Probability and Random Variables/Processes for Wireless Communication
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Module 2
Lecture No 7
Bayes Theorem and Aposteriori Probabitiesr

Hello, welcome to another module in this massive open online course on probability and random variables for wireless communications. In the previous model we have looked at various concepts of probability. In this module, let us start looking at another new concept or a new result which is very important in the context of communication. This is the Bayes Theorem.

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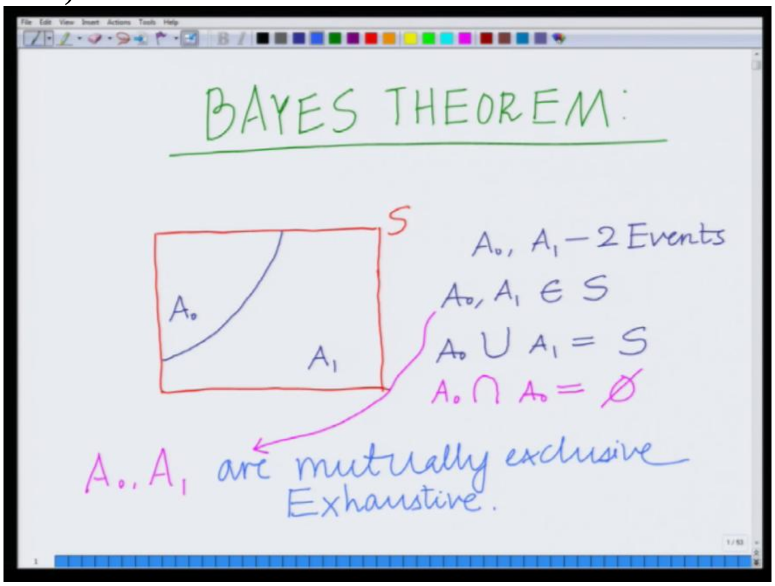

So let us look, in this module, let us start looking at Bayes Theorem. For Bayes Theorem, what we would like to do is we would like to consider the sample space, S. I would like to consider 2 events. Consider 2 events, A0 and A1, these are basically 2 events. And these, both A0 and A1 belong to the sample space S such that

$$A0 \cup A1 = S$$

And A0, A1 are mutually exclusive. That is,

$$A0 \cap A1 = \emptyset$$

So we are considering 2 events, A0 and A1 in our sample space S such that A0 ∪ A1 that is the union of these 2 events is equal to the entire sample space, S and these 2 events A0 and A1 are mutually exclusive that is A0 ∩ A1 is the null set or null event, ∅. Such events, A0 and A1 are known as mutually exclusive and exhaustive.

So these events, A0, A1 are mutually exclusive and exhaustive. Exhaustive meaning, their union spans the entire sample space, S and while their intersection is the null event. Therefore these events are mutually exclusive and exhaustive.

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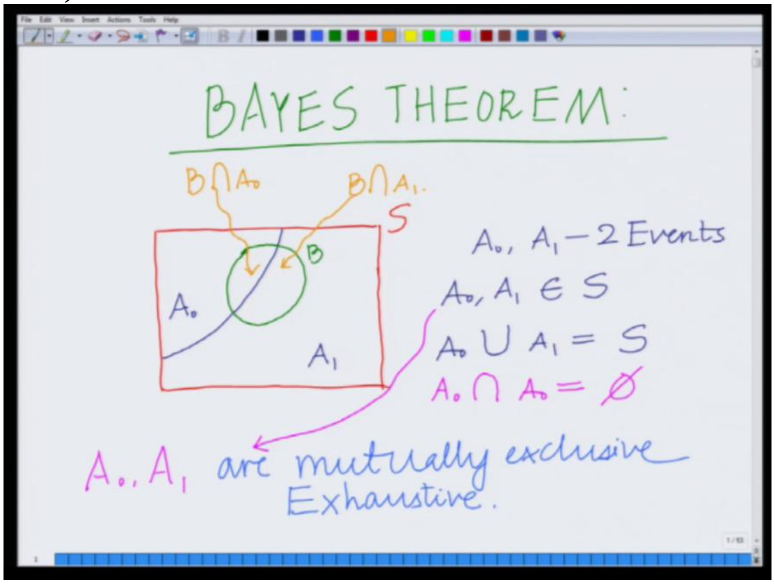
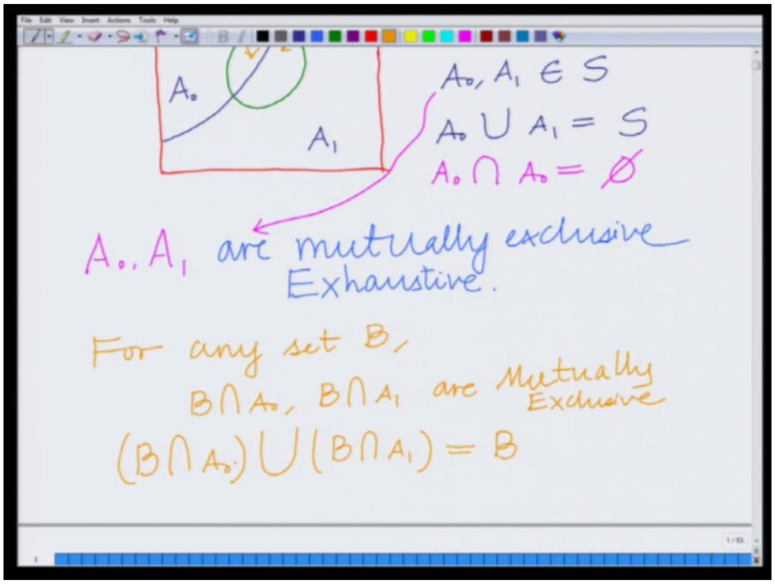


Now let us consider another event B. Let us consider an event B in the sample space, S. Now you can see, this part is B ∩ A0 and this part is B ∩ A1. ~~A0~~ now you can see clearly, B ∩ A0 and B ∩ A1 are also mutually exclusive. Further, B ∩ A0 and B ∩ A1 together span B or together, the union of these 2 is B.



Alright! So we can see, for any set B, $B \cap A0$, $B \cap A1$ are mutually exclusive events that is their intersection is $\emptyset$ or the null event. Further,

$$(B \cap A0) \cup (B \cap A1) = B$$

Alright ! So we are saying that for any event B, we have $B \cap A0$ and $B \cap A1$ are mutually exclusive. That is, their intersection is a null event and $(B \cap A0) \cup (B \cap A1) = B$.

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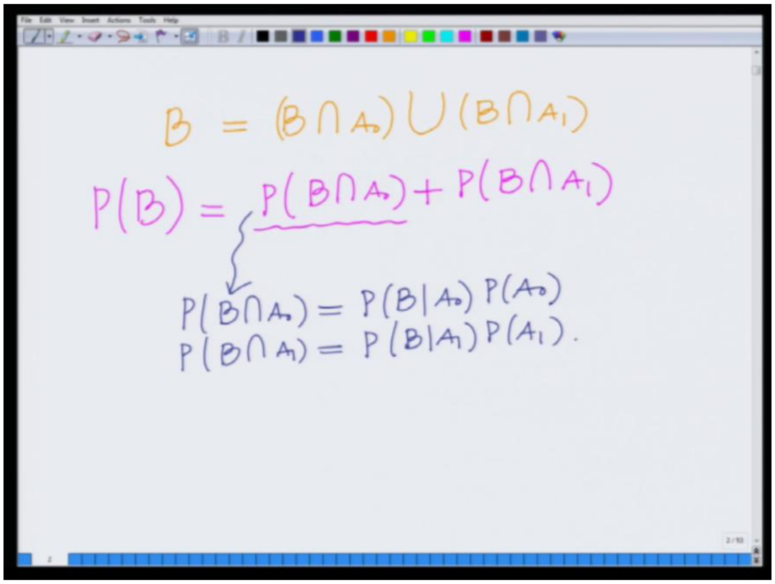

Therefore, if I now look at the probability of B, so now,

$$B = (B \cap A0) \cup (B \cap A1)$$

Therefore,

$$P(B) = P(B \cap A0) + P(B \cap A1)$$

because $B \cap A0$ and $B \cap A1$ are mutually exclusive. Now again we know that from the basic definition of conditional probability, we had already shown that,

$$P(B \cap A0) = P(B|A0).P(A0)$$

$$P(B \cap A1) = P(B|A1).P(A1)$$

We know from the definition, from our conditional probability module that, the probability of B intersection -

$$P(B \cap A0) = P(B|A0).P(A0)$$

$$P(B \cap A1) = P(B|A1).P(A1)$$

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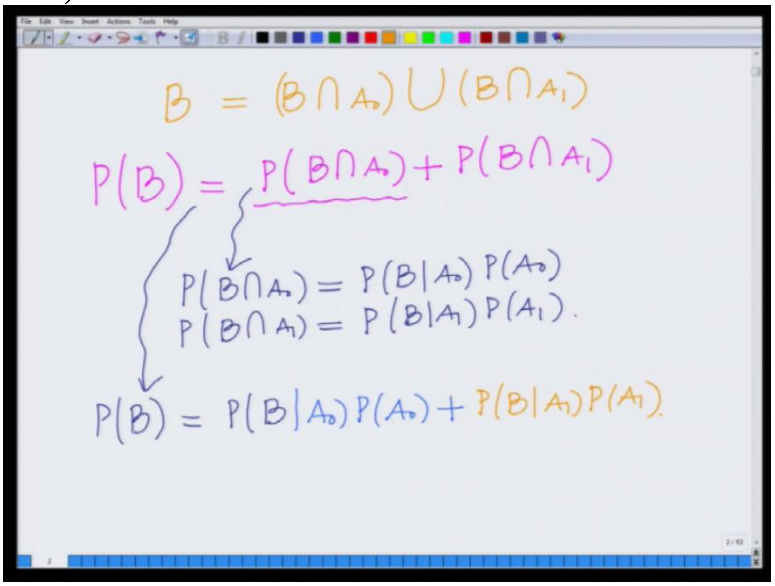

Therefore, substituting these in the expression above, we have the total probability,

$$P(B) = P(B|A_0).P(A) + P(B|A_1).P(A_1)$$

So the probability of B is the probability B given A_0 times probability A_0 plus probability B given A1 times the probability of A1.

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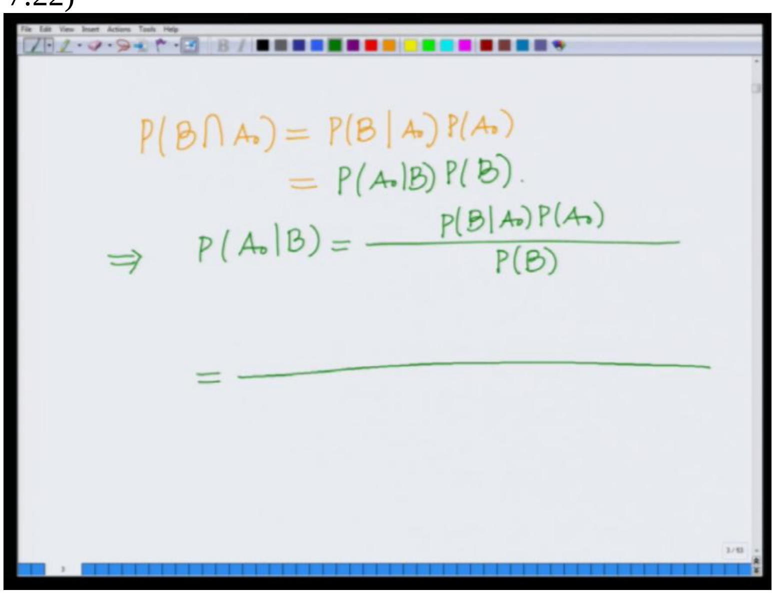

Now, we also know again that,

$$P(B \cap A_0) = P(B|A_0).P(A_0)$$

$$= P(A_0|B).P(B)$$

Therefore from this, what I have is interestingly I have from this. This implies that,

$$P(A_0|B) = \frac{P(B|A_0).P(A_0)}{P(B|A_0).P(A_0)+P(B|A_1).P(A_1)}$$

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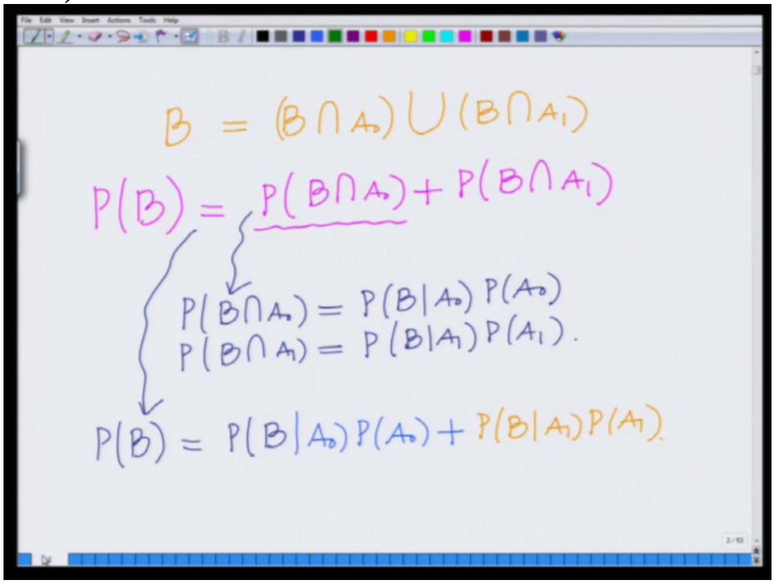

Now what I am going to do here is I am going to substitute the expression of probability of B from the previous page and therefore what I have is this is equal to

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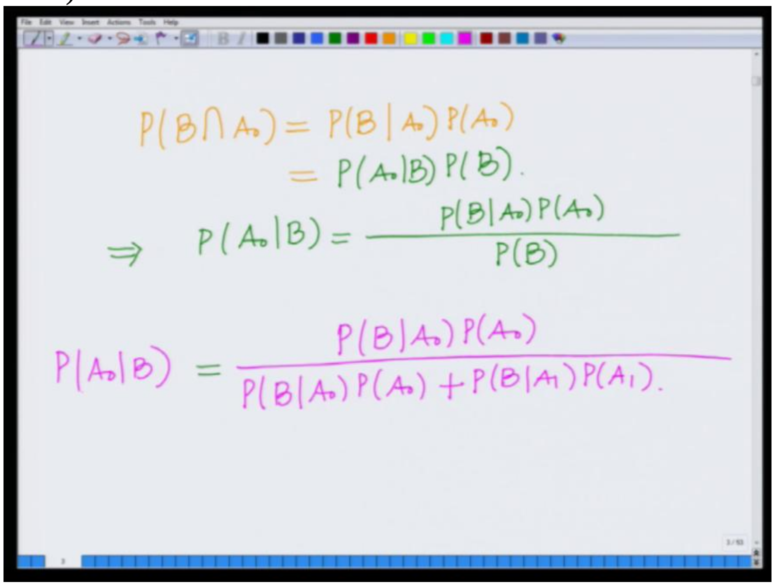

…

$$P(A_0|B) = \frac{P(B|A_0).P(A_0)}{P(B|A_0).P(A_0)+P(B|A_1).P(A_1)}$$

This is the expression for $P(A_0|B)$. So we have, probability of A_0 given B equals probability of B given A_0 times probability of A_0 divided by the probability of B given A_0 times $p(A_0) + p(B)$ given A1 times the $p(A_1)$. Right?

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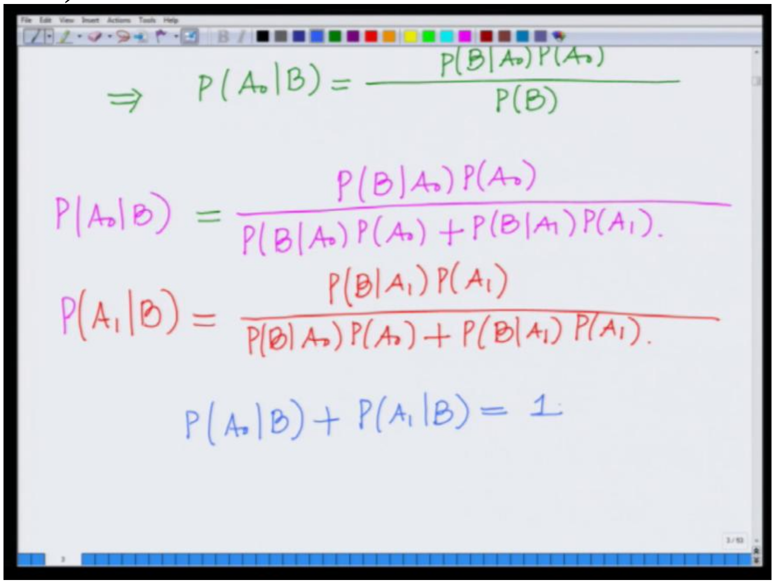

Similarly, I can derive the expression for $p(A_1 | B)$. It is given similar to as above,

$$P(A1|B) = \frac{P(B|A1).P(A1)}{P(B|A0).P(A0) + P(B|A1).P(A1)}$$

As A_0 and A_1 are mutually exclusive and exhaustive events. Therefore

$$P(A_0|B)+P(A_1|B)= 1$$

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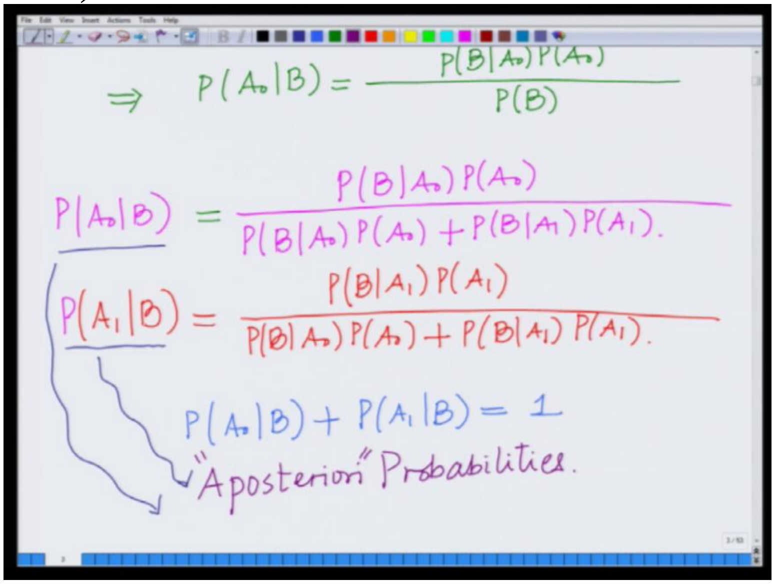

And these quantities here, these quantities that we have calculated here, so this is basically the Bayes theorem. Right? Now these quantities here, the probabilities of A0 given B and the probabilities of A1 given B: these quantities are very important in the context of communication. These are known as Aposteriori probabilities. These quantities are very important in the context of communication.

The probabilities $P(A_0|B)$ and $P(A_1|B)$, are known as Aposteriori probability. And we are going to introduce an example later which will clarify the application of these. Aposteriori probabilities can now be computed using Bayes Theorem.

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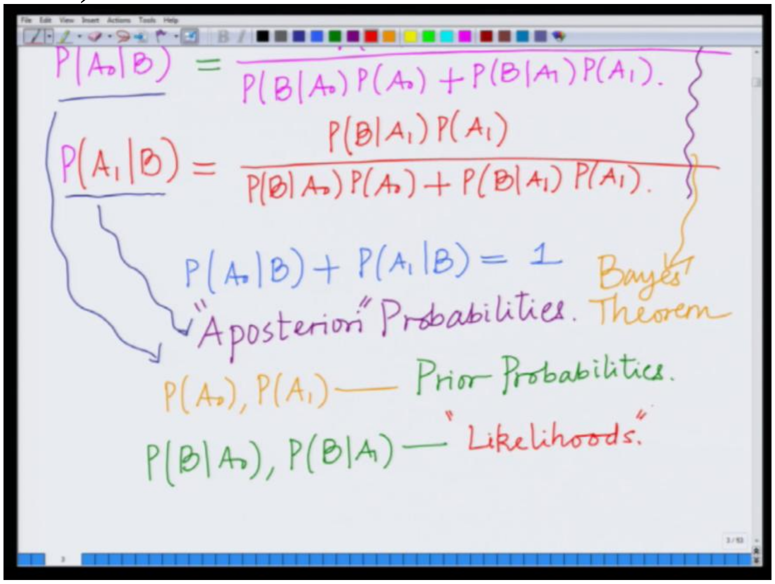

The expressions that you see for the Aposteriori probabilities that you see, this is nothing but this is our Bayes Theorem. Our Bayes theorem gives an expression for the Aposteriori probabilities. The quantities, $P(A_0)$, $P(A_1)$: are known as the prior probabilities and the quantities, $P(B|A_0)$ and $P(B|A_1)$, are called the likelihoods.

All of these are important terminologies in the context of communication or wireless communication. So if A0, A1 are mutually exclusive and mutually exhaustive, the Bayes gives us a very useful relation to calculate the Aposteriori probabilities, $P(A_0|B)$ and $P(A_1|B)$ in terms of the prior probabilities $P(A_0)$, $P(A_1)$ and the likelihoods, $P(B|A_0)$ and $P(B|A_1)$.

And this is a very important result and we are going to demonstrate an application of this shortly in the context of the MAP principle or the maximum Aposteriori probability receiver but before we do that, let us extend this Bayes Theorem to a general case with N mutually exclusive and exhaustive events, A_i.

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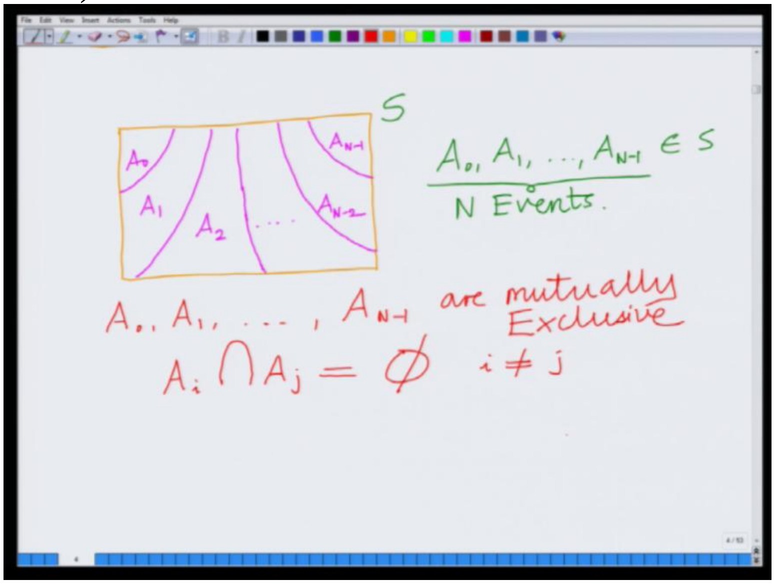

So now, let us extend this to a general version of the Bayes Theorem. So let us now state a generalised version of or a general version of Bayes Theorem where we now consider a sample space, S. Let us now consider the sample space, S which is divided into N mutually exclusive and exhaustive. So what I have over here is, I have my sample space, S and A_0, A_1 up to A_{N-1}. These are N events which belong to S. Further, A_0, A_1 up to A_{N-1} are mutually exclusive.

$$\Rightarrow A_i \cap A_j = \emptyset, \forall\ i \neq j$$

That is, we have the property that, I have N events,

A_0, A_1 …. A_{N-1}, these are mutually exclusive implying that if I take any 2 events, Ai and Aj, Ai intersection Aj is the null event that is phi.

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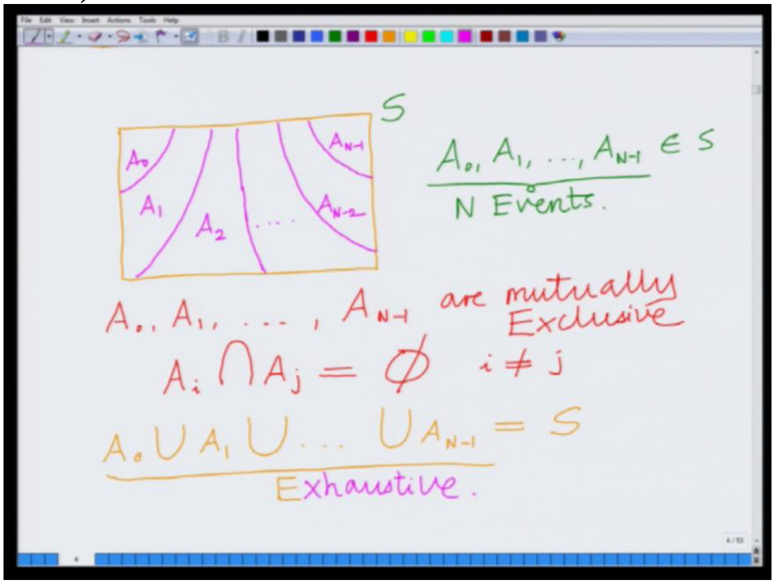

Further,

$$A_0 \cup A_1 \cup \ldots \cup A_{N-1} = S$$

This is the exhaustive property.

Exhaustive implying that the union of all these events is this entire sample space and mutually exclusive implying that if you take any 2 events, A_i and A_j, their intersection is the null event.

Basically, in terms of set theory, we say the sets of events, A_0, A_1 …. A_{N-1} are a partition of the entire sample space, S. That is, their union's spans the entire sample space, S and these different events or sides are disjoint. That is, if I take any 2 sets, their intersection is the empty set. These are mutually exclusive and exhaustive events.

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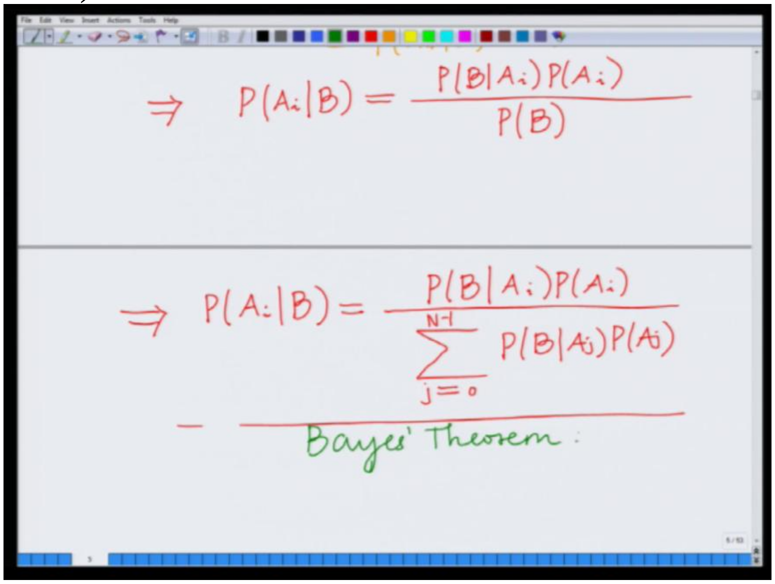

And now the Bayes Theorem for the Aposteriori probabilities, Bayes theorem gives the Aposteriori probability similar to what we have seen before, P of A_i given B is given as

$$P(A_i|B) = \frac{P(B|A_i).P(A_i)}{\sum_{j=0}^{N-1} P(B|A_j).P(A_j)}$$

and this is the result for the Bayes Theorem.

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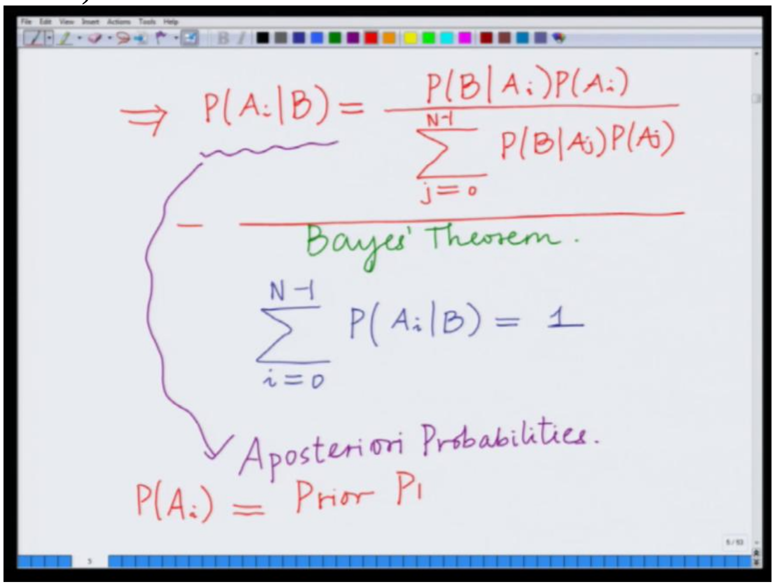

And also, therefore, as we have also seen before, summation of, it is easy to see that the summation of $j=0$ *to* $N - 1$ or $i=0$ *to* $N - 1$, for $P(A_i|B)$ is equal to 1. Further, these quantity's, $P(A_i|B)$, are the Aposteriori probabilities. Remember these are the Aposteriori probabilities. The probabilities of $P(A_i)$ are the prior probabilities.

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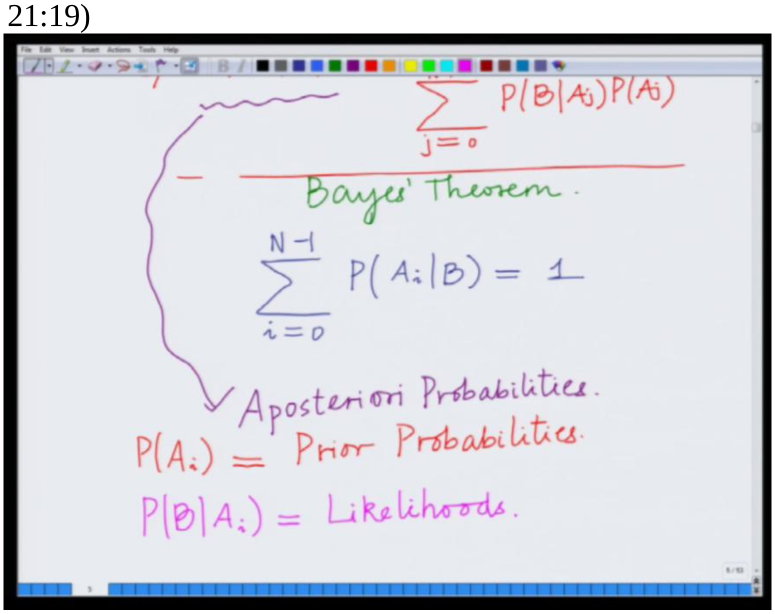

And the $P(B|A_i)$, are the likelihoods.

So this is a general expression for the Bayes Theorem. It expresses for the Aposteriori probabilities in terms of the prior probabilities and the likelihoods. We said that this Bayes Theorem or this Bayes result is a very important principle in communications. This is used to construct what we call that as the maximum Aposteriori, the MAP receiver which is something that we are going to look at in our next module. So I would like to conclude this module here. Thank you very much.