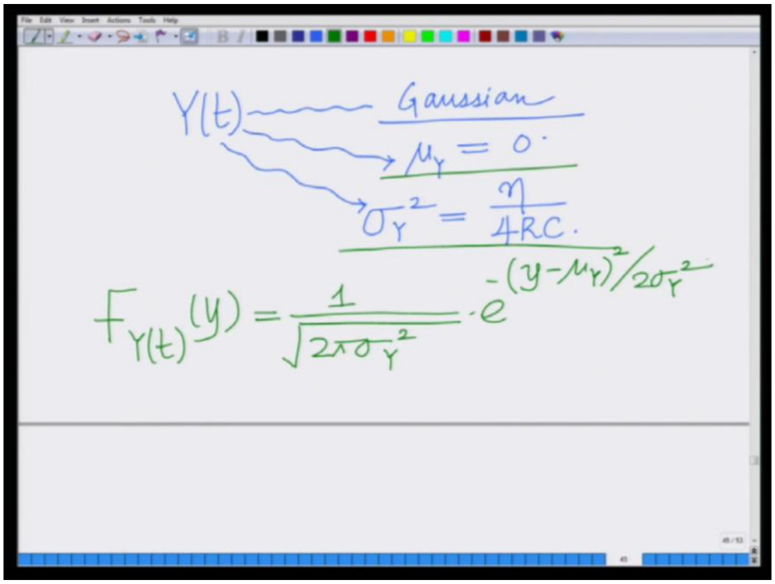

And therefore, the probability density function at any combination of times is Gaussian. Therefore if you take, at any particular time, t, the probability density function of Y(t) is Gaussian. Therefore the probability density function $F_{Y(t)}(y)$, now I can write,

$$F_{Y(t)}(y) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-(y-\mu)^2}{2\sigma^2}}$$

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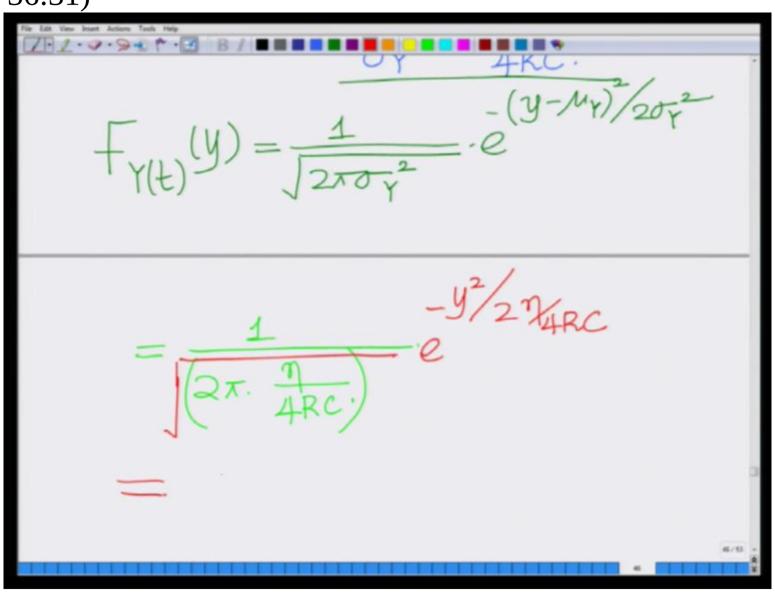

… now if I substitute the values for these parameters that is equal to –

$$= \frac{1}{\sqrt{2\pi\frac{\eta}{4RC}}} e^{\frac{-(y)^2}{2\frac{\eta}{4RC}}}$$

So now, if I substitute this what I have…

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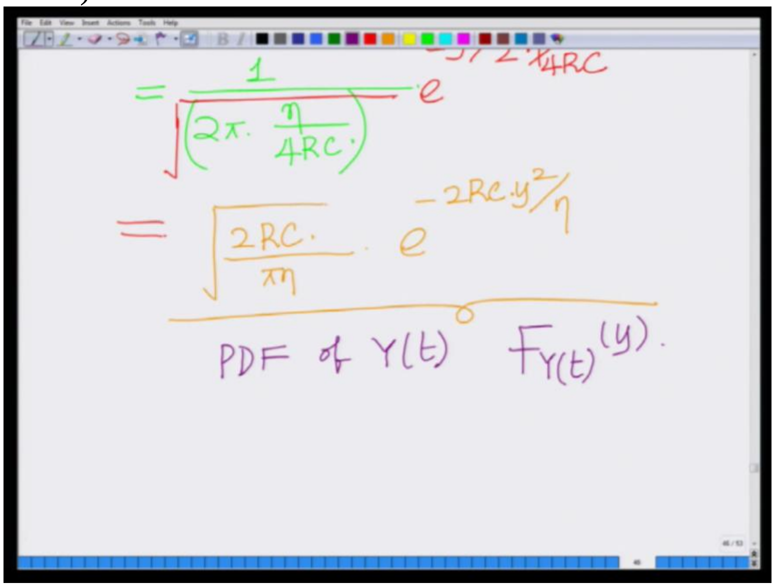

… is basically,

$$= \sqrt{\frac{2RC}{\pi\eta}}\, e^{\frac{-2RC(y)^2}{\eta}}$$

This is the PDF of your output random process. So this is the PDF of Y(t) at time instant T. That is the PDF, what is PDF? $F_{Y(t)}(y)$, PDF of the random variable, Y(t) at time instant t. So this illustrates an interesting, a very interesting example in the context of communication systems. That is what happens at the receiver when you process, that is you go through different stages of processing of the signal. At each stage, you have both outputs. You have the signal output and you have the noise output.

So at each processing stage, how is the noise output modified. So what we have seen is a simple example. That is when we filter this signal, remember, the signal has noise added to it. So when we pass through the processor or through this filter, the noise is also passing through the filter and what we have derived today is basically something very interesting. If we have white Gaussian noise of power spectral density $\frac{\eta}{2}$ at the input of this lowpass RC are the filter, then the output is also wide sense stationary Gaussian noise. It is no longer white because you are passing it through a filter. Right? And in fact, we have also derived the expression for the power spectral density. It is no longer white. The power spectral density of this noise at the output of this RC filter, the power of this Gaussian noise at the output of the RC filter is $\frac{\eta}{4RC}$.

And we also derived the probability density function, the Gaussian probability density function at a time instant T for this output Gaussian noise. So this comprehensively explains or illustrates several aspects of wide sense stationary random process. That is a wide sense stationary random process, a whiter random process, a Gaussian random process, a white Gaussian 0 mean random process and what happens when you transmit this through an LTI system. How does the LTI system affect this noise at the output. So, such processing or such knowledge or these results play a very important role in the analysis of digital communication systems and also wireless communication system in particular. So we will end this module here. Thank you very much.

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