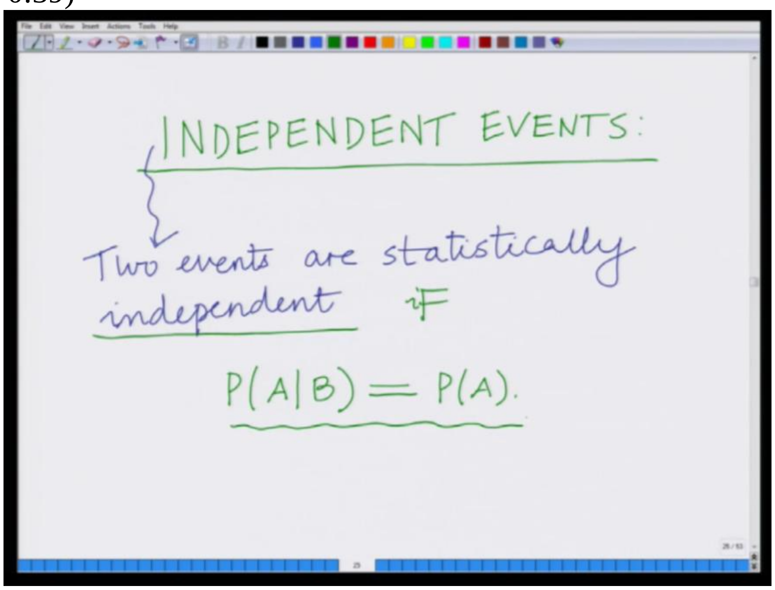
Probability and Random Variables/Processes for Wireless Communication
Professor Aditya K. Jagannatham
Department of Electrical Engineering
Indian Institute of Technology Kanpur
Module No. 1
Lecture 4
Independent Events-Mary-PAM Example.

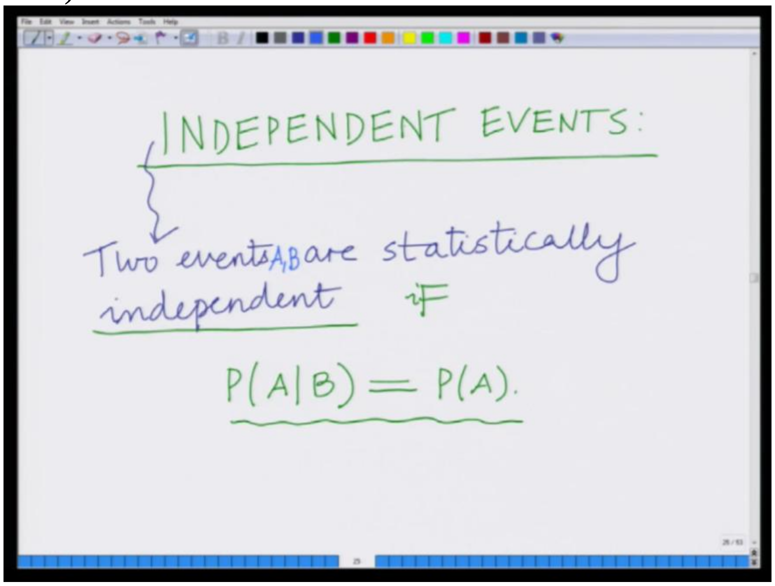
Hello everyone, welcome to another module in this massive open online course on principles of probability and random variables for wireless communication. Alright, so in the previous model, we have seen the concepts of condition the probability. Let us now look at another aspect, another very important property that is known as independence. All right?

(Refer Slide Time: 0:39)



So today, let us look at the concept of independent events. And this is a very important aspect of probability. That is, let us define this. This is a very key concept in probability. We say, 2 events are statistically independent if the probability of A given B.

(Refer Slide Time: 2:10)

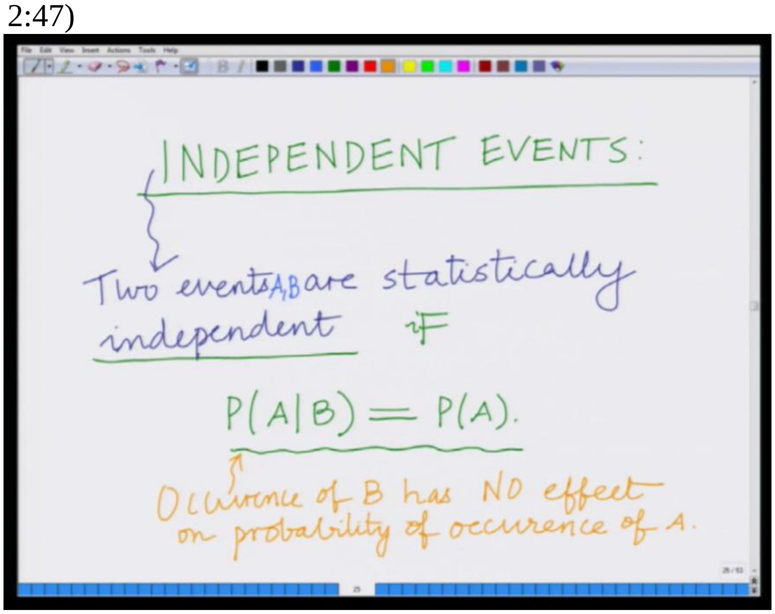


That is 2 events, A and B are statistically independent if

$$P(A|B) = P(A)$$

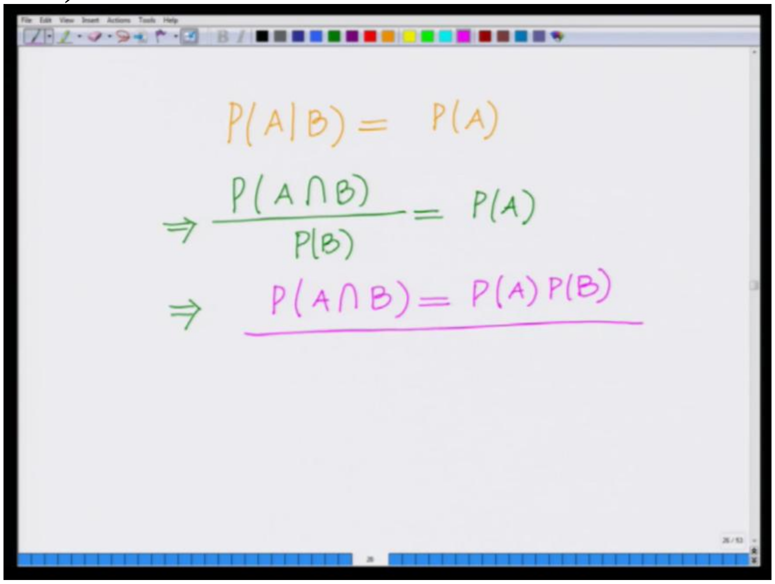
That is, what are we saying? That is, the probability of A given B that is we say, 2 events A, B are statistically independent if P(A|B) or the probability of A conditioned on B is simply P(A). That is, given that event B has occurred does not in any way affect the probability of occurrence of A. We say that A and B are independent. If given that B has occurred does not in any way affect the occurrence of the probability of A. Therefore, the probability of A conditioned on B is simply the probability of A because that is unaffected.

(Refer Slide Time: 2:47)



So let me again write this. This implies, the probability of occurrence of B has no effect that is occurrence of B has no effect on probability of occurrence of A. Therefore in that scenario we can say A and B are independent.

(Refer Slide Time: 3:41)



Let us simplify this further. So we say that A and B are independent if

$$P(A|B) = P(A)$$

But we have also defined,

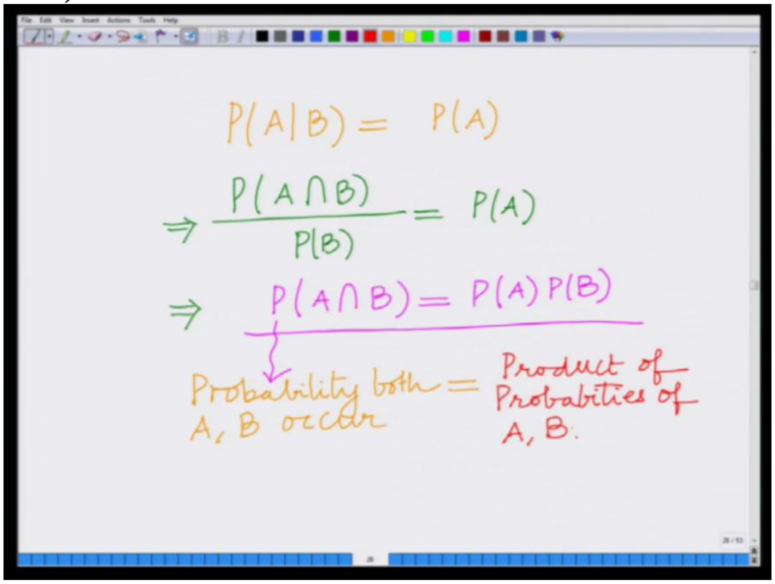
$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$$\Rightarrow \quad \frac{P(A \cap B)}{P(B)} = P(A)$$

$$\Rightarrow \quad P(A \cap B) = P(A).P(B)$$

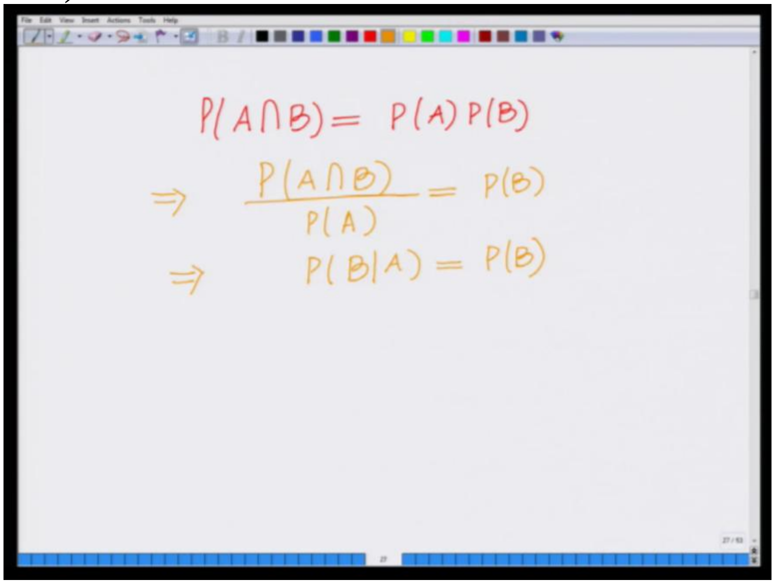
Right, so the probability, which, if you can see this, this is a very important probability. So for Independent events A, B, the probability of A intersection B that is the probability that both events A and B occur is simply the product of the probabilities A and B.

(Refer Slide Time: 4:48)



So the product, so the probability both A, B occur is simply equal to the product of probabilities A, B. That is, the probability that both A and B occur that is $P(A \cap B) = P(A).P(B)$ if A and B are independent. That is the probability that both A and B occur is simply the product of the probabilities of events A and B.

(Refer Slide Time: 5:45)



Similarly. One can also define independence the other way. That is the probability of…now similarly let us start with this probability for Independent events, we have,

$$P(A \cap B) = P(A).P(B)$$

$$\Rightarrow \quad \frac{P(A \cap B)}{P(A)} = P(B)$$

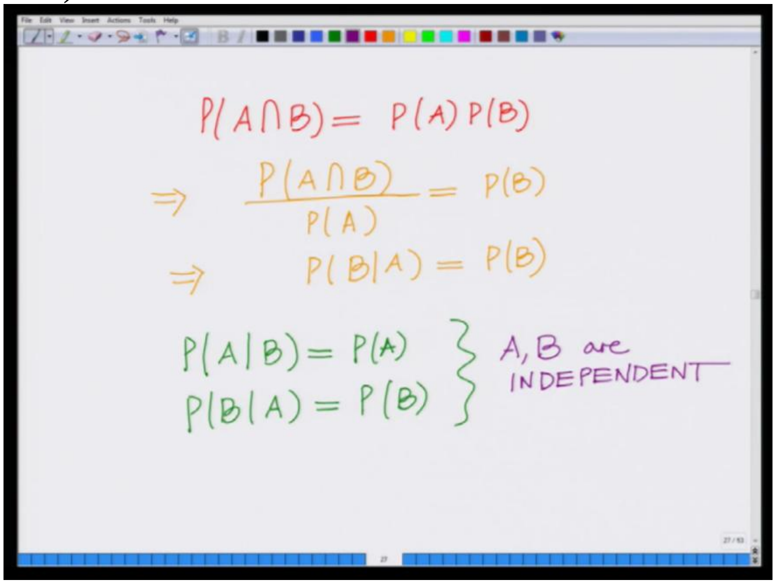
$$\frac{P(A \cap B)}{P(A)} = P(B|A) = P(B)$$

Therefore, if A and B are independent, we have said that $P(A|B) = P(A)$.

Similarly, we have also derived from that property that $P(B|A) = P(B)$.

Right? It works both ways. If A is independent B then B is independent of A.

(Refer Slide Time: 7:11)

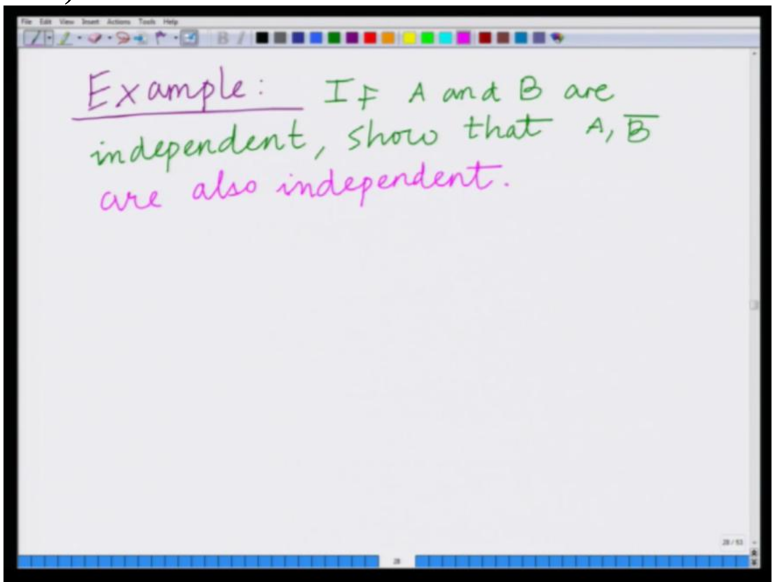


Therefore, we can say now, that the condition for Independence is that the $P(A|B) = P(A)$ or both. And that also implies that $P(B|A) = P(B)$. This is, if A, B are independent. If A, B are independent events, we are saying that the probability of A given B is equal to the probability of A, probability of B given A is equal to the probability of B and further

$$P(A \cap B) = P(A).P(B)$$

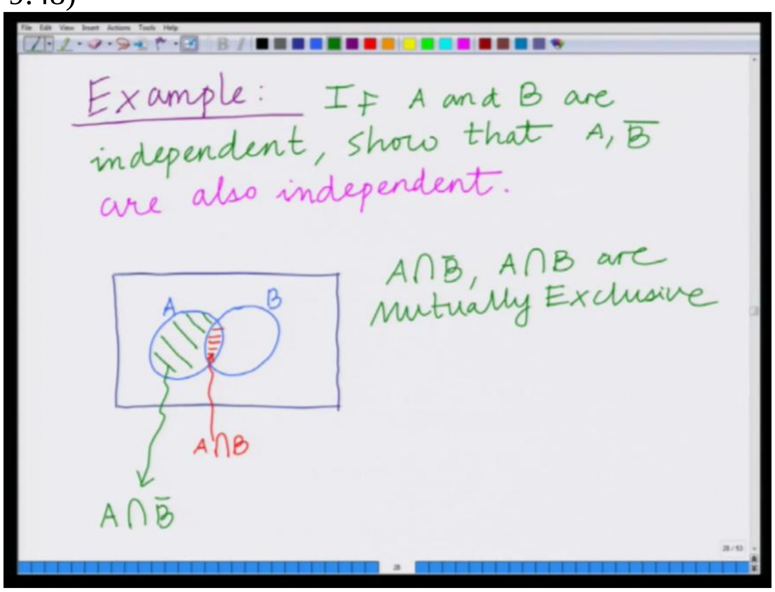
All right? This is a simple technical definition of independence and intuitively we have also said that this means that the occurrence of event B has no bearing on the probability of occurrence of A. Also, the occurrence of event A has no bearing on the probability of occurrence of event B. That is when events A and B are independent. Ok?

(Refer Slide Time: 8:37)



Let us look at an example. Let us prove an important property as an example. If A and B are independent, show that A, $\overline{B}$ are also independent. That is, if 2 events A and B are independent, then we want to show that A and $\overline{B}$ are also independent. That is the event A is also independent. That is, if event A is independent of event B then we want to show that event A is also independent of the complement of event B.

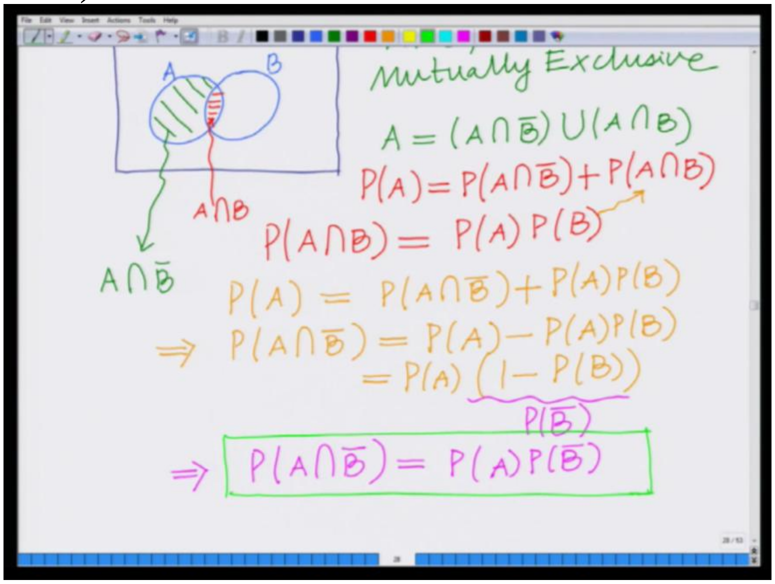
(Refer Slide Time: 9:48)



And this can be seen as follows for instance we have already seen yesterday. From our pictorial, from our diagrammatic representation, we have already seen yesterday that if we have 2 events A, B, this is the region $A \cap B$ and this is the region $A \cap \overline{B}$. And you can see that $A \cap \overline{B}$ and

$A \cap B$ are as we had seen yesterday, these are mutually exclusive. We can see that the events $A \cap \bar{B}$ and $A \cap B$ are mutually exclusive. That is, their intersection is the null event. Further, the union of these 2 events is the event A. Right?

(Refer Slide Time: 11:15)



So,

$$A = (A \cap \bar{B}) \cup (A \cap B)$$

Therefore, using the 3rd axiom of probability, we can write,

$$P(A) = P(A \cap \bar{B}) + P(A \cap B)$$

However, we are given that A and B are independent. Therefore, we have,

$$P(A \cap B) = P(A).P(B)$$

Therefore, now using this property, we can say, substituting this property over here, in place of $P(A \cap B)$, we have,

$$P(A) = P(A \cap \bar{B}) + P(A).P(B)$$

which implies,

$$P(A \cap \bar{B}) = P(A) - P(A).P(B) = P(A).(1 - P(B))$$

And now you can see that basically,

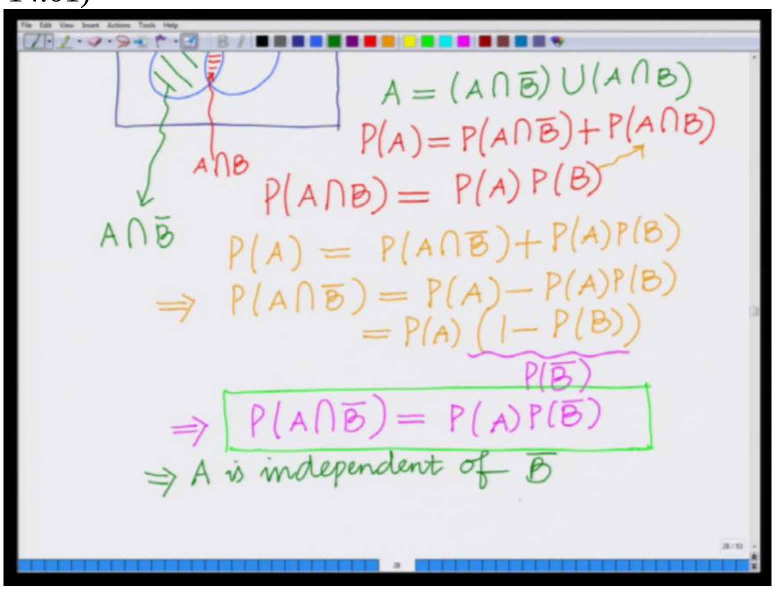
$$1 - P(B) = P(\bar{B})$$

which basically implies,

$$P(\mathrm{A} \cap \bar{B}\ ) = \mathrm{P(A)}.\, P(\bar{B}\ )$$

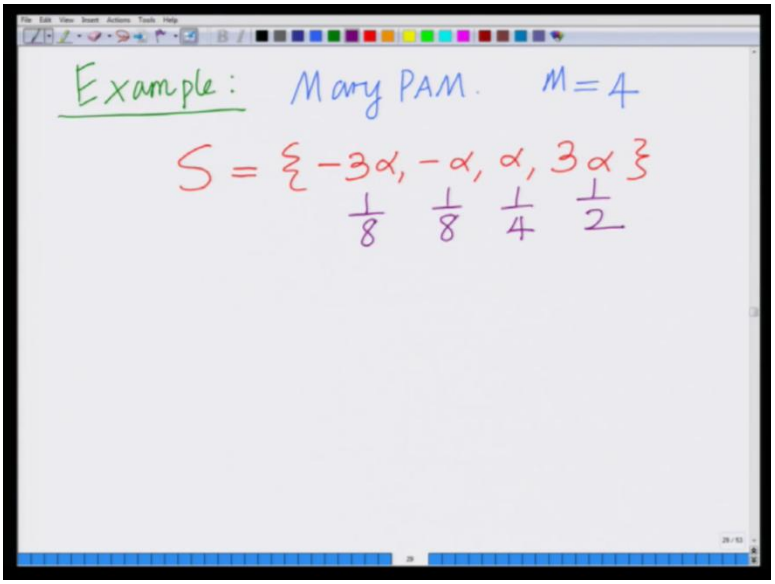
Therefore what we have demonstrated is that given A and B are independent, we are able to show that probability of A intersection $\bar{B}$ is equal to the probability of A times the probability of $\bar{B}$. Therefore A is also independent of $\bar{B}$. Remember, the condition for independence is the probability of ~~B~~ the intersection or the probability of joint event is equal to the product of the probabilities of the individual events. Therefore, we have shown that since $P(\mathrm{A} \cap \bar{B}\ ) = \mathrm{P(A)}.\, P(\bar{B}\ )$, A is independent of $\bar{B}$ as well if A is independent of B.

(Refer Slide Time: 14:01)



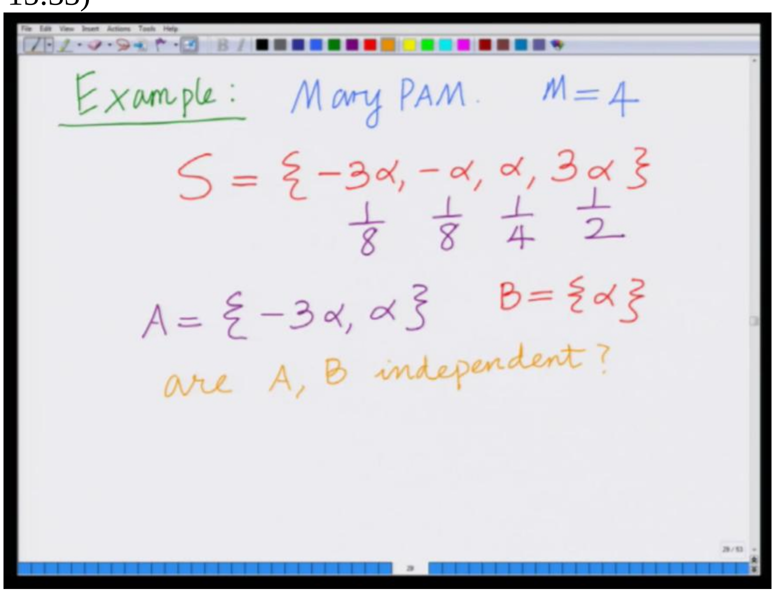
So this shows, this implies A is independent of B. So this shows that A is independent of B complement.

(Refer Slide Time: 14:28)



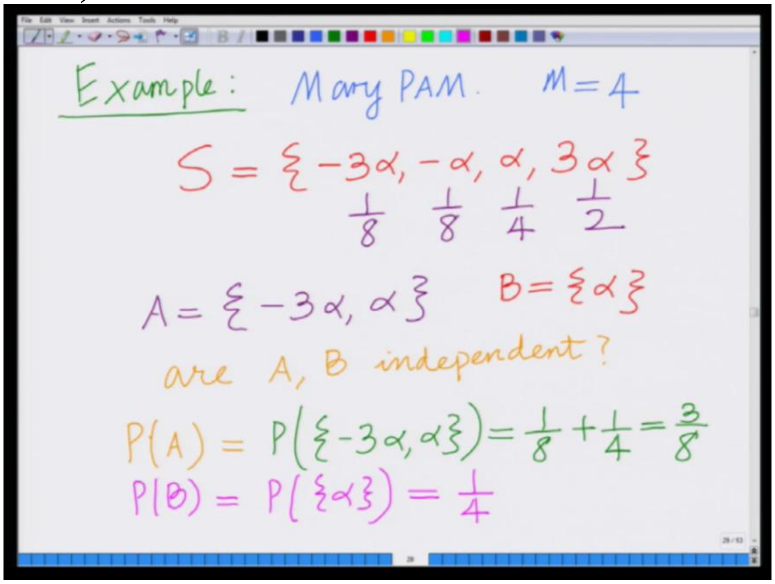
Let us look at another example. Let us go back to our M-ary PAM to understand this. Remember we are looking at M-ary PAM where M = 4. PAM stands for pulse amplitude modulation. Our sample space is **-3α, -α, α, 3α**. And I have the probabilities, 1/8, 1/8, 1/4 and 1/2. So we are looking at M-ary PAM constellation, pulse amplitude modulation with M = 4. I have the symbols, **-3α, -α, α, 3α**. The probabilities are 1/8, 1/8, 1/4 and 1/2 respectively.

(Refer Slide Time: 15:33)



And let us also look at the events, A which we have already defined previously. A is equal to **-3α, α** and B equals simply the event **α** corresponding to the symbol **α**. Now, let us ask the question, are events A, B, are these events independent? So let us use our definition for independence. So we have defined A as the event corresponding to the…similar to what we had done before, A is the set. A is the event **-3α** containing the sample points, **-3α, α**. So A basically corresponds to observing the symbol is either **-3α or α**. And B, the event B corresponds to the symbol **α**. And therefore, now we are asking the question, are these 2 events independent? Let us use our definition of independence to check if these 2 are independent.

(Refer Slide Time: 16:44)

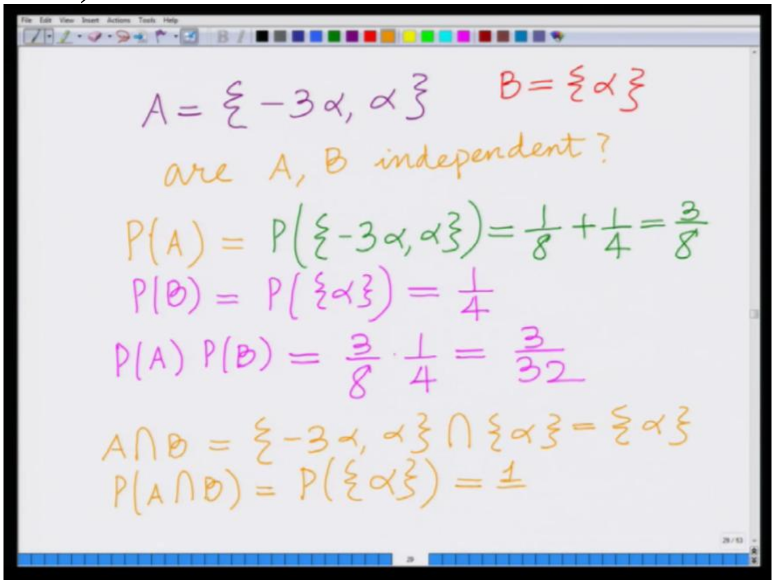


Now we have P(A) which we have already derived before is simply,

$$P(A) = P(-3\alpha\ ,\ \alpha\ ) = \frac{1}{8} + \frac{1}{4} = \frac{3}{8}$$

P(B) is equal to the probability of simply a single sample point **α** which is equal to 1/4. ~~Therefore the probability of A intersection B.~~

(Refer Slide Time: 17:34)



Therefore,

$$P(A).P(B) = \frac{3}{8} \cdot \frac{1}{4} = \frac{3}{32}$$

Now let us look at the event (A ∩ B).

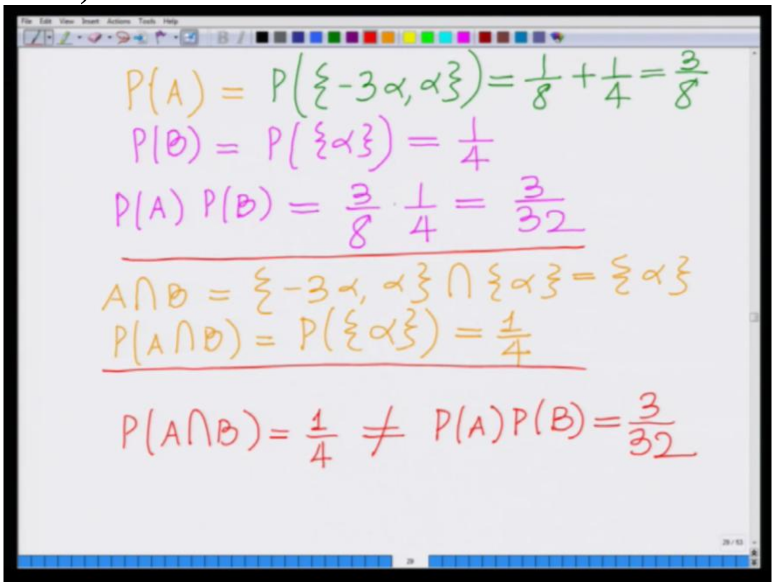
$$A \cap B = \{ \mathbf{-3\alpha}, \boldsymbol{\alpha} \} \cap \{ \boldsymbol{\alpha} \}$$

which is basically the single sample point **α**. Therefore,

$$P(A \cap B) = P(\{ \boldsymbol{\alpha} \}) = \frac{\mathbf{1}}{\mathbf{4}}$$

Now you can see the probability.

(Refer Slide Time: 18:25)



Now you can see from both of these. This is probability of A times probability of B. This is probability of A intersection B. Now you can see,

$$P(A \cap B) = \frac{\mathbf{1}}{\mathbf{4}} \neq P(A).P(B) = \frac{\mathbf{3}}{\mathbf{32}}$$

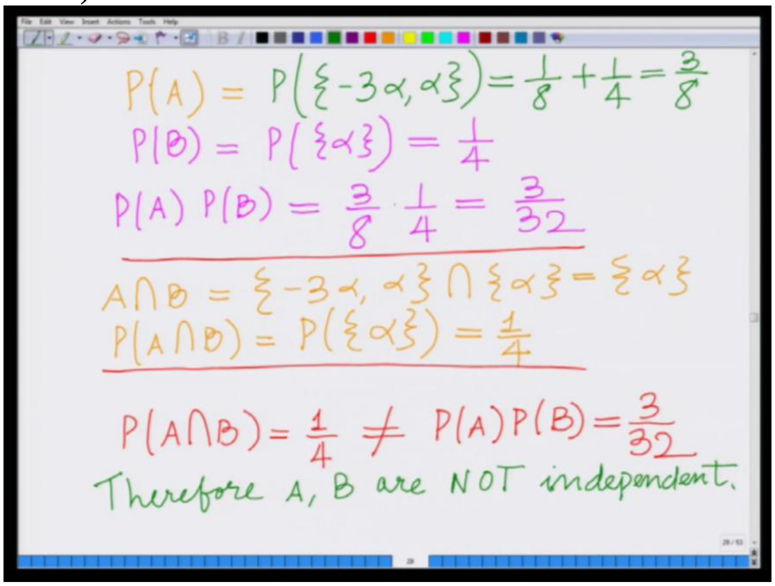
So we are saying that,

$$P(A \cap B) = \frac{\mathbf{1}}{\mathbf{4}}$$

$$P(A).P(B) = \frac{\mathbf{3}}{\mathbf{32}}$$

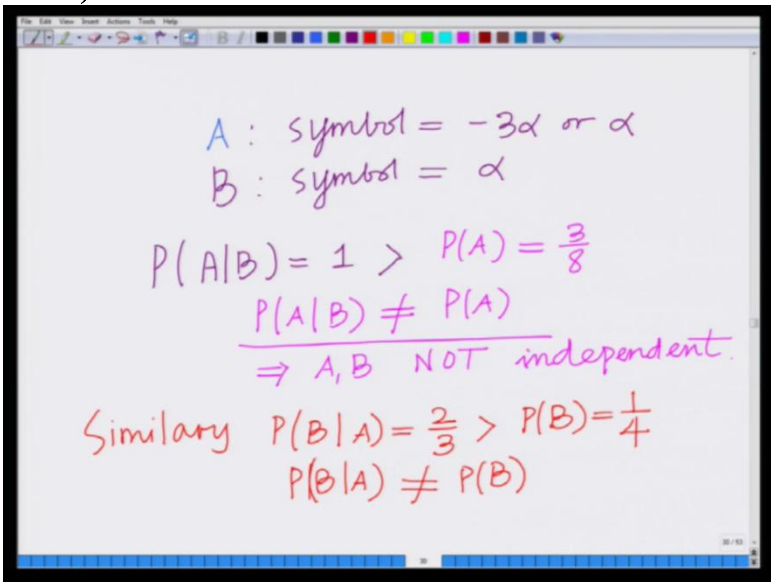
Therefore P(A ∩ B) that is the probability that both events occurring is not equal to probability of A times the probability of B that is the product of probabilities. Therefore, A and B are not independent.

(Refer Slide Time: 19:22)



Therefore, we are saying something interesting, therefore A and B are not independent events. And this is also natural because A is the symbol **-3α or α**. A denotes the events observing the symbols **-3α or α**, B denotes observing the symbol **α**. Therefore, if one has told you that he has observed B that is he has observed symbol **α**, then definitely, you can conclude that event A has happened because because event A is either **-3α or α**. So given B conveys a lot of information about event A. In fact, given event B has occurred, one can conclude that event A has definitely occurred. Therefore, observing event B has an effect on observing, on the probability of occurrence of A. Therefore, B and A are not independent.

(Refer Slide Time: 20:40)



Remember, what we are saying is B, our A equals minus 3 alpha equals observing that is our symbol **-3α or α**. B equals, let us put it,

A : symbol = **-3α or α**

B : symbol = **α**

So if one has told you that B has occurred that the symbol is **α**, then definitely A has occurred. That is, the symbol is either **-3α or α**. And therefore, the occurrence of B has an effect on the occurrence of A. Or B affects the probability of occurrence of A. Therefore, by definition they are not independent.

Therefore we can say that the probability…and remember, we had seen this previously,

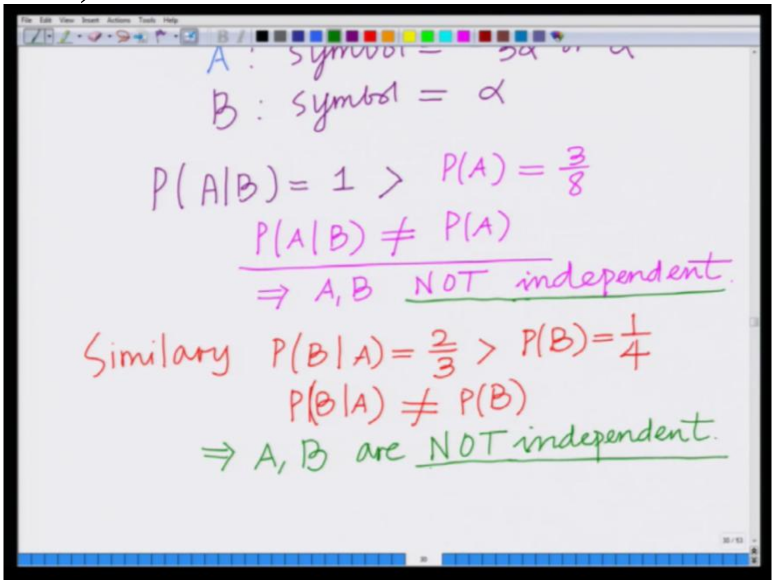
$$P(A|B) = 1 > \text{P(A)} = \mathbf{\frac{3}{8}}$$

Therefore, $P(A|B) \neq \text{P(A)}$, which basically also again implies A, B are not independent. Similarly,

$$P(B|A) = \mathbf{\frac{2}{3}} > \text{P(B)} = \mathbf{\frac{1}{4}}$$

Therefore, $P(B|A) \neq \text{P(B)}$, which also basically once again implies that…

(Refer Slide Time: 23:13)



… All of these imply the same thing, A, B are not independent. Correct? So this is the key aspect here. That is what we are saying is, these 2 events, A, B where A denotes the event containing the sample points **-3α and α**, B denotes the event containing the sample point **α**, these 2 events are not independent. Because the occurrence of one has an effect, has an impact on the probability of occurrence of the other. Given that B has occurred has an impact on the probability of occurrence of A. Given that A has occurred has an ~~effect~~ impact on the probability of occurrence of B. Therefore A and B are not independent.

We have also verified this just using our definition of independence that is the probability of joint event, $P(A \cap B) \neq P(A).P(B)$.

Therefore these events are not independent. We will look at some other examples of independence in subsequent modules. Let us stop this module here. Thank you very much.