Introduction to Logic

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Lecture - 28
Axiomatic Propositional Logic

Welcome back. In the last few lectures, we talked about many important decision procedure methods, starting with 2 table method and then semantic tableaux method and we of come off with semantic resolution reputation method etcetera and all. We solved 1 particular problem with respect to all these methods. And then we have seen which method to we need to employee and all. It is all depends upon our convenience and all. So, for we talked about different decision procedure methods, with which you will come to know, whether a given well from formula it is a valid formula or tautology or when 2 groups of statements are consistent to each other etcetera.

So, today, I will be talking about another way of doing it, which comes under the category of syntactic method. So, that is the axiomatic propositional logic. I will be talking about the axiomatic propositional logic. In that I will be talking about 2 different axiomatic systems; one is due to a famous Russell and Whitehead real 2 mathematicians and all. This axiomatic system is due to Russell and Whitehead, which is called as Russell and Whitehead axiomatic systems, which the propos it in the historic book, that is, principia mathematica.

So, I will be taking selective portion of that principia Mathematica. And I will be talking some of the proofs which are already there in that book. So, this comes under 1 particular chapter, which is based on deduction in propositional logic. So, I am restricting my attention to, my focus on propositional logic.

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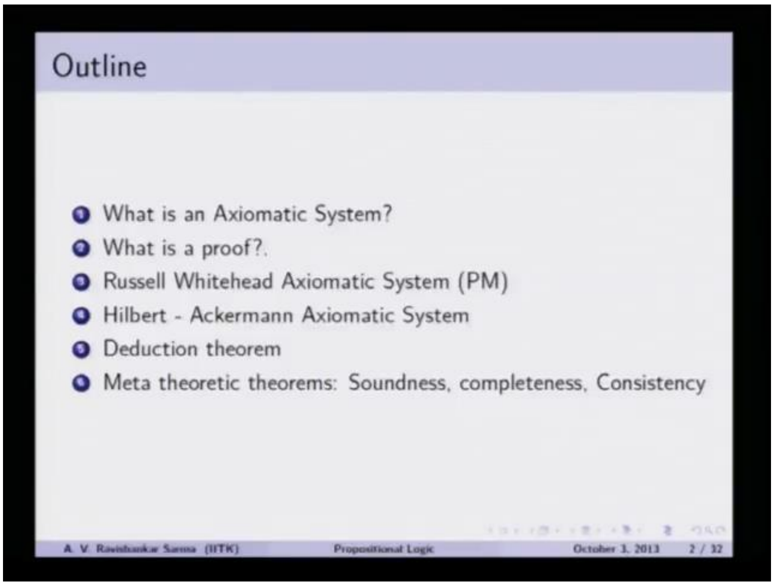

So, we will be mostly talking about Russell and Whitehead axiomatic system. So, now, in this lecture, what I will be doing is; these are the things which I would be doing. So, first I will talk about; what I mean by axiomatic system, how it has originated etcetera. And then the axiomatic system what occupies the central position is a proof, a proof of what? A given theorem.

So, will be talking about these specific key terms such as, axiom proof what you mean by deduction, what you mean by saying that a particular a proposition, particular statement is consist to be a lemma or conjecture or Carolyn etcetera. These are things which you commonly come across within the proof theorem. So, usually a proof is considered to be finite sequence of steps, which ends in finite intervals of time. So, if you proof ends finite steps in finite intervals of time, then usually it is considered as an effective proof.

So, how did we how did we come to this particular kind of regress proofs and all. So, what is wrong with the proofs that are already a existing in the Euclidean geometry? Euclidean geometry is also considered to be one of the important axiomatic systems, but still we do not treat it as a as regress as this, that we find it in the principia Mathematica. So, what is the reason for that particular kind of thing, I will talk about it in a in an action. And then I move on to a different kind of axiomatic system, which derives is motivation from axiomatizing geometry. Principia Mathematica is motivated by, motivation as come from the arithmetic whereas Hilbert Ackermann axiomatic system, the motivation comes from the geometry.

So, then we will see with the help of deduction theorem, will we simplifying some of the proof that are there in either in Russell Whitehead axiomatic system or in the Hilbert Ackermann axiomatic system. And then we will talk about some of the important meta theoretic theorems, such as soundness. So, that is whether all things that you are proving are going to be true or not. Usually in the proof, the last step is considered to a proof which is obvious; obviously, considered to be true.

If it is a true or tautology then it has to find a some kind of decision procedure, we need to find a decision pro, with the help of decision procedure method we should be in a position to check, whether it is a valid formula or not. The soundness ensures that, all the things that you are derived are true and the completeness assures as that all the true propositions, finds some kind of proof. And consistency is natural property, which is in given point of time, you cannot derive both x and not x.

So, in this lecture, I will be focusing my attention on the origin of axiomatic system. And then partly I will be talking about Russell Whitehead axiomatic system and some of the important proofs, within that axiomatic system due to Russell and Whitehead.

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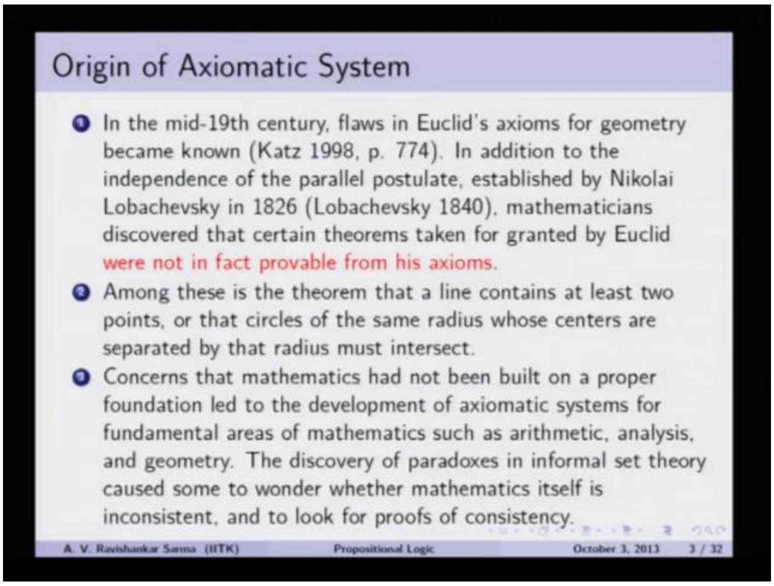

So, the origin of axiomatic system is usually like this, usually we find axiomatic system either in Aristotle in framework or even in a Euclid's, in the Euclid's in the works of Euclid's that is elements. So, Euclid has come off with 5 postulates and then their few definitions and then their some common notions. And with the help of these common notions definitions and axioms, he use it as a postulates. With that all the other true propositions are true propositions in geometry are derived.

So, in 19 century, mathematicians started finding flows in the Euclidean axioms. So, there is fifth postulate which is objectionable. So, the flows in the Euclidean axioms a in the geometry let to these particular kind of axiomatic systems. So, the work of Katz in 1998, he found flow in the Euclidean axioms is the postulates. Usually the fifth postulate is highly in on, whether are not to replace that particular kind of postulate, or change it change it in such way that, you can talk about a different kind geometry.

So, after in the mid 19 century, different geometries have coming to an existence; alignment geometry has come off with another kind of thing etcetera. All these comes under the category of non Euclidean geometry. So, in addition to the independence of the parallel postulates established by another Russian mathematician; Lobachevski mathematician discovered that, certain theorems take in, certain theorems in Euclidean geometry or taking for granted by Euclid, but not in fact probable by using his own axioms.

So, there are 2 problems with this Euclidean geometry. The first thing is that, the fifth postulates postulate has created problem. And people have come off with various other kinds of geometry and all. And the other thing is that, so many commonsensical notions, direct observation etcetera, which are not part of part and parcel of your proof, that already present in the proof and all. There are certain things which are outside the proof and unfortunately they are also part of the proof and all.

So, we need separate all these all these common. If you want to have a regression kind of proof, everything needs to be stated explicitly. And they should not be any notion which is which is not part of the proof, that should play a role in deriving some kind of theorems and all. So, among this these a theorem that this is the fifth postulates, which tells us that, a line contents at least 2 points, are that circulars some same radius, whose centers are separated by that radius must interest.

So, this has created problem in all and then it led to different kinds of whether, the angles total sites of the angles of a triangle, angle of a triangle is consider a 180 or greater than 180 etcetera, questions arrows. So, this is 1 thing, which led to thing in a different way; the non Euclidean geometry or making the proofs more regress and all that is also Hilbert was taking up in his axiomatic system. And the other thing is it is; their concerns that mathematics had been built on a not been built on proper foundations built it, built axiomatic system.

So, that led to some kind of paradoxes. We usually mathematics was rested on set theory, but set theory is played by important paradoxes, which was discovered by Russell and Whitehead. And this paradoxes played this kind of axiomatic system, that also led to the development of regards axiomatic system, that is, what you find it in principia Mathematica. There are but in Russell and Whitehead motivation, has come from the paradoxes at arrows in the work. A set theories played with paradoxes at and there was issues in the geometry and then Hilbert Ackermann took geometry into consideration and axiomatic geometry and all. These discovery of paradoxes in informal set theory, cause someone to wonder whether mathematic itself is consistent or not and to look for proofs of consistency etcetera and all.

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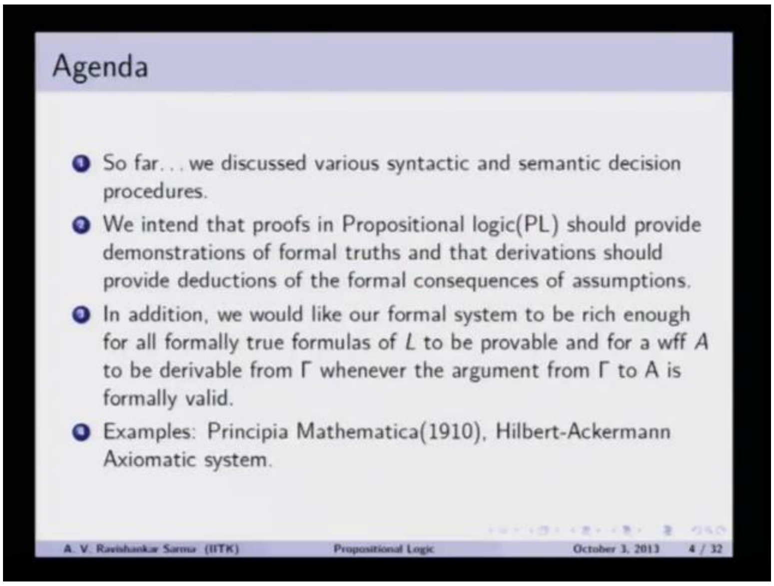

So, there are 2 things which we need to know; the developments of Euclidean geometry, led to some kind of regress kind of axiomatic system, which is little bit different from what Euclid as done, that is, in your proof, if you if your proof has to be regress, all the commonsensical notions direct observations etcetera, should not be ... it might be mistake and all. We might, it might be very well be recuse that, our intuitions might be misleading you know. So, in that sense, everything has to be treated explicitly and then from that, all true things can be derived. So, for that he need to come off with a few axioms has as much as possible and then few transformation rules etcetera. And then you derive all the other true things.

So, this is what we discussed so far, we discussed syntactic and semantic decision procedures. Semantic decision procedures are truth table and semantic tableaux etcetera and all, syntactic procedures, we have natural deduction etcetera and all. And other method added to that, we have is axiomatic propositional logic, where we are not interested in talking about meaning of this formulas and all, but you will be interested in saw the patterns etcetera and all. So, if given the axioms, we will be deriving some kind of theorems.

So, now, in this, this is axiomatic propositional calculus comes under category of syntactic kind of tools and all, which will be talking about in great and detail. So, in this we intend that, proofs in propositional logic should provide demonstrations of formal truths. That means, your deriving all the true propositions from a given set of we can call it as self evident truths or things which cannot be proved etcetera and all, there all axioms. And derivation should provide deductions of the formal sequences of assumptions and all. If that is a case than that what we are trying to do.

Let me see in an action all the true propositions so that, you come across all the valid formulas. Now we are trying to generate proofs for these things. It is not that, they came just like that, but as an outcome of some of the axioms and the transformation rules, you will generate these theorems. In addition, we would like our formal system to be rich enough for all formally true formals of l to be provable. That means, all the true propositions that you come across in your formal axiomatic system, have to be provable, it has to find a proof. We say that, something is true and he do not have a proof that works and all.

So, he need a kind of proof and all the proof propositions that encore’s that, all the true propositions will find a proof. And for a well form formula, a to be derivable from a set of formula gamma, whenever the argument from gamma to a, is usually considered as formally valid. All the true propositions also have to be valid. So, now, these are some of the examples of axiomatic systems; principia Mathematica and Hilbert Ackermann axiomatic system. There may be many other axiomatic systems, but I will be talking about mainly these 2 axiomatic systems.

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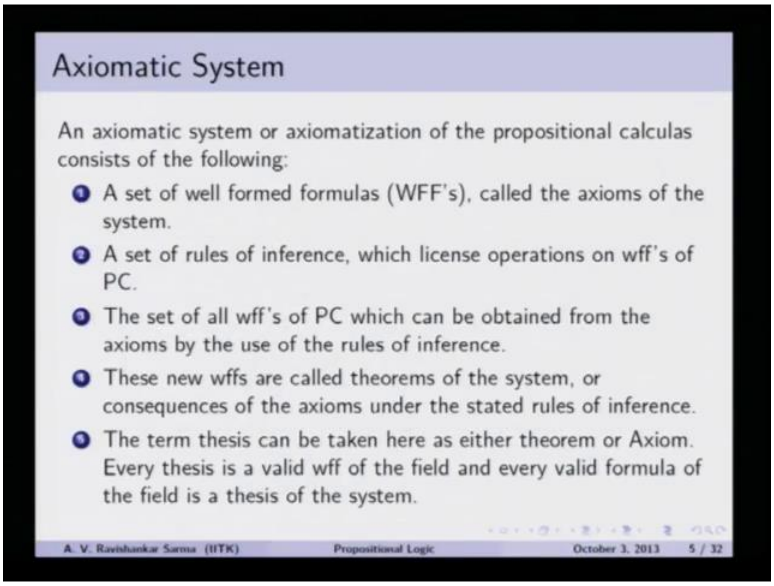

So, to begin with, what do you mean by an axiomatic system. So, an axiomatic system either he can take it as Hilbert Ackermann axiomatic system or principia Mathematica are that is, Russell Whitehead axiomatic system, which is defined as follows. At least these are the 4 file things, which essentially any axiomatic system should consist of. The first thing which is that, a set of well formed formulas are usually called as axioms. These axioms needed have to be proved at all. So, there are considered to be self evident truths which are obviously, true always true.

Then these axioms will also serve as schemas, schemas in a sense that you substitute anything into it uniformly, you will generate a theta. So, that is the greatness this axioms and these things does not require any proof that is why they are well self evident kind of truths. In addition to that, there are some set of rules of inference, which license some kind of operations on well formed formulas of propositional calculus. That means, what even you intend to have as an operation, will not have work and all, but there are some set of rule like rules of inference such as modus ponens, transformation rules such as if you uniformly substitute some formula into the formula, then also you will generate theorems from the axioms.

So, this is a second thing, he need to have transformation rules such as modus ponens etcetera and all. As minimal as possible in any given, axiomatic system, these rules would be very minimal. And the third thing he need to have a set of well form formulas in the propositional logic, propositional calculus, which can be obtained from the axioms, by means of rules of inference, that is, usually even you 1 rule of inference is usually used that is the modus ponens rule.

So, when you have a plus b and a, a gets detached and whatever you whatever follows is b. That is the main rule of inference which we commonly use in all the, in throughout the axiomatic systems, whatever axiomatic system you are trying to considered. The fourth 1; these new well form formulas are usually called as theorems of the system. That means, you started with the axioms and then you transformed those axioms, I would trim those axioms in such way that, by applying transformation rules and truth preserving rules such as modus ponens, you generated another kind of inference.

So, our path is like this that, our parties like each and every steps starts with the truth, that is axiom which is; obviously, true. And then you if you transformation rules on this axioms and that is well form formula is also going to be true. And then that gets transform into another well form formula, it is also going to be true, each step is consider to be true. So, in that sense, the last step of your proof is also going to be true. If, all the step that had their deduction process, is already true in the last step of your proof, which is usually called as theorem which is; obviously, called as a true.

So, establishing the validity is another kind of issue, but deriving kind of true formula from a given a sequence of formulas is another issue. So, lastly that we use particular kind of term, that is, what we call it is thesis, thesis is like some kind of it can be take in as either axiom or it can be even take an as a theorem. So, in order to say that any well form formula is an axiom or even a theorem, sometimes we use this word thesis.

So, every thesis is; obviously, considered be a well valid well form formula of a given field. And every valid formula of that field is also considered to be a thesis of the system. Either it should be an axiom or it should be theorem, all the valid formulas, it should be 1 of the 1 of the thing should be true. Either it should be an axiom, which does not require any proof or it can be proved by a reducing the given axioms by using transformation rules and rules of inference to another kind of theorem. So, this is what we mean by an axiomatic system.

So, in an addition, an axiomatic system should consist of at least set of axioms to start with, either you can have 3 or you can have 4 or you can have 5, but as minimal as possible to start with. And then you apply transformation rules, substitution rule etcetera and all. You transform these axioms into another thing, which are also considered to be theorems. And then you need to have some kind of rule of inference, that is, usually we use it modus ponens. That is all you want to derive all the true proposition that a cursing you are formal logical system.

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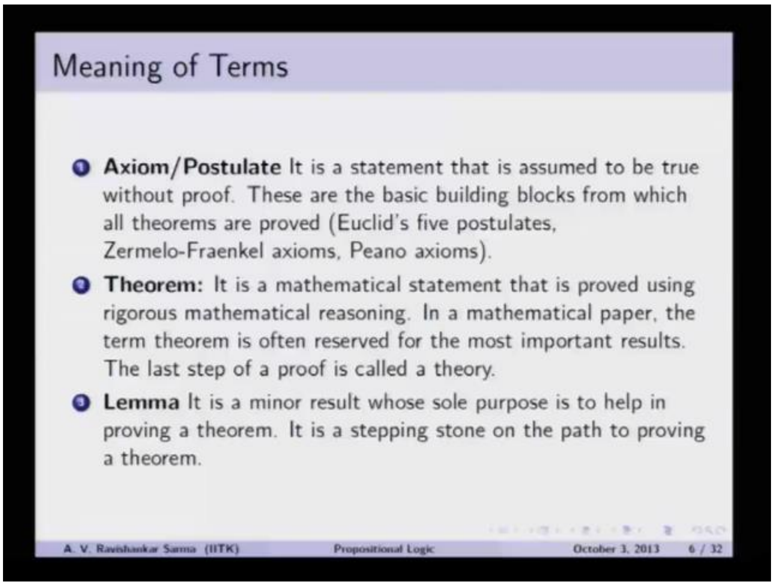

So, now, there are certain terms which need to be defined clearly before moving further, otherwise it will create some kind of confusion. So, the movement axioms, the word axiom, postulates, etcetera, they are 1 other same. Euclid has use this particular kind of face postulates, small or less means the same thing as axiom, that the, that there is what we trying to use in the modern mathematics in particular. The axiom is considered to be statement, usually a statement is consider to be as a true or false, that is assume to be true without any proof and all, it does not require any proof.

For example, we will be presenting some this axiomatic system Russell and Whitehead axiomatic system. Suppose if you says that these are my 5 axioms; that means, any 1 of this axioms cannot be deduced from any other axiom and all. So that means, 1 axiom should not be reduce to another 1. So, it so happen that Paul Burness logician which he was come later, contemporary to Britton Russell invited, he came up with proof of his fifth axiom of Russell Whitehead axiomatic system, can be reduced from other axioms and all. In that sense, it loses his status of axiom, it will become a theorem.

So, when Russell invited formulated their axiomatic system, it has 5 axioms and all. Suppose, if it is proved, any axiom is proved from any other thing, then it is consider to be a theorem, it will lose his axiom at a status. So, coming up with these axioms is the most difficult kind of thing. So, axiom is a sentence which does not require any proofs, they are like self evident kind of truths. In order to start any branch any field of enquiry, either you are talking about physics and mathematics and anything, 1 needs to start with some kind of metaphysical assumptions.

For example, when you was getting your physics in particular, concept of matter and all these things, it will be asking several questions and all; how the matter has coming to existence, then you will date you will go back to some billions of years ago and then you start saying that, was a big bang and then from then due to the big bang and it has resulted in some kind of hot planets etcetera and or to things and all. It is started cooling and all and different planets formed etcetera and all. Or universe has originated by means of big bang and all. So, now, if you go back and ask; how this big bang took place and all, your all silent all in this cases.

So, we assume that, there was a big bang found that, all the universe has originated. So, in the same way, when you are talking about etiology or something like that, you will have your metaphysical assumptions such as, God has created his universe, in 6 days and 7th 8 took rest etcetera. So, you need start with some kind of metaphysical assumptions to do anything and to start any kind of enquiry and all.

So, in the same way is axiomatic system, we need to start with axioms which does not require any proof and all, their self evident kind of statements which are; obviously, true.

It is like 2 plus is equal to 4 which can we question and all. Of course, 1 can even question the foundations of any area and all you can start questioning the axioms, but usually it is the case that, they are assume to be true without any proof. So, these are also considered to be basic building blocks, from which all the other things can be constructed. Just like you are constructing a building, you need to have a bricks. And bricks are arrange in certain way and you will be nice structures and all. Example could be Euclid's 5 postulates Ermelo Fraenkle axioms or Pianos axioms in arithmetic etcetera, Ermelo Fraenkle axioms in the set theory.

So, now, what you mean by a theorem. A theorem is a mathematical statement that is proved using some kind of rigorous mathematical reasoning that is what very spoke about it in an extensive detail, that is, a deduction. A deduction is considered to be a, deductive argument is considered to be an argument in which, if you a premises are in certain, their absolutely true etcetera and all. Shall we what we mean by deduction. In a mathematical any mathematical paper that you come across, the term theorem is often reserved for the most important results.

It is considered any mathematical research paper and all, yes often considered as 1 of the important results that is considered to be theorem and that needs to have a proof. In mathematics, if you say that I have come off with the theorem. And then if I do not show, I mean that it is it can be proved in all that is that is levels are with a theorem. You need provide a regress proof for whatever result that you are trying to talk about in a mathematical paper.

So, in the last step of your proof is what is considered to be a 2 statement that is considered to be a theory a theorem. So, now, most of the time, you will be using these words lemma, whenever you reading some kind of mathematical mathematics books and all, you will be seeing all these axioms, theorems, lemmas, corollaries, etcetera and all. What do you mean by lemma? It is a minor result, whose sole purpose is to help in proving a particular kind of theorem, their supporting kind of theorems.

So, it is a stepping stone on the path of proving certain kind of theorem. This is a minor kind of importance step that you will be taking in proving the major theorems. So, that is what we call it as lemma.

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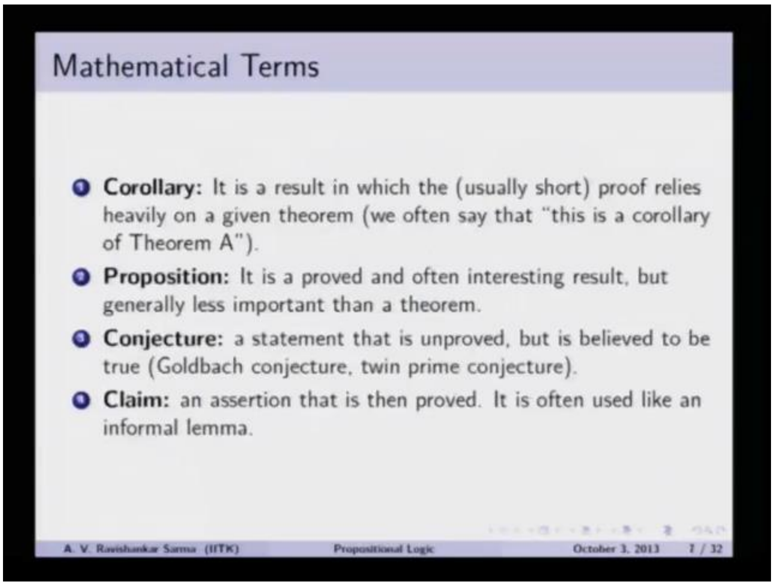

And there are some other term which are also very important. So, they are like this. So, what this is the terms which you commonly come across when you are reading a mathematical paper or any other mathematical text book. It corollary; a corollary is an important result, which the proofs relies heavily on a given theorem. So, we often say that, this is a corollary of a theorem A etcetera and all. This is a some kind of thing which comes as an outcome of your particular kind of theorem.

So, this is what is called as a corollary. And proposition is the 1 which we have been discussing right from the beginning of this course, that is, any sentence which is usually considered to be as a true or false, that is a called as a proposition. So, it will, it can also in ma in the context of mathematics, it is proved and often interesting result, but generally less important than a theorem. A theorem is consider to be the most regress kind of a statement and all in your enquiry. And of course, axioms are all which does not require any proof and all, there absolutely true and all which does not require any proofs.

So, now, the other term which you come across is the conjecture. So, conjecture is considered to be a statement, that is yet to be proved and all that is unproved as of now, but it is believed to be true and all. So, there are many such conjectures, which are our gut feeling says that their; obviously, true and all their all true statements, but as of now they find any proof. In a same way when Fermat has propose Fermat’s last theorem, he took almost 2 to 300 years to prove as from as last and all, till that time it remain as a conjecture, till somebody some mathematician as come off with a regress proof the Fermat’s last theorem.

Conjectures are yet to be proved kind of theorems and all we are or that feelings tells as that they are; obviously, true, but still I did not find any been a position to find a regress proof for that. So, this is what we called as a conjecture. So, all these conjectures can be refuted or may be can be proved also. So, till we find a regress proof for this conjecture, we will not accept it as a theorem. But all conjectures may be potentially capable of becoming theorems, once it finds regress proof. The last thing that we will be using in the in this context is the claim. A claim is an assertion in any mathematical paper, you will be coming across this particular kind of thing.

So, in any research paper etcetera and all, you will be claiming something. So, that is considered to be an assertion that is then; obviously, eventually it is call to be proved. It is slightly difference from the conjecture. Conjecture is the 1 which all gut feeling let us that so; obviously, true. It is often use like some kind of informal lemma, it is not lemma as such, but it is considered to be a kind of informal kind of lemma. So, that is what is called as a claim.

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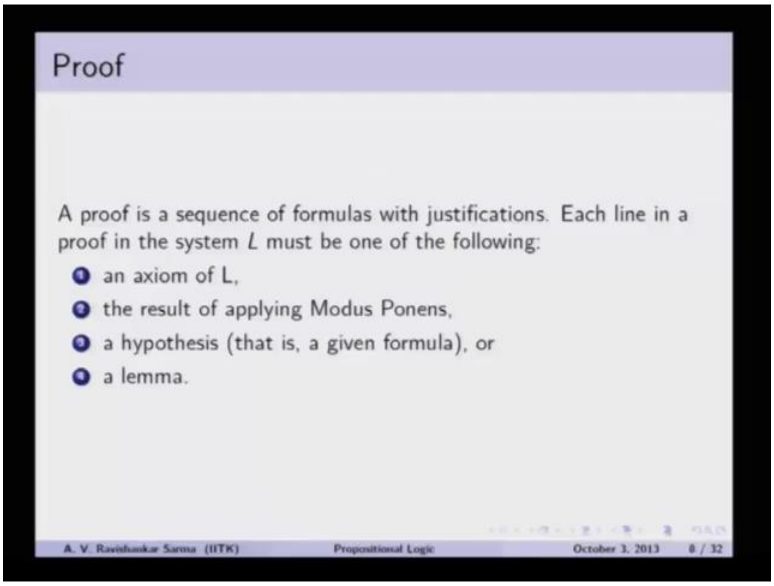

So, now, given this set of key words etcetera and all, what is that we are essentially trying to do, this is that, we are presenting 1 particular kind of axiomatic systems, which is in which, the proofs are considered to be regress, as well as there are no other kind of hidden kind of assumptions, which will become part and parcel your proof. I mean just as in the case of Euclidean axioms, there are certain extra assumptions etcetera, which has come, which also played a crucial role, in formulating the proofs within the geometry. You want to be you want avoid such kind of thing. So, our proof have proofs have to be regress.

So, what do you mean by a proof. So, now you have axioms, theorems, lemmas corollaries, conjecture etcetera and all claims propositions. Now, with using this things what do you mean by a proof. So, usually a proof is considered to be a sequence of formulas. And all these formulas, will find some kind of justification. So, each step needs to be justified by some kind of a statement, that is, either a whether you used modus ponens rule or used transformation rule or it has come from axiom or it has it is just another kind of theorem, all these things needs to be addressed.

So, each line of the proof in the axiomatic systems L, it should have this particular kind of things. So, in a proof, either that particular kind of thing has to be an axiom; that means, it does not need any proof and all. You just write it and a like that, you say that that is an axiom, that itself is a since they does not require any proof and all. So, that itself will be some kind of thesis and all, thesis is the 1 which have introduced earlier. In that context, we say that it is a theorem, it is as well as an axiom also because, the last step of your proof is considered to be a theorem. So, if you are given axiom and all, it would not require any proof and all then; obviously, it is considered to be a thesis. So, that is an exceptional kind of thing and all it; obviously, tautology etcetera and all.

So, an axiom does not require any proof. Suppose, if you with the proof that, 1 if you lose its axiomatic status. So, that is a first thing that you need to have. And the second thing is that, the next step whatever step at you are trying to considered, whether or not it is a result of a playing some kind of transformation, a playing some kind of rules of inference. 1 rule of inference that we will commonly using is the modus ponens rule, that is, a a in plus b and b follows from that particular kind of theorem. Or you can begin your proof by assuming some kind of thing, like it can be considered as a hypothesis, which is already part of you the given formula that need to be prove.

So, we discussed in the context of natural deduction method. Suppose, if you are trying to prove a in plus b b in plus c and a in plus c. So, now, what you will do is; you will list out all the hypothesis a in plus b b in plus c. And in addition to that, you will assume the antecedent of the conclusion that is a. And from these 3 whether or not c can be reduced is the 1 which we are trying to see. Either it should be a hypothesis which comes from the given formula, or it must be some kind of a lemma, which is supporting kind of theorem. It is also considered to be true.

So, our journey starts from truth, that is, a axiom and trans apply transformation rules to it, it is also true. And then we are using truth preserving rules, such as modus ponens, etcetera and all. It is a deductive kind of reasoning. So, that is also obviously, absolutely true, etcetera and are each step is considered to be truth. So, that is why our journey ends with the final step of your proof that is considered to be theorem.

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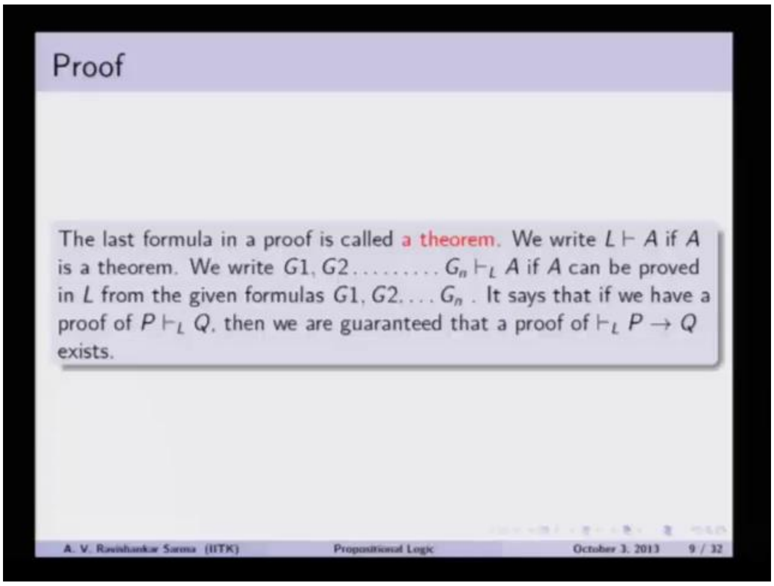

The last step of your proof is usually considered to be a theorem. So, you need to know that, 1 can come to the destination in finite steps and all. Some sometimes you can come of with 15 steps to find the final kind of reach the destination, that is, the theorem. Or sometimes you might prove the same thing, within some 10 steps and all. So, the proof which consist of 10 step if; obviously, considered to be a regress kind of or effective kind of proof, when compare to a proof which consist of 15 steps are made more in that.

So, as far as possible or in excessive information should not be there to maintain information economy. In order make this proofs effective, your proof has to be, the steps of your proof has to be as minimal as possible. So, when we write this thing, L single tungs10 A; that means, A is considered to be reduced from l. Since, A is considered to be the last step of your proof. So, he is called as a theorem.

So; that means, we write G 1 G 2 G n etcetera, this is a conjunction of all these formulas. Usually it can be called as premises etcetera and A is considered to be the conclusion. And that is reduced within the formal system L, if an only if A finds a proof in a given formal system from a given set of formulas G 1 G 2 G n etcetera. It says that, if you have a proof of this thing q from p then we are guaranteed that we have a proof of p in plus q a well.

So, this is a sought deduction theorem which was later introduced by her brand etcetera and all; so will talk about particular kind of deduction theorem little bit later. So, what essentially we are trying to do is from q if q is obtained from p, then you discharge you assumptions like p q etcetera and all. And you will start of new word p in plus q, rather than just p and just q.

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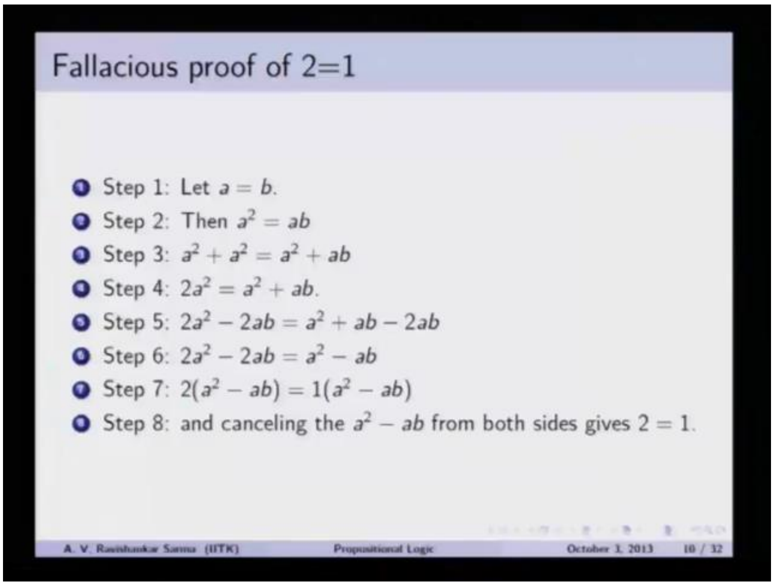

So, now, there are in the elementary, I mean off course if you go back to our elementary schooling etcetera and all. In the, in our childhood, we might have derived certain proofs which are considered to be bogus proofs and all. But it might have convinced us at that stage. So, let us assuming, let us assume that whether or not is constitute to be genuine proof or it is a bogus proofs or it require some kind of fixing etcetera and all fixing the problem that arises in 1 this steps and all.

So, what is considered to be a proof? So, for we have discussed that, each step has to be true and it should find a justification. If you have justification is wrong, then you cannot move to the further step. So, let me some there is proof is considered to be defective or some problem with the proof. So, now, observe this particular kind of proof to be equivalent to 1. So, for that, you start with some assumptions, let us assume that a is equivalent to b, then you multiply a both sides, then the second step you will get a square is equal to a b. Now you add a square to both sides and all a lecture n r h s. Then it will become the third step a square plus A square is equal to A square plus a b because, you have added a square to the both sides a lecture n r h s.

So, now, then the step 3, a lectures will become 2 a square equivalent to a square plus a b. So, now, you subtract minus 2 a b both sides, this is what you get. So, that is 2 a square minus in the 6 step 2 a square minus 2 a b and a square minus a b. So, now, you take 2 common of these things, 2 into a square minus a b and 1 into a square minus a b.

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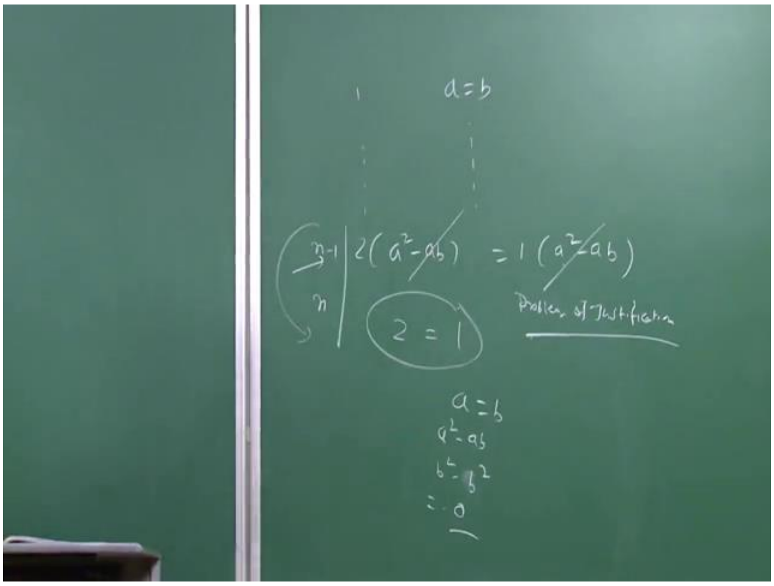

So, it is like 2 into a square minus a b is equal to 1 into a square minus a b. So, this is what you got it from a is equal to b. All the down, you got this particular kind of step. Till here, it does not seem to be little bit problematic and all because we are started with all these assumptions etcetera and all we cannot just it done anything etcetera, except that we are added subtracting both sides etcetera and all.

So, now, the problem real problem comes from is that, the movement you cancel these 2 things then suppose if you say that. So, this is the n minus 1 step and this is the n step, you cancel a square minus a b both sides and then you will infer that, 2 is equal to 1. You can say that a square minus a b and a square minus a b it will become 2 is equal to 1. But one of the important principles and the logic is that, you cannot cancel in such a way that, in a square minus a b from both sides because, when especially when a is equal to b, then a square minus a b usually it will become... So, when a is equivalent to b, it is substitute for a b and all becomes, b square because it is b into b square. So, now, it will become 0. So, 0 cannot be cancel with another 0 and all.

So, it is in that sense, our principles of mathematics will not permit us to move from this, it do not allow us to cancel this particular kind of step and all. So, now, this will not a lead us to the next step because, reverse from problem with the justification. Our justification is defective and all because, 0 cannot be cancel by 0. So, that is not permitted in at least in the principles in the arithmetic.

So, that will not let us allow that 2 is equal to 1. Of course, if you do not notice it properly and all, the side line this particular kind of thing, it appears that you know it appears to be a wonderful or nice proof and all. But you can you can stop it step number 7 and you can question; how did you cancel a square minus a b both sides and all 0 cannot be cancel with 0, 0 cannot be cancel with 0.

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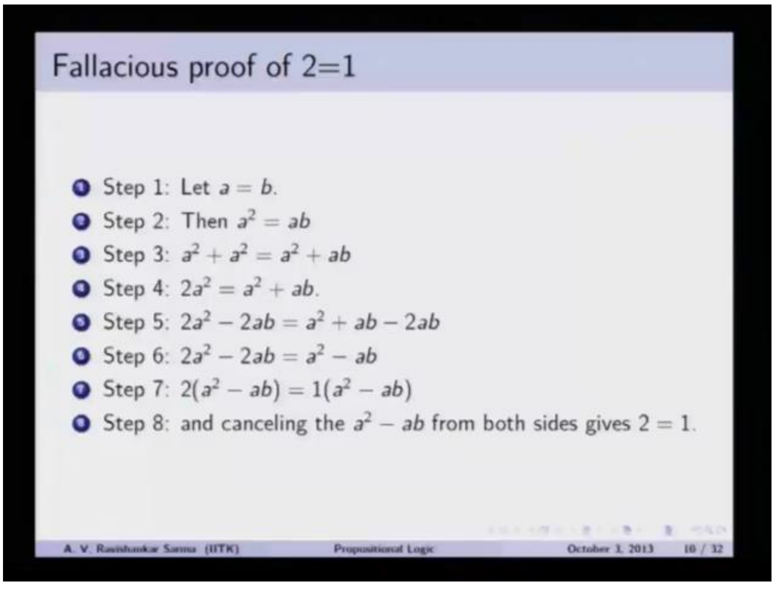

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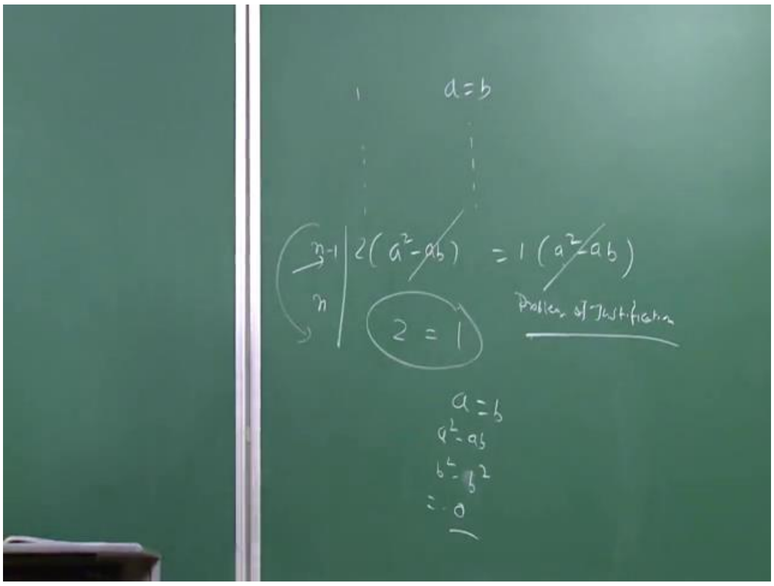

So, this step is not allowed. So, that is the reason why this itself the proof itself starts stops here and then you need not have to this particular kind of thing. So, now, each step of you proof needs to be justify. So, here there was a problem with justification, it goes against the principles of dramatics. So, you will not generate a proof for 2 is equal to 1. If, you generate a proof for 2 is equal to 1 and then say that this is your effective proof and all, then this is considered to be an effective proof, it is a bogus kind of proof.

So, then the next question that comes to us is that, what constitution effective proof. So, as it appears that, from this particular kind of thing, what we can length from this particular kind of proof is that, if you have proof has to be effective etcetera and all, each step should find a justification. That should be according to the principles of logic etcetera. In the same way, when we are talking about axiomatic systems, which is based on set of axioms and transformation rules etcetera, that thing has to when each step needs to be justified, either by using the axiom or transformation rules applied on a axiom or it should be, it should result in from by applying some kind of modus ponens principle or I mean any one of these things should be there, then on needed the next step is going to be justify.

But here the seventh step is a problematic step, your proof end there itself. And you can you can clearly say that, 2 cannot be equivalent to 1.

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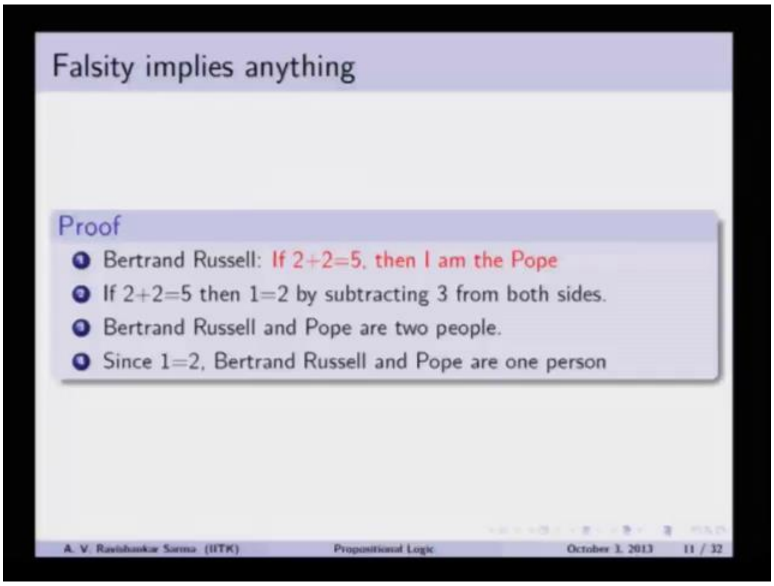

So, there are certain things which are very important. Mathematicians would always be interested in starting with the tautologies; that means the 2 statements. And mathematicians will never begin with contradiction, like contradiction is a 1 a sentence, which a can be spoken as both true and both false, x and not x is kind of inconsistent statement is unsatisfiable also considered to be contra dictate to each other p and not p's contra dictate to each.

So, why mathematicians thing that, this the contradiction are going to be considered to be as a hell. So, here is 1 of the interesting and funny kind of for example, which is given by a famous mathematicians, again they cannot since, we are talking about Russell Whitehead axiomatic system. So, this example is also due to Russell and Whitehead Bertrand Russell. In 1 of the parties which he attended, he funnily proved that, a contradiction impress anything. So, that is the reason why you know 1 mathematician’s will always be loving this tautologies etcetera and all, from true you generate true’s etcetera and all.

But when you start with the contradiction, you can derive anything. So, this is what is the thing, you just gives a funny kind of proof. So, that is like this is what better in Russell says. If 2 plus to 2 is equal to 5, that is false statement, he false statement impress anything, then he is trying to prove any strange kind of proposition, that in all that is going to a p o, he wants establish himself that if 2 plus is equal to going to be 5, then is going to establish that I am going to be a p o I means in better.

So, now, this is the proof that he tries to give, it is a funny kind of proof and all this is not be take an very seriously and all. But usually, mathematicians would love to start with true's tautologies are rather than the contradictions, it is contradiction leads to help. So, now, the proof goes like this; if 2 plus 2 is equal to 5 then if you subtract 3 from both sides, so that is like this. First you take the antecedent for this particular kind of thing. So, now, our assumption is that 2 plus 2 is equal to 5. So, now 2 plus 2 minus 3, you substituted minus 3 from both sided subtract are substitutes subtract 3 from both sides. So, this is what you get.

So, now, 4 minus 3 is 1 and this is equivalent to 2. So, this is the proof and all. So, this is your hypothesis 2 plus 2 is equal to 5, you assuming that that is true. If that is true, then the next step 2 plus 2 minus 3 5 minus 3 is also has to be true. That means, 1 is 2 is also true. Now, the third step is that, is funny he has used it, it is not a regress kind of proof and all, it is the only is this is this proof is only useful sake of fun. So, now, Bertrand Russell says that, Bertrand Russell and pope are 2 different people.

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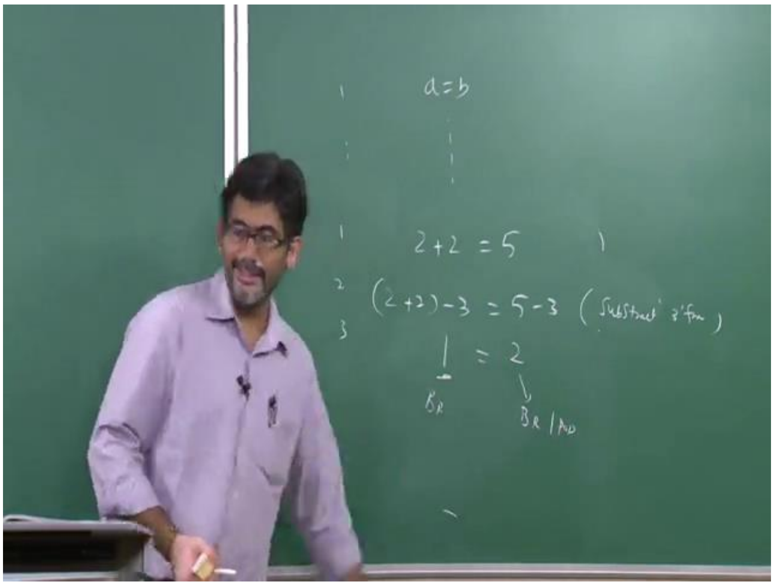

Bertrand Russell, suppose 1 is taken as Bertrand Russell and 2 means that, Bertrand Russell and pope are 2 different things. So, now, this equation tells us that, Bertrand Russell and Bertrand Russell and pope in different way, there 1 are the same. Since, 1 is already equivalent to 2, Bertrand Russell, Bertrand Russell and pope also has to be same. Even if there is a another person exist and all, pope the 2 has to be equivalent to 1.

So, it is in that sense Bertrand Russell and pope are considered to be 1 person so; that means, he has to be none other than pope himself. So; that means, you proved that he is the pope. So, this is a funny kind of example and all, so which this is come from falsity in place anything. So, this is the same thing which can be proved in classical logic in a different way, by using the principles of logic. So, that is a reason why we begin with contradictions in a given proof.

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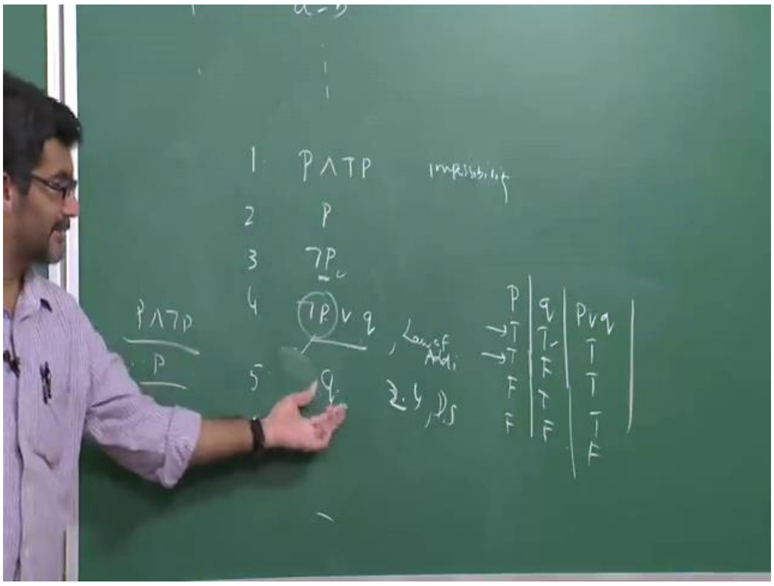

For example, you would start with P and not P. So, this is we know that it is an apparent contradiction, a statement cannot be both true and both false. So, now, this is what is given, if you assume that this is true and even this are also true because, of this particular kind of principles. So, this eliminate this conjunction, than p will be come true and in p and not p not p is also going to be true. So, what we have done is, we have eliminated this conjunction. So, this is what is called as impossibility are something which is I contradictions.

So, now, fourth 1; since not P is already true, this is also always considered to be true, then irrespective of whether or not q is true or q is false, the whole statement is going to be true because, of this particular kind of formula. This we know that, semantics of dissection is always that if it is F and F T F T F and all, then it is going to be false only in this case in all of the cases is going to be true.

So, now, we know that this is already true, so that means, these are the 2 things which you need to take into consideration instead P we have not P here. So, now, irrespective of whether q is p whether q is false the p or q is going to be true only. So, in that sense, this statement is also true. So, now, in the fifth step, you have p here not p are q now.

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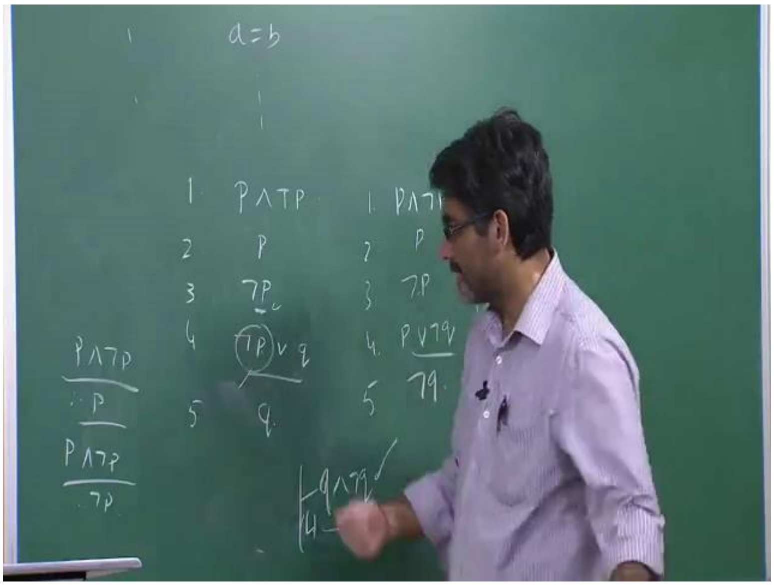

So, these 2, so how did we come to this 1, this is law of addition. So, now, this and these 3 and 4 2 and 4 disincentive syllogism leads to this 1 q so; that means, any particular kind of strange proposition 1 can prove. So, starting from the contradictions, suppose if you say that it is raining and it is not raining and this statement is going to be pics lies and all, q is considered to be pics lies. So, this is 1 way of proving this thing.

So, now, the problem here is that, we can prove either q and even you prove not q also. So, how can you get this not p? Again the same proof. First step you take the same assumptions and 4. So, instead of this 1, what we try to do is; since p's already true, you can add any strange kind of proposition q and this is also going to be true. So, now fifth 1. So, this is law of addition as is the case of this particular kind of thing. So, now, observe these 2 3 and 4 disincentive syllogism, will the 2 not q.

So, now, with the contradictions you proved not q and you with the contradictions even you proved q and all. So, that is makes your system inconsistent, since you are derived q and not q is part of your formal system, whatever system you are trying to talk about l. So, it is in that sense your formal. Logical system is going to be inconsistent. So, that is what we are trying to avoid. So, this eventually led to different kinds of theorems, which are little bit counter into like paradox of material implication, which we talk about it after we introduce Russell Whitehead axiomatic system. So, now with this particular kind of note will try to end it. And the all other proofs etcetera, we will try to do it in the next lecture.

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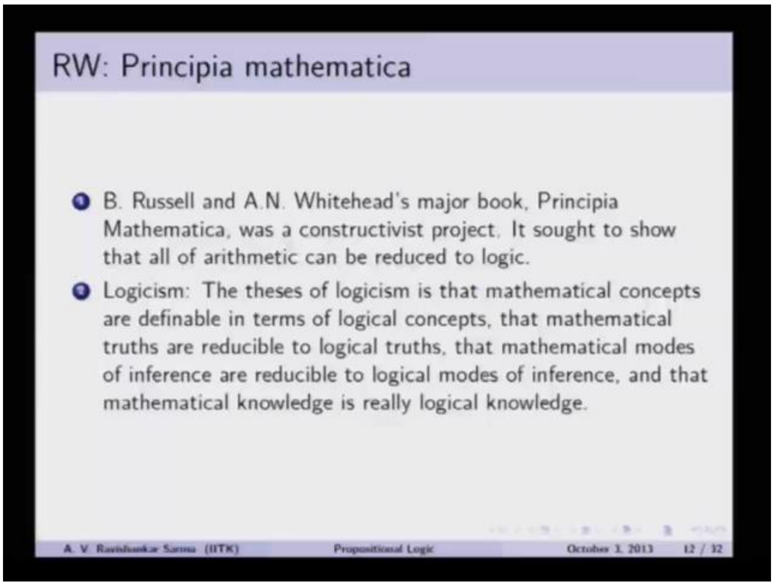

So, now, what is that Russell and Whitehead as achieved in the book principia mathematica, lots of things which he did, it is voluminous kind of work and all, which in which consist of 3 books arranging from 400 to 500 pages, each book has 400 pages. So, Bertrand Russell and White in the book principia mathematica, which is considered to be path breaking kind of book, 1 of the greatest books of this twentieth century. In that, it is kind of some kind of constructivist project, constructivist proofs etcetera. It show that, all arithmetic can be reduced to logic. That means, there are 2 things which are important here; when whenever you talk about some statement in the arithmetic; that can find an appropriate translation in the logic. That means, if you if you utter any statement in the arithmetic, it will have its corresponding language, you can discuss everything with the help of only this axioms transformation rules and other things.

So, this grand program is what is considered to be a program, which is called as the logic. So, logicism is like this. So, the thesis of logicism is that, all the mathematical concepts are definable in terms of logical concepts. So; that means, you express any kind of plus operation or anything in the arithmetic etcetera, that as it is corresponding logical operation. That means, it is in that sense, you can talk all the mathematical truths in terms of logical truths. Logical truths are axioms etcetera and all.

So, any kind of mathematical statement can be converted into appropriately into a corresponding logical statement. And all the mathematical modes of inference are reduce to logical modes of inference. And it is that sense, all mathematical knowledge is nothing, but a real logical knowledge. So, the basic idea here is that, mathematics is a branch of logic, are all the mathematical concepts can be reduced into the concepts of logic and all. For example, if you, if you cover with and a axiomatic system which consist of only set of axioms and transformation rules and the rule of inference, as minimal as possible rules are as minimal as possible. And then you start talking everything, in the language of this axiomatic system.

So, what is there in our language of axiomatic system with respect to principia mathematica? We have some Russell and Whitehead has 5s axioms. And then there is a transformation rules and substitution rules and then the modus ponens principle. That is it, that is all we have in all the truths of mathematic arithmetic etcetera, they can be properly translated into this is one of this axioms and all. All the p in plus q in plus etcetera, will become statements of arithmetic and all.

If you can do that particular kind of thing, then it attains his regret because, whenever you are proving certain kind of theorem, such as simple theorems like p in plus p or etcetera and all, that can be this p’s q’s r’s can be some the statements in mathematics. So, now we are generating all the mathematical theorems; we are trying to show that, they will come as a logical theorems and all. So, mathematical knowledge will now tends out to be a logical knowledge because, we are using only axioms and modus ponens etcetera and all.

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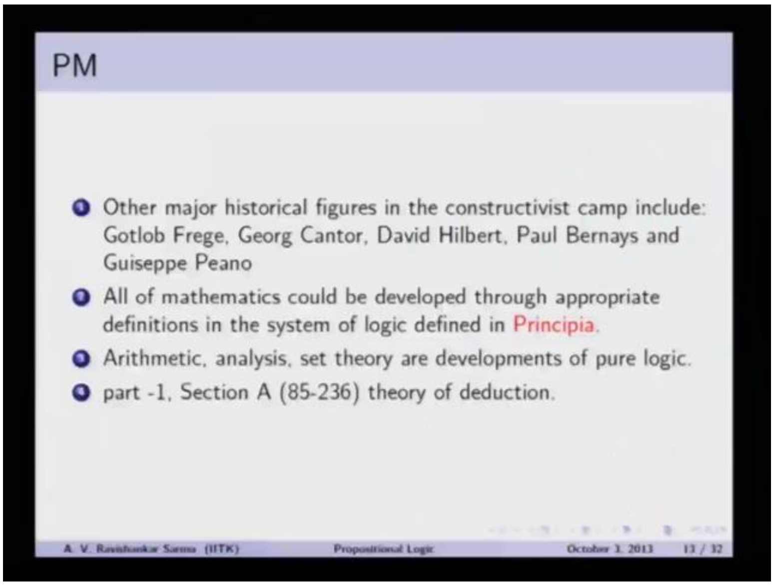

So, this is what we do in the principia mathematica. So, mathematics can be propelled reduce to logic. So, now, there are other things which Russell and whitehead is trying to achieve. The other major historical figures in the constructivist camp, so how did they construct? They constructed various kinds of theorems on few set of rules, which are few set of axioms rules and the transformation substitution rules a lots of other axiomatic systems which are already there.

Before Russell invited you have Frege's axiomatic system, in cantor and off course after that David Hilbert were trying to axiomatic geometry and he has come on with axiomatic system David Hilbert’s axiomatic system. And Paul Barnes especially, he showed that 1 of the axioms of Russell Whitehead, can no longer have that particular kind of status. We are different axiomatic systems and piano is arithmetic, also comes under this particular kind of category. All the mathematics should be developed through appropriate definitions in the systems of logic, defined in ... If you can achieve that particular kind of task and all then your set have you have reduced mathematics to logic. In the sense, mathematics is considered to be a branch of logic.

So, all the arithmetic, analysis, rather things which you commonly come across in the mathematics; set theory etcetera, if this can be reduced to a set of concepts of logic, then it can be called as developments of pure logic, rather than the development of mathematics. So, in the next class what will be doing is; we will be focusing our attention on the main book were principia mathematica of a due to Bertrand Russell and Whitehead, were which is a chapter specific chapter on deduction, where using his set of axioms, set of axioms, he showed that for example, of contradictions or no contradictions, law of excluded middle, law of identity etcetera, can be derived from this set of axioms and all.

So, you should note the we should note that, any formal axiomatic system that you are going to come off with, at least you know we are to ensure that, 1 should be in a position to derive if the minimal things such as, law of excluded middle, that is a sentence is either true or false or law of identity such as p in plus p etcetera, and all. All these things should come as theorems in your axiomatic system.

In the next class, using the principles, using the axioms of Russell and Whitehead, will be proving certain important theorems such as law of identity, law of excluded middle, law of contraposition, all these things are valid theorems. So, now, what is that we are trying to achieve in an action, this is that, we know that certain kinds of valid formulas existing our formal logical system. So, now, we are trying to find a regress proof for his valid formulas, valid well form formulas. So, that is what we are trying to achieve and this will constitute part of the proof theory.

So, once we introduce this axiomatic systems, then we will discuss about whether or not your when a given axiomatic systems is going to be consistent; that means, when you will be in a position a whether or not you are in a position to derive both x and not x. If you are derive, if you are able to derive x and not x as your theorem, in your axiomatic system, that axiomatic system is going to be inconsistent or we will be talking about all the valid formulas whether it finds proof etcetera and all.

So, that is all the true statements are provable and all the provable statements are true that is a soundness property or whether you are a formal axiomatic system is going to be complete etcetera and what are the limitations etcetera. All these things we are discussing the forthcoming classes.