Partial Differential Equations
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Lecture – 5.1
Qualitative Analysis of Wave Equation - Parallelogram Identity

Welcome to this lecture starting from this lecture, we are going to study a qualitative analysis of wave equation. So, far we have done the quantitative analysis for the wave equation namely we have solved Cauchy problems initial boundary value problems associated to the wave equation. So, in today's lecture we are going to discuss a special property of solutions to wave equation in 1 dimension it is known as parallelogram identity.

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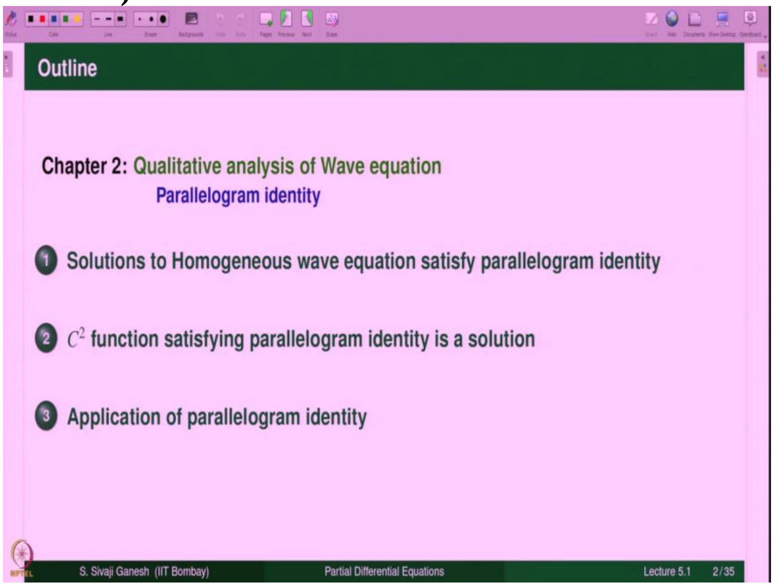

The outline for today's lecture is first we show that, solutions to homogeneous wave equation satisfy a parallelogram identity then we show that C 2 function satisfying parallelogram identity is need a solution to the homogeneous wave equation and we apply parallelogram identity and solve a few problems.

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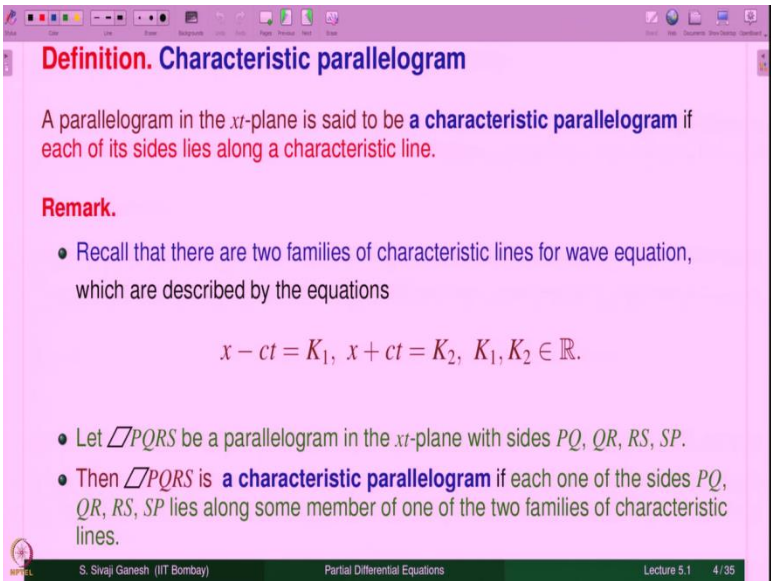

So solutions to homogeneous wave equation satisfy parallelogram identity. Definition of a characteristic parallelogram, a parallelogram in the xt-plane is said to be a characteristic parallelogram if each of its sides lies along a characteristic line. Recall that there are 2 families of characteristic lines for wave equation they are x - ct = constant and x + ct = constant these are the 2 families.

So let parallelogram PQRS be a parallelogram in the xt-plane with sides PQ, QR, RS, and SP then parallelogram PQRS is a characteristic parallelogram, if each one of the sides PQ, QR, RS, SP lies along some member of one of the 2 families of characteristic lines.

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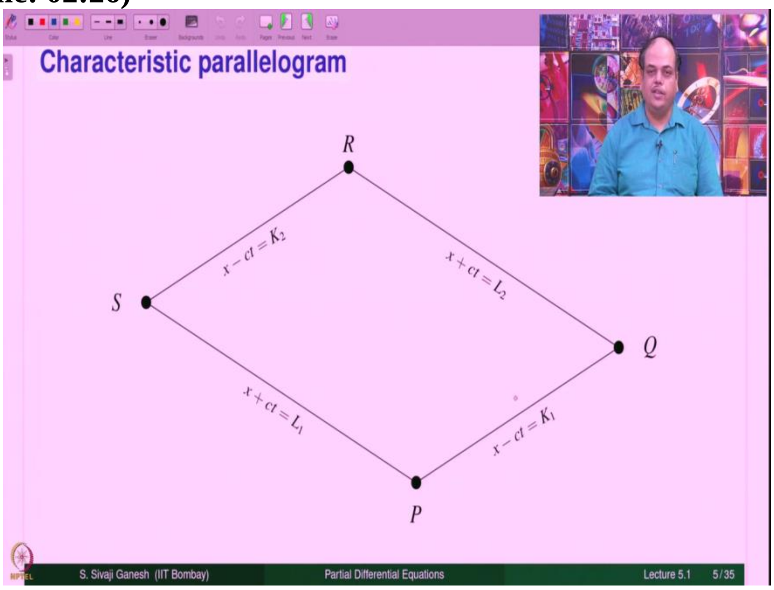

The picture is here P Q R S the side PQ lies on this line x - ct equal to constant, QR lies on x + ct equal to constant, RS lies on x - ct equal to constant and SP lies on x + ct equal to constant. So, this is a characteristic parallelogram because each of its sides lies on some characteristic line. So, we have this theorem. Suppose PQRS is a characteristic parallelogram with the line segments PR and QS as its diagonals they just to fix this kind of a picture PR and QS are diagonals.

So in principle, Q can be here and S can be here, but we are going to say that without loss of generality, let us assume PQRS are described in this anti clockwise manner, the vertices are the parallelogram after all these only a description naming.

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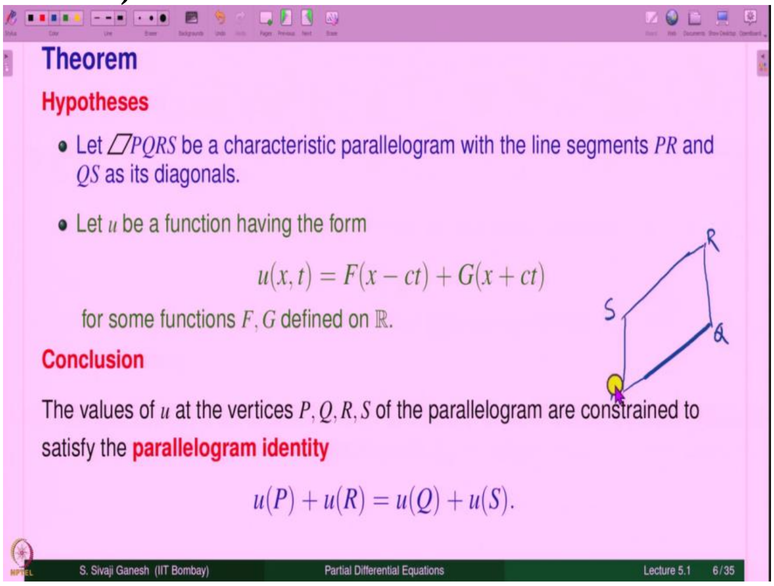

Let u be a function having this form u of x t = F of x - ct + G of x + ct for some functions F,G defined on R. So no assumptions on F and G. What all we need is F and G are just functions define on R, then this automatically defines a function on R 2 u of x t, for x t belongs to R 2 conclusion is the values of u at the vertices P, Q, R, S of the parallelogram that is a characteristic parallelogram, they satisfy the parallelogram identity u of P + u of R = u of Q + u of S. So this is how the characteristic parallelogram looks like. So u of P + u of R = u of Q + u of S.

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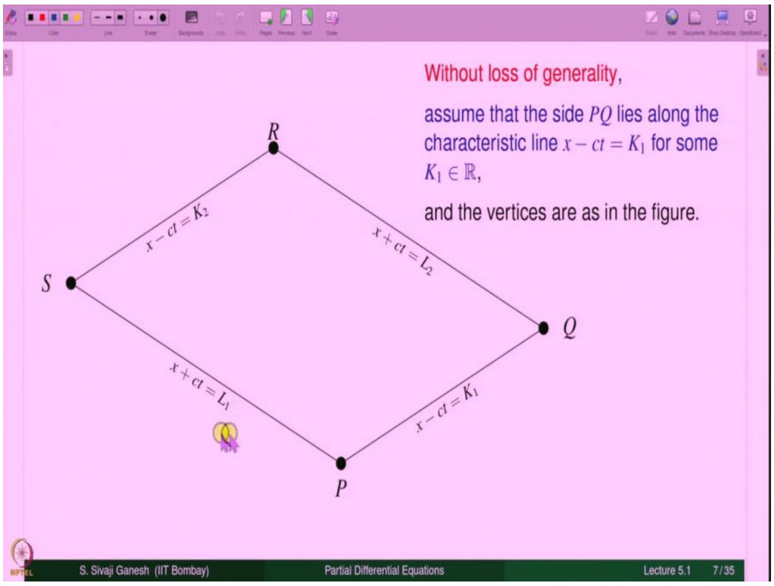

So without loss of generality, assume that the side PQ lies along we have to set up some notations. So PQ lies along this characteristic line x - ct = K 1 on the vertices are as in this picture, namely, they described in this anti clockwise manner just to set up notations and therefore, QR lies on some member of the characteristic lines family or of course, it has to be from other family x + ct = L 2, there is some number L 2 there is some number K 2 such that RS is along this line x - ct = K 2 the number L 1 such that SP lies along x + ct = L 1.

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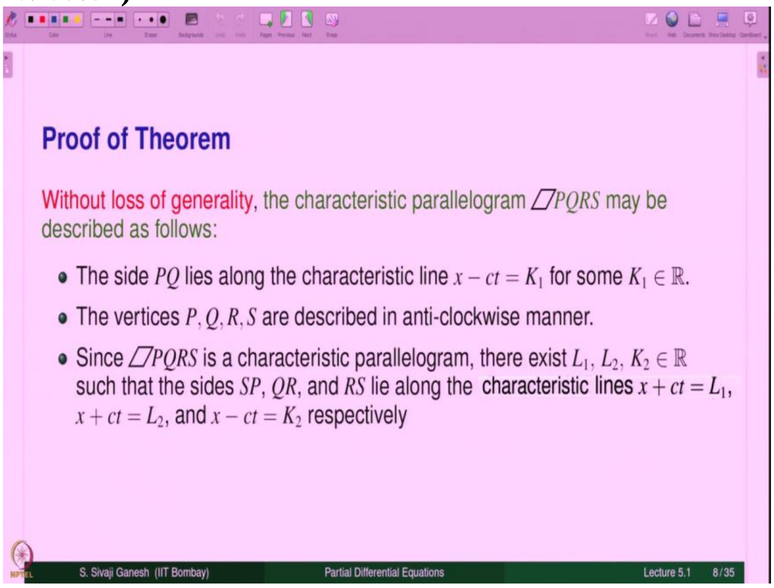

That is what precisely we are assuming. So, without loss of generality, the characteristic parallelogram may be described as follows, the side PQ lies along the characteristic line x - ct = K 1 for some K 1 the vertices are described in the anti-clockwise manner PQRS is a characteristic parallelogram. Therefore, there are characteristic lines along with the sides of PQ RS lie. In other words, there are numbers L 1, L 2, K 2 such that the sides SP, QR, and

RS lie along the characteristic lines which are described here; we already saw this in the picture.

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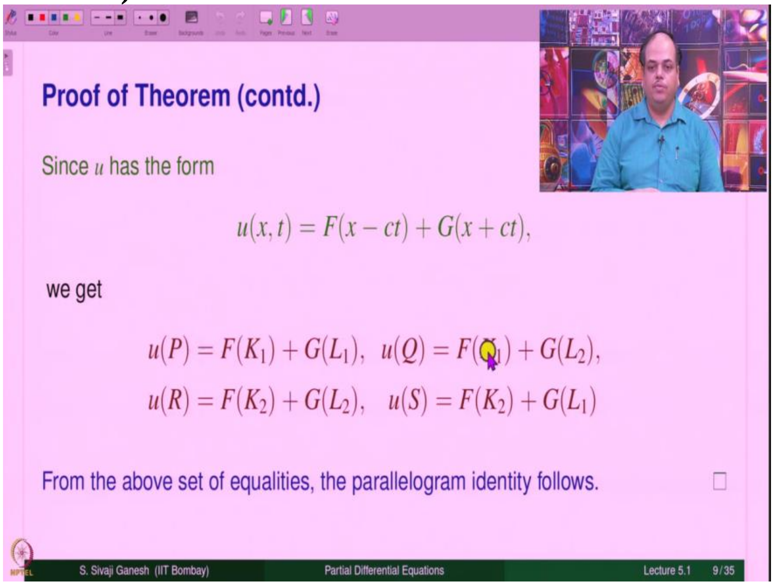

Since, u has this form that u of x t = F of x - ct + G of x + ct we get, u of P = F of K 1 + G of L 1 because P lies on x - ct = K 1 and x + ct = L 1. Similarly, u of Q = FK 1 + GL 2, u of R is F of K 2 + G of L 2 and the FS = F of K 2 + G of L 1, from the above set of equality is the Parallelogram identity follows, you can easily check that u of P + u of R = u of Q + u of S.

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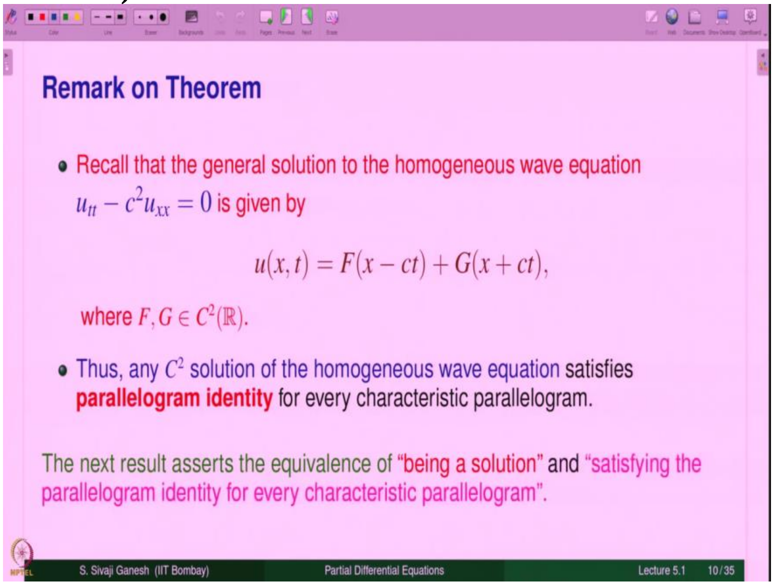

So, as a remark on a theorem, recall that the general solution to the homogeneous wave equation u tt - c square u xx = 0 is given by u x t = F of x - ct + G of x + ct where F and G are, C 2 functions defined on R. Therefore, any C 2 solution of the homogeneous wave equation satisfies parallelogram identity for every characteristic parallelogram recall parallelogram identity stated only for characteristic parallelograms. So, the next result asserts the equivalence of being a solution to the wave equation homogeneous wave equation and satisfying the parallelogram identify for every characteristic parallelogram.

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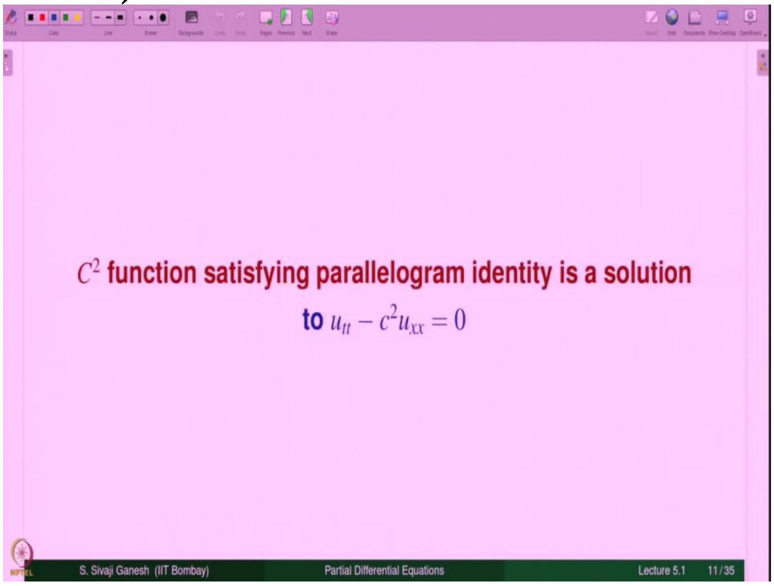

In other words, C 2 function satisfying parallelogram identity is a solution to the homogeneous wave equation.

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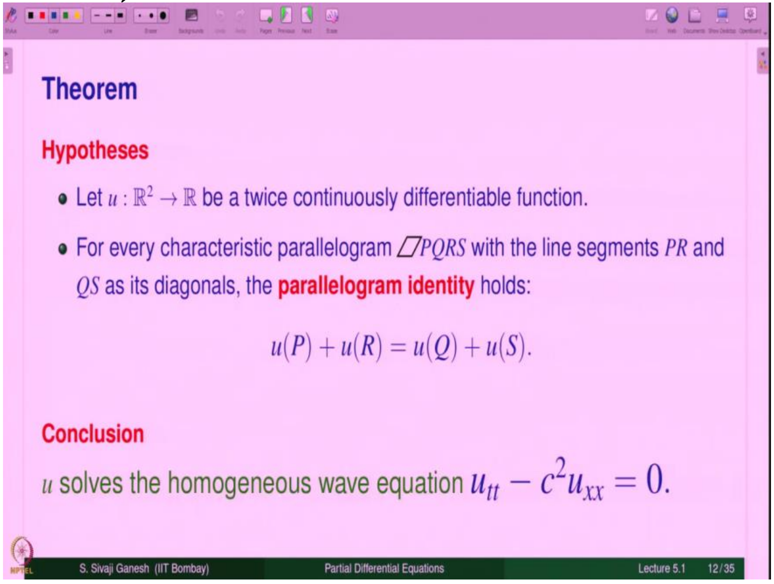

So let u from R 2 to R be a twice continuously differentiable function for every characteristic parallelogram PQRS with the line segment PR and QS as its diagonals the parallelogram identity holds. Conclusion u solves the homogeneous wave equation, which is u tt - c square u xx = 0.

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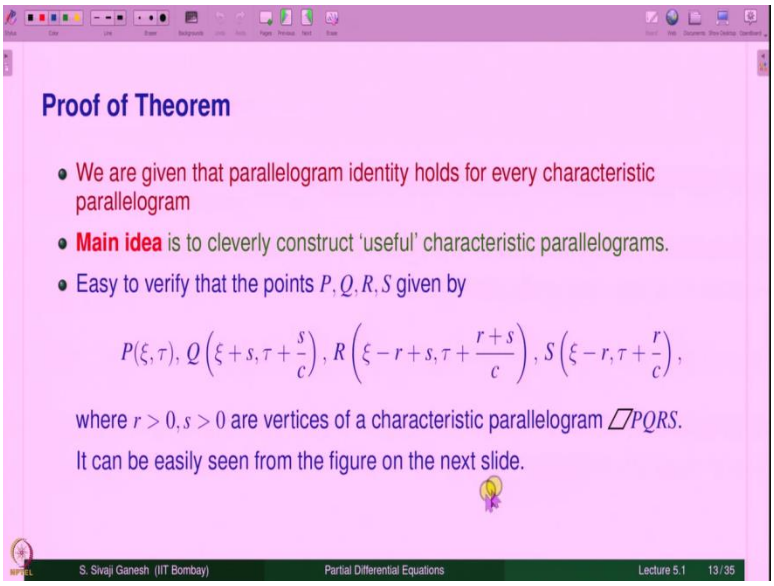

So, proof of the theorem we are given that parallelogram identity holds for every characteristic parallelogram. Main idea is to cleverly construct useful characteristic parallelograms this is a standard idea in mathematics whenever you are given wealth of information, like here, something holds for every characteristic parallelogram if you want to use it, you are really you exploit it by cleverly making choices.

So it is easy to verify that these point PQRS are vertices of a characteristic parallelogram. In fact, we have derived these points how they should look like and then wrote down here. So it is easier if you look at the picture.

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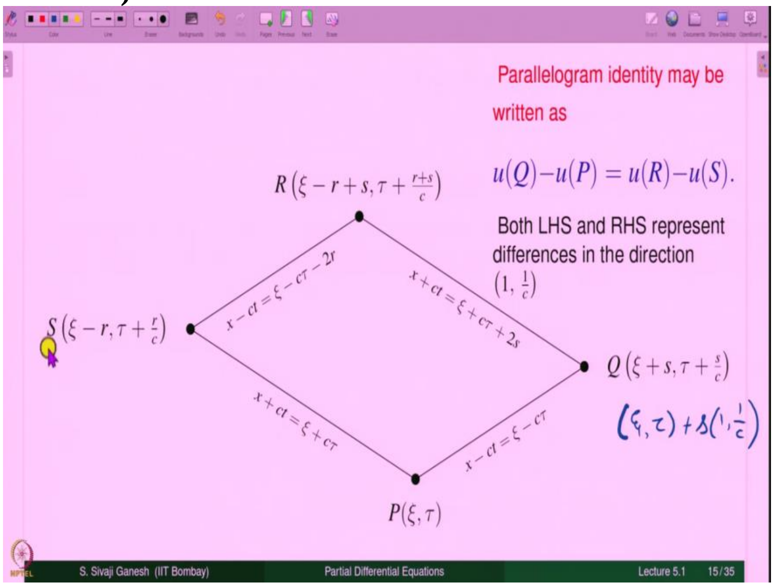

So let us take the point P as xi tau, and this line is given by x - ct equal to constant since this point lies on it, that constant has to be xi - c tau. So at this now I am going to consider Q which is of type xi plus s, something, I can determine what that point is using this equation that is why I get Q. Similarly, I propose S is like xi – r, something and that something can be determined by using this equation x + ct = xi + c tau I get S.

Once I know S, I can write down the equation of this characteristic line passing through this point which is in this equation. Similarly, from Q I can write down the characteristic line passing through Q the other one, one I already know so, this is from the other family and I see where they intersect I get this point that is how the vertices were determined. So, why is it a clever choice, we will see it on the next slides in the proof.

So, this is the picture that we have for PQRS and they lie they are actually vertices of a characteristic parallelogram. So, we are now in a shape to apply the parallelogram identity u of P + u of R = u of Q + u of S. So, it can also be written as or rewritten as this u Q - u P = u R - u S. Now, notice what is u of Q - u of P it is a value of u at this point what is this point it is actually xi, tau + S into 1, 1 / c and when S = 0 I am at P.

So, this is a some kind of difference of u values of u along this direction 1, 1 / c. Similarly, this u of R - u of S you see S and S / c. So, the point R is nothing but this point S plus S times 1, 1 / c. So, that is also a variation or difference in this direction 1, 1 / c. So, if you divide these differences with S divide by small s which is this s and then take the limit as s goes to 0 what we get is a directional derivative of u in this direction 1, 1 / c at the point P. Similarly, if you look at u R - u S divide by with small s this s then we get the directional derivative of u at this point S in the direction 1, 1 / c.

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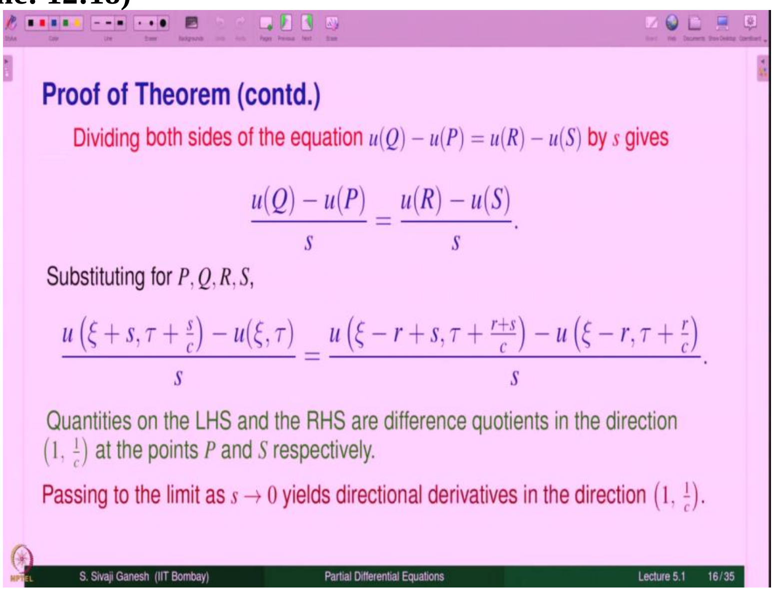

So, just substituting for PQRS we get this now, you look at this this is a difference quotient when you are trying to compute the directional derivative of u in the direction 1, 1 / c at the point xi tau. Similarly, this also when you are trying to compute the directional derivative of u at this point, xi – r, tau + r / c which is the point denoted by capital S in the paragraph in the direction 1, 1 / c. So passing to the limit yields directional derivatives.

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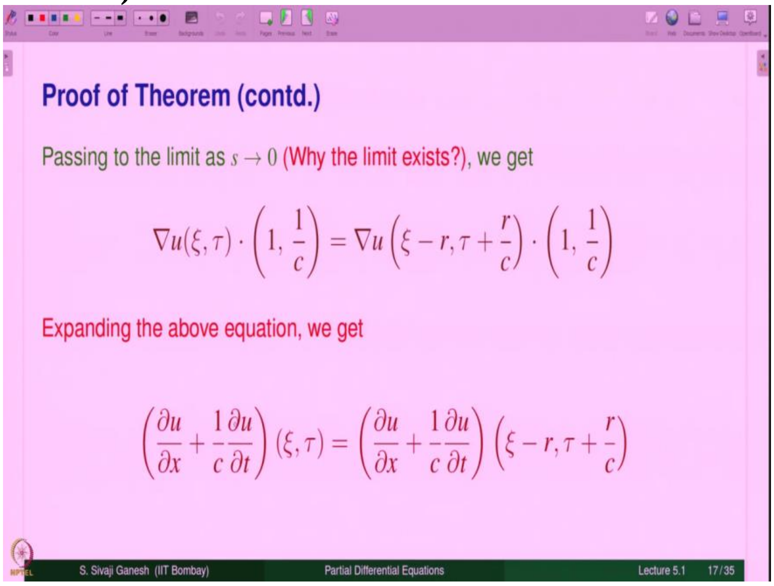

Why the limit exists? Because we are given that the function is C 2 therefore, all partial derivatives of order 1 exist. So, we can compute using any formula that you like. So, we get this. So, if you expand what is this grad u is u x, u t u x, u t is the gradient dot 1, 1 / c that will give u x + ut / c. So, getting this at the point xi tau similarly, the RHS. So, derivative in the directional derivative in the direction of 1, 1 / c is nothing but this particular combination of the partial derivatives.

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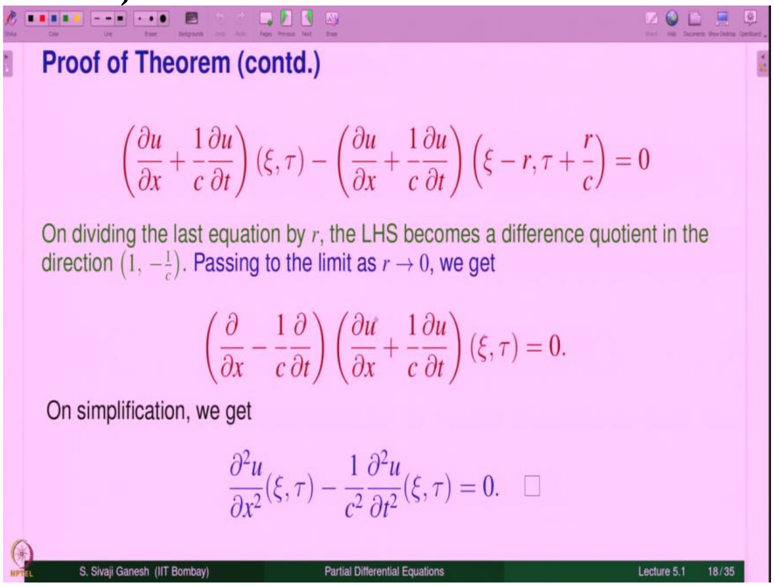

So, rewriting what we have here we get this. Now, if you look at this is the point P, this is the point S, this is also like a difference quotient once you divide with r, but in which direction this suggests xi tau, this is xi tau + -r, r / c, r is positive therefore, +r, -1, 1 / c as far as the direction goes it is 1, -1 / c. So, passing to this limit as r goes to 0, we get the part of the directional derivative in the direction 1, -1 / c which is here the first one here.

This is the directional derivative in the direction 1, -1 / c of this quantity which is there here for which we have the difference quotient here. When you divide with r, so once you expand, you get dho 2 u / dho x square -1 / c square dho 2 u / dho t square at xi tau = 0 xi tau is an arbitrary point. Therefore, u satisfies the wave equation at every point. Of course, here we have used that, when you expand, you will get dho 2 u / dho x dho t and dho 2 u / dho t dho x, they get cancelled because u is a C 2 function mixed partial derivatives are equal.

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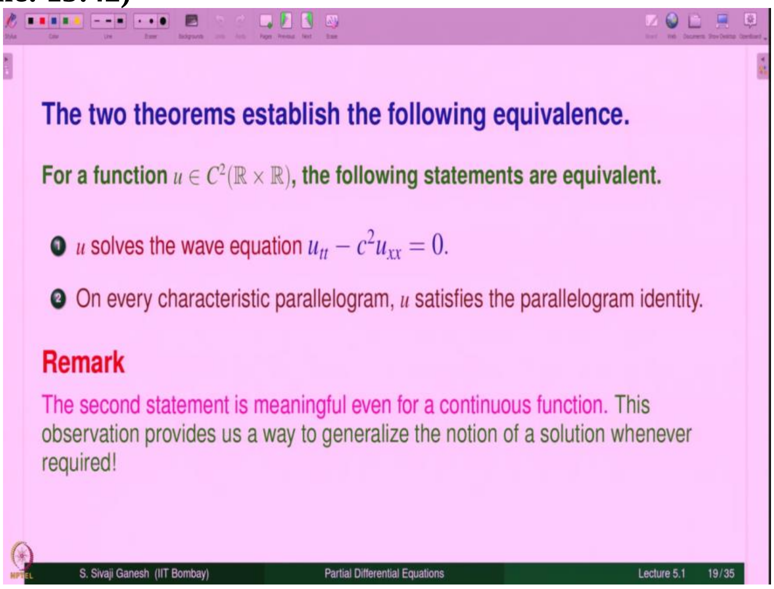

So the 2 theorems established the following equivalence, for a function which is C 2 of R cross R, in fact, the same proof works with C 2 of R cross 0 infinity, the following statements are equivalent, u solves the wave equation is same as saying that on every characteristic parallelogram, u satisfies the parallelogram identity. The second statement is meaningful, even for a continuous function.

In fact, you do not even need continuity, but I am just putting together something nice to have, of course, we can even take discontinuous functions. That is another thing, what I am saying is that this second part, namely parallelogram identity, makes sense, this statement makes sense without any requirement of differentiability on u. This observation provides us a way to generalize the notion of a solution whenever required; we are going to discuss them later on.

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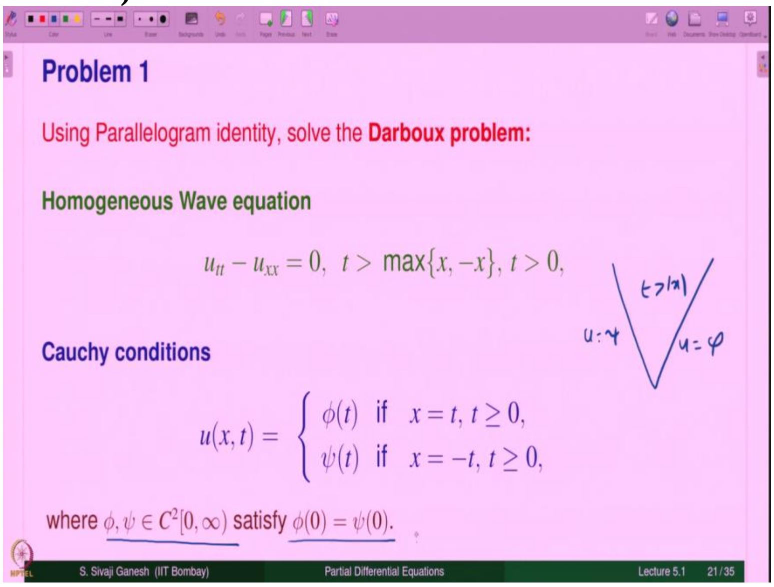

So let us look at some applications of parallelogram identity which is in solution of an IBVP with Dirichlet boundary conditions. So, using parallelogram identity, solve the Darboux problem, what is Darboux problem u tt - u xx = 0 there is a wave equation posed in which domain t bigger than maximum of x, - x that is nothing but this t is greater than mod x maximum x, -x is precisely mod x. In this domain, we have to solve a wave equation and we are given Cauchy conditions u is given to be.

So, u is given to be phi here and u is given to be psi here and these are domain in which we have to solve and we are given phi and psi to be C 2 functions satisfying phi of 0 = psi of 0.

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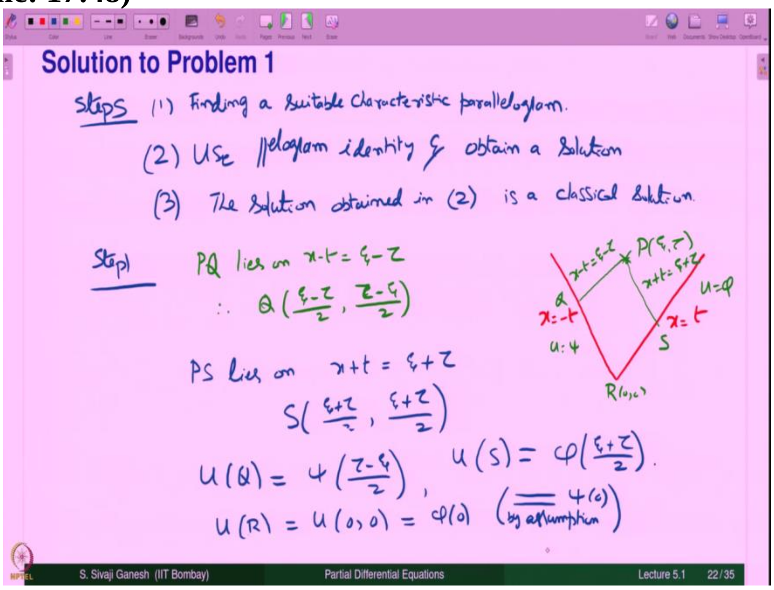

So let us solve this problem. So what are the steps involved? For first step is finding a suitable characteristic parallelogram suitable means useful, second is use parallelogram identity and obtain a solution. Of course, third thing still remains that we have to check that the solution that we have obtain into is indeed a classical solution, let us look at the step 1 here first. Step 1 is to find a suitable characteristic parallelogram. These are the lines x = t and x = -t both of them are characteristic lines.

So let me pick up a point P here I name it as psi tau and not x and t, because I would like to use this notation of x and t in describing the lines. So this line is x - t = xi - tau and this line is x + t = xi + tau. So, we call that as P let us call this as Q, R is the origin and this is the S and here we are given u = phi and here we are given u = psi. Therefore, u at P can be obtained very easily what we need to know is what is Q what is S? So, let us find out what is Q and S. So, PQ the line PQ the side PQ lies on x - t = xi – tau.

Therefore, Q coordinates are given by xi - tau / 2, tau - xi / 2, PS lies on x + t = xi + tau but S also an x = t. So, x component t component must be same therefore, S is xi + tau / 2, xi + tau / 2. So, we know the coordinates for Q and S, r of course is 0 0 therefore, u of we have to find what is u of Q. So, u of Q is here it is given in terms of psi. So, psi of tau - psi / 2 and u of S is given in terms of phi that is phi of xi + tau / 2 and what is u of R u of R is u of 0 0 and that is equal to phi of 0 of course, we have assumed that is equal to 0 psi of 0 by assumption you may call compatibility condition.

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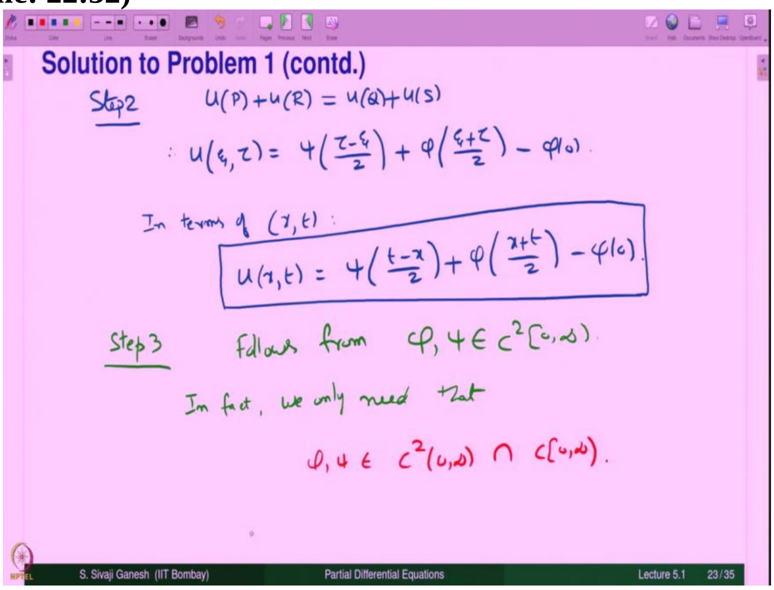

So, step 2 is to get a solution apply parallelogram identity we already computed u of R u of Q u of S. So, therefore, u of xi tau = psi of tau - xi / 2 plus phi of xi + tau / 2 - phi 0, so, in terms of x t simply replace psi tau with x t. So, u x t = psi of t - x / 2 plus phi of x + t / 2 -phi 0. So, we obtained the solution now what remains is step 3 or to check that u given by this box the formula is actually a classical solution to the given problem.

And that follows from our assumptions from phi and psi we assumed this in fact, we only need the following. In fact, we only need that what do we need phi and psi should be C 2 functions because I should be able to differentiate the expression for you 2 times and I need continuity only up to 0 of course, one can check this problem is also well posed.

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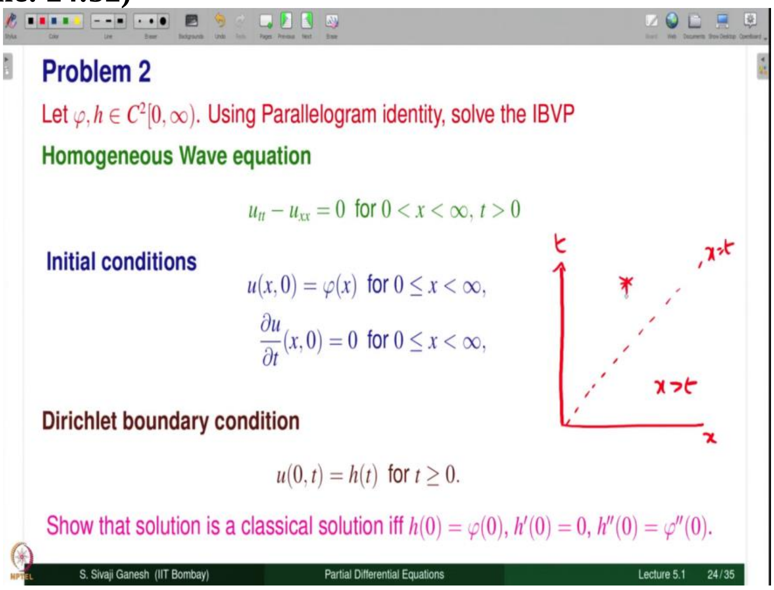

So, let us look at problem 2 here we are supposed to solve homogeneous wave equation with initial conditions and possibly a nonzero boundary condition general function h of t. We will use parallelogram identity to solve in some region of this domain, the domain in which we are interested in solving is this x positive, t positive from our prior experience we do know that in this domain which is determined by the line x = t namely x bigger than t, the D'Alembert's formula holds for the solution. So, essentially we need to solve at a point which is above this line, let us say point here using parallelogram identity.

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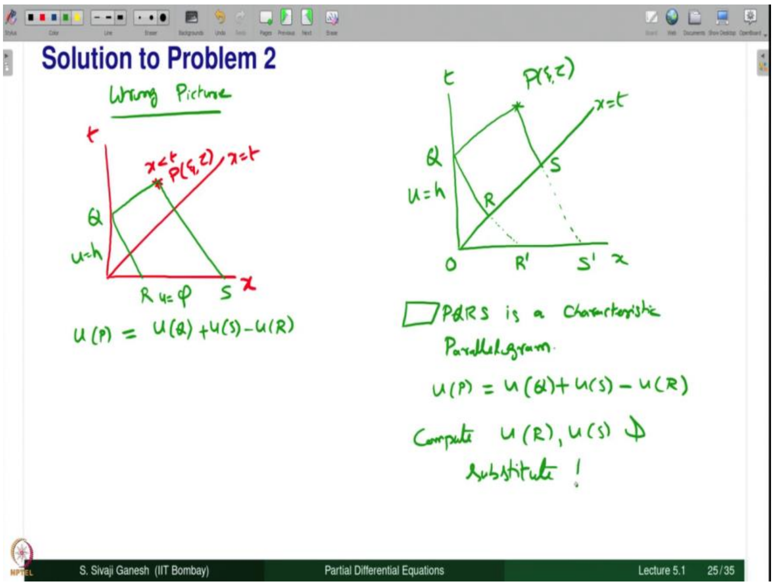

Therefore, let us look at how to solve that in this region where x is less than t this region. So, let us this is the line x equal to t these are the access let us take a point P let us denote by xi tau because we are going to use x and t to describe the equation so the characteristic lines. So, what we do is just take this line which is parallel to this characteristic and this is the other one and this is other one.

So, Q R S so, we know parallelogram identity gives us that u of P = u of Q + u of S - u of R, what is u of Q it is determined in terms of h because u = h what is what is u of S. u of S is given in terms of phi because u = phi here and you have R also so, it looks like it does not depend on psi. So, it means we are airing somewhere then when we look back PQRS is only a trapezium, it is not a parallelogram forget about being a characteristic parallelogram.

So, these are wrong picture tempting, but wrong picture. So, what is the correct picture? It is the line x = t start at a point P which is xi, tau. So, this has to be a characteristic this is the other characteristic now, it looks Q this point is R this point is S, we will determine what these point R. We know u at Q because u is prescribed here as h but we do not know what u of R is and u of S that needs to be determined once again using the D'Alembert's formula because for which D'Alembert's formula holds.

So, if we call this as origin O let us call this r dash u of R is given in terms of 0 O and R dash. Similarly, this also so, in which case we have to find out what are these points R dash and S dash to get solution at these points. So, it is a 2 step process. So, at this in this picture, what we have is PQRS is a characteristic parallelogram. Therefore, u of P = u of Q + u of S - u of

R by parallelogram identity. So, now what we have to do is compute U of R, u of S and substitute in this formula.

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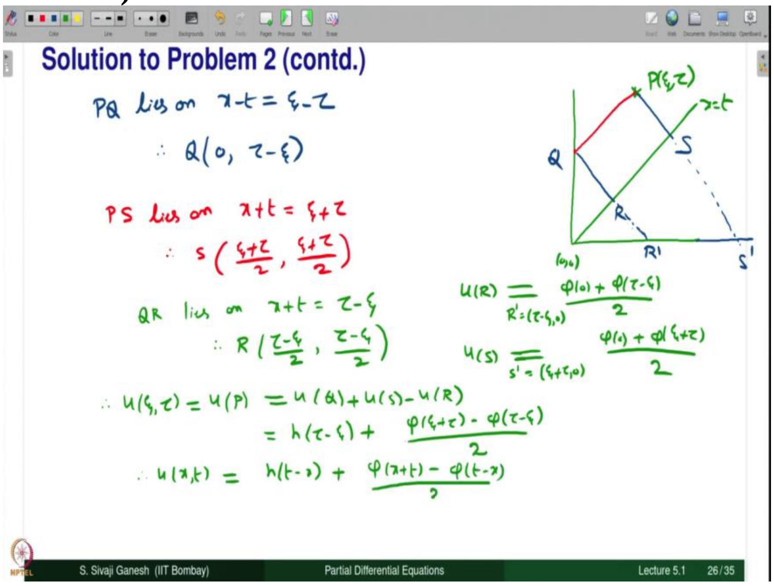

Q R S R dash S Dash. So PQ the side PQ lies on x - t = xi – tau line. Therefore, Q is where x is 0, therefore, 0 when x is 0 t is tau - xi. Now, let us look at PS that lies on x + t = xi + tau therefore, the point S is xi plus tau / 2 xi + tau / 2, because the point which lies on the line x = t. So, x and t coordinates are same, let us look at QR it lies on x + t = tau - xi because it passes through the point Q.

Therefore, the point R is given by tau - zi / 2, tau - xi / 2, because R is also a point which is lying on the line x = t. Therefore, we can write down what are the values let us write one by one, what is u of R, u of R is given by phi of 0 + phi of tau - xi / 2, I am using the D'Alembert's formula for us psi 0. So, the coordinates of R dash are tau - xi, 0. Similarly, u of S is equal to because S dash is xi + tau, 0 the value is phi of 0 + phi of psi + tau / 2.

So, we got R and S. So, therefore, u at the point xi, tau is u of P by parallelogram identity it is u of q + u of s - u of r. So, that is nothing but h of tau - xi + phi of xi + tau - phi of tau - xi / 2. So, now let us switch to x t instead of psi tau because now we have a formula. So, u of x t is nothing but h of t - x + phi of x + t - phi of t - x / 2.

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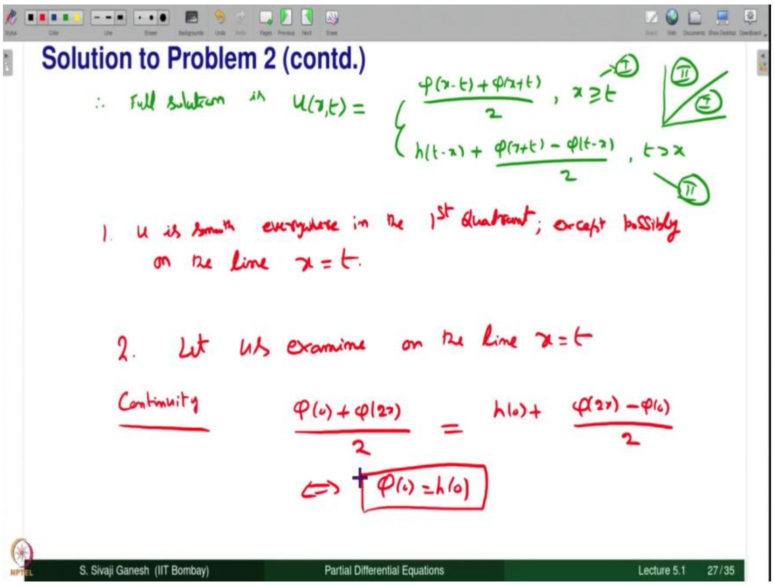

Therefore, let us write down the full solution. Full solution means we write down what a solution in x less than t, x greater than t in one place is u of x t = phi of x - t + phi of x + t / 2 if x is greater than or equal to t h of t - x + phi of x + t - phi of t - x / 2 if x t is bigger than x. So, this is region 1, region 2. So, this is for the region 1 and this is for the region 2. In region 1 it is given by D'Alembert's formula.

Now, let us make some observations. First point is that u is smooth everywhere as smooth as the given function phi and h R in each of these region 1 and 2 everywhere in the first quadrant except possibly on this line x = t. So, let us examine what happens on the line x = t. So, first part is continuity is it continuous at points of x = t. So, from region 1, what we get is phi 0 + phi of 2 x / 2 and from region 2, what we get is h 0 + phi of 2 x - phi of 0 / 2. So, this is a same as phi of 0 = h of 0. So, there is one compatibility condition that we get. So the continuity of this function demands that phi 0 must be equal to h 0.

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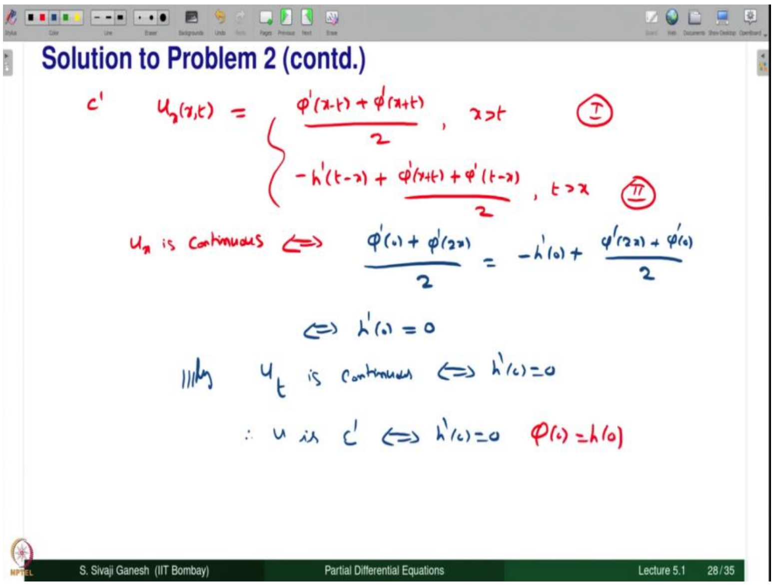

Similarly, let us check for C 1 x for that we need to write what is u x of x t. Of course, u t of x t as well. So, u x of x t is phi prime of x - t + phi prime of x + t / 2 in the region x bigger than t h prime of - h prime of t - x + phi prime of x + t + phi prime of t - x / 2 in the region t bigger than x. So, this is the region 1 is the region 2. So, therefore, u x is continuous if and only if the limits from both the regions 1 and 2 as we approach x = t coincide.

So, what we have is phi prime of 0 + phi prime of 2x / 2 should be equal to - h prime of 0 + phi prime of 2x + phi prime of 0 / 2 and this happens if and only h prime of 0 = 0. So, similarly, u t is continuous if and only h prime of 0 is 0. So, the same compatibility condition, therefore, u is c 1 if and only h prime of 0 is 0 phi of 0 = h of 0.

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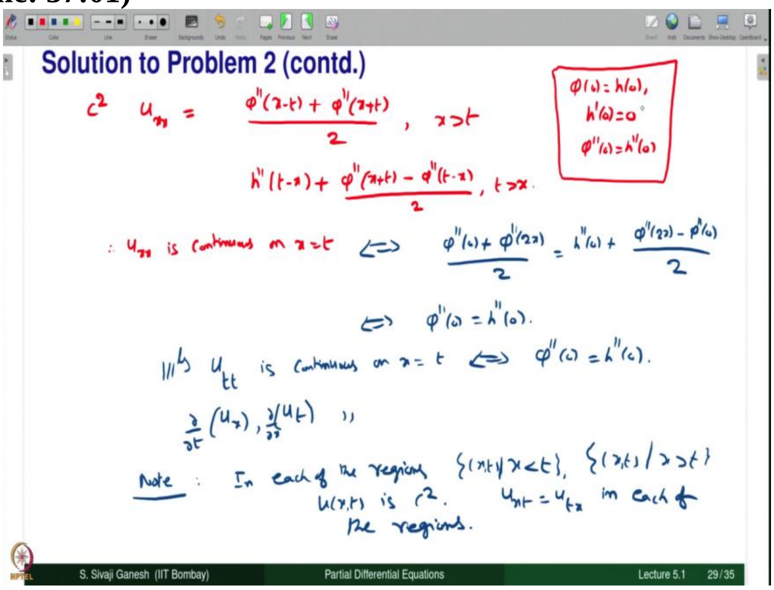

Let us look at c 2 x for which we need the formula for u xx in both regions. So, therefore, u xx is continuous at the points of the line x = t if only if phi double dash of 0 + phi double dash of 2x / 2 = h double dash of 0 + phi double dash of 2x - phi double dash of 0 / 2 and that is if and only if phi double dash of 0 = h double dash of 0. Similarly, u tt is continuous on x = t under the same conditions, no new compatibility conditions are required and you can easily check that you take u x and differentiate with respect to t.

Similarly, take u t and differentiate with respect to x they are also continuous on x = t under the same conditions in fact note there is you have to check for one of them because in each of the regions 1 and 2 what are the regions set of all x t such that x is less than t and set of all x t such that x is bigger than in each of them, u of x t that we have defined is a c 2 function. So, therefore, u xt is same as u tx in each of the regions.

Therefore, the u that we have obtained is a classical solution if and only if the following compatibility conditions are satisfied. These 3 compatibility conditions are satisfied. This is because we do not have the psi in the problem of psi is 0. If psi was there we would have got more such conditions and this will be different actually h prime of 0 will be in terms of psi. We have not check the existence of u x, u t, u xx, u xt, u tt at the points on the line x = t. It is left as an exercise to you to check this using the definitions.

Assuming these compatibility conditions which are written on the top of this page namely phi of 0 = h of 0, h prime of 0 = 0, phi double dash of 0 = h double dash of 0.

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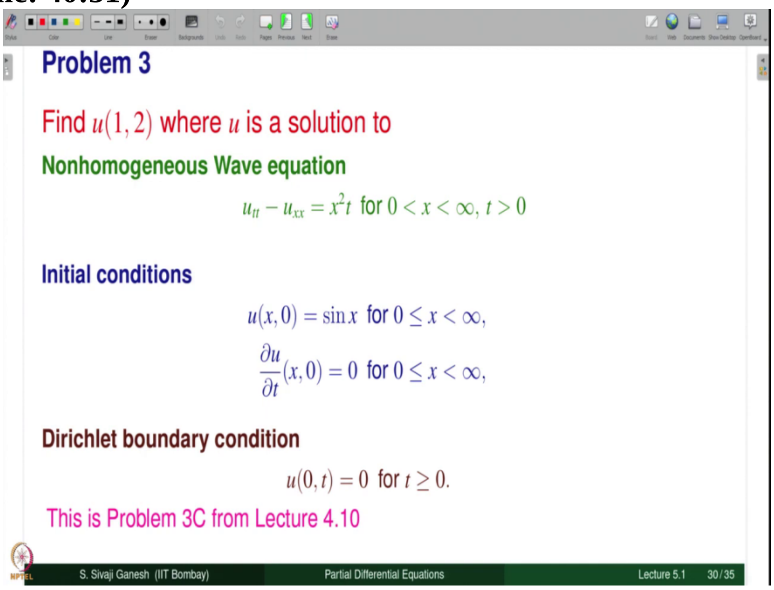

Now, let us look at the problem 3 now, we are asked to find u of 1 2, we have a non-homogeneous equation and the usual Cauchy data and Dirichlet boundary condition which is

0. This problem can be solved using many techniques. One of them is you make this 0 that is all homogeneous problem with the same these conditions and then non-homogeneous term is handled using D'Alembert's principle that is one that we have already explored.

Another idea is to extend this problem to hole up are here x is positive it is posed only for x positive extend this problem to x in r that means extend this function, these functions so, that you have a problem Cauchy problem for R then you use the D'Alembert's formula and you get a solution. And there is a third approach; sometimes we are lucky that we can spot some special solutions which satisfy this equation, the non-homogeneous part. If you notice this problem we have already considered in lecture 4.10. Here we solve it again. But we use parallelogram identity?

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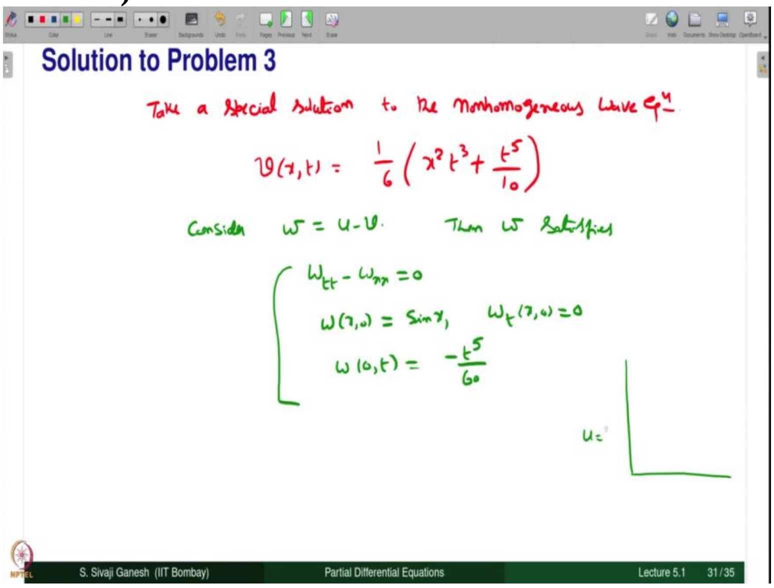

So take a special solution. Usually we get this by inspection, particularly if the right hand sides are simple functions, then it is easier to guess not always possible to guess. But it is a trick after all. So, take a special solution to the non-homogeneous wave equation, we are not talking about any other conditions only equation and that v of x t = 1 / 6 x square t cube + t power 5 / 10. There could be other functions also. But you have to figure out at least 1 function then we are on the road to solve this problem.

So, now consider w = u - v then w satisfies w tt - w xx = 0, because both u and v solve non-homogeneous problem therefore, the different solves homogeneous equation. What is w of x 0, w of x 0 is u x 0 – v x 0 luckily v x 0 is 0 when you put t = 0 v of x 0 is 0. So it is u x 0 which we want it to be sin x. Similarly, w t of x 0 is 0. But now the problem is the boundary condition that turns out to be a nonzero function, but we do not bother because we have parallelogram identity with us, which will give a solution to problems like this even if the data here is h and h nonzero.

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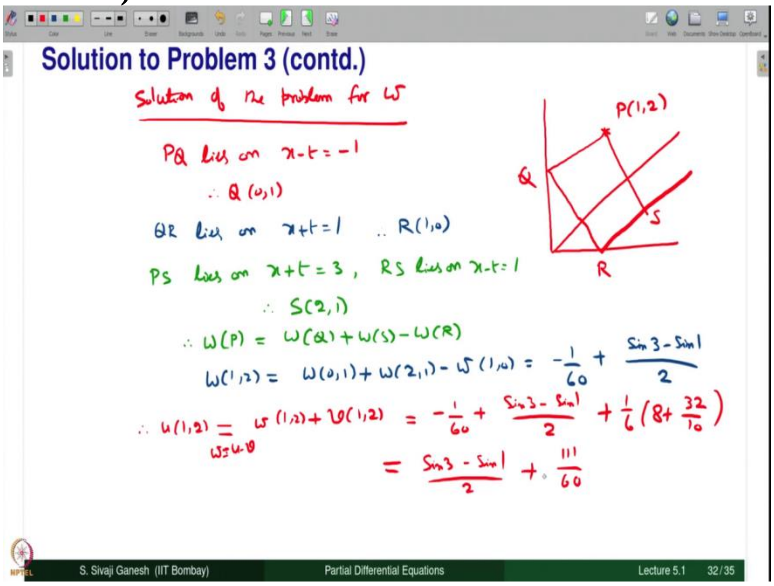

Solution of the problem for w. Remember, we want to solve w of 1, 2 we want to find let us draw this line and 1, 2 actually comes in this region if this is 1 unit 2 unit will be much higher somewhere here. So, this is a point P we have 1, 2. Now, let us draw the characteristic parallelogram first you could stop as before here or you could also go down and take this line and see where it hits.

So this is Q this is R this is S. PQ lies on x – t = -1. Therefore, Q is 0, 1 QR lies on x + t =1 therefore, R is 1, 0. PS lies on x + t = 3. RS lies on x – t = 1, therefore, S is 2, 1, therefore, w of P = w of Q + w of S – W of R by parallelogram identity and we get w of 1, 2 = w of 0, 1 + w of 2, 1 – w of 1, 0 and that is nothing but -1/ 60. That is a first term other things as sin 3 – sin 1 / 2.

Therefore, u of 1, 2 = w of 1, 2 + v of 1, 2 by the definition of w because w was u – v and that is equal to -1 / 60 + sin 3 – sin 1 / 2 + 1 / 6 into 8 + 32 / 10. This upon simplification becomes sin 3 – sin 1 / 2 + 111 / 60. This is exactly the same solution that we obtained earlier.

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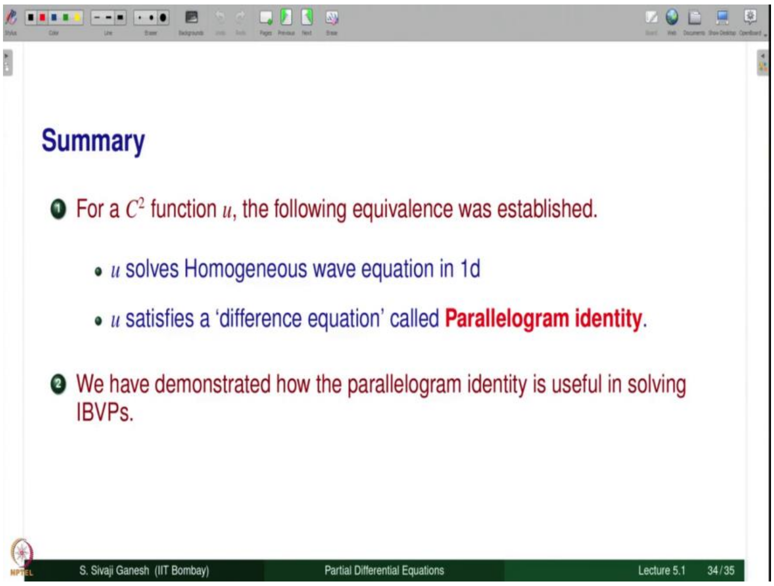

To summarize for a C 2 function, u the following equivalence was established that is u solves homogeneous wave equation in 1 dimension, if and only if it satisfies parallelogram identity on every characteristic parallelogram. Then we have demonstrated how the parallelogram identity is useful in solving initial boundary value problems. Thank you.