Fourier Analysis and its Applications

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22 The Airy's function

Integral representation of Airy's function: We would like to subject Airy's equation to Fourier transform. But the main question is whether the equation has solutions which are amenable to Fourier transforms? Since Airy's equation (after a trivial change of variables $x \mapsto -x$)

$$y''(x) + \frac{1}{3}xy = 0. \tag{4.15$'$}$$

does not contain the y' term, the Wronskian of any two linearly independent solutions is a non-zero constant. So if one solution decays at infinity along with its derivative then the second linearly independent solution must grow rapidly at infinity. So *at most one solution* (upto scalar multiples) can be subjected to Fourier transforms. We leave aside the question of whether there is such a solution and proceed formally. Taking the Fourier transform of (4.15) we get the ODE:

$$\xi^2\widehat{y} - \frac{1}{3i}\frac{d\widehat{y}}{d\xi} = 0 \tag{4.16}$$

Integrating this linear ODE we get

$$\widehat{y} = \exp(i\xi^3)$$

Now we use the Fourier inversion theorem and obtain, ignoring multiplicative constants,

$$y(x) = \int_{\mathbb{R}} \exp(ix\xi + i\xi^3)d\xi \tag{4.17}$$

which can be written as

$$y(x) = \int_{\mathbb{R}} \cos(x\xi + \xi^3)d\xi \tag{4.18}$$

The problem is that the integral (4.18) is a conditionally convergent integral and the steps leading to (4.18) are suspect. We could however directly try and verify that the integral (4.18) is a solution of the ODE but that too is problematic since differentiation under the integral sign is not easy to justify. Let us now make the change of variables $x\xi + \xi^3 = s$. The function $x\xi + \xi^3$ is monotone increasing on say $[r, \infty)$ and on this interval the change of variables is licit. The integral transforms into

$$y(x) = \int_R^\infty \frac{\cos s ds}{x + 3\xi^2} \tag{4.19}$$

where ξ is a function of s. Clearly $\xi(s) \to \infty$ as $s \to \infty$. Let us integrate by parts the integral (4.19) and we are led to discussing the convergence of

$$6\int_R^\infty \frac{\xi(s)(\sin s)\xi'(s)ds}{(x+3\xi^2)^2} = 6\int_R^\infty \frac{\xi(s)(\sin s)ds}{(x+3\xi^2)^3} \tag{4.20}$$

Exercise: Write down the boundary terms arising from integration by parts.

We have the unpleasant job of expressing $\xi/(x+3\xi^2)^3$ in terms of s which would be quite intractable but this is needed only for an exact evaluation of the integral. Since all we need here is a proof of convergence of the integral, we can get by with a rough estimate.

From the equation $x\xi + \xi^3 = s$ we infer $s/\xi^3 \longrightarrow 1$ as $\xi \to \infty$ whereby,

$$\frac{1}{2} < \frac{s}{\xi^3} < 2, \quad \text{for all } \xi >> 1.$$

and so $c^{-1}\xi^2 < s^{2/3} < c\xi^2$ whereby $3\xi^2 + x > \frac{3}{c}s^{2/3} + x$. But $\xi >> 1$ forces $s >> 1$ and x is fixed so that

$$\frac{3}{c}s^{2/3} + x > s^{1/2}, \qquad \text{say.}$$

Hence we get the estimate

$$\frac{\xi}{(3\xi^2 + x)^3} < \frac{cs^{1/3}}{s^{3/2}} = cs^{-7/6}$$

which suffices to conclude that the integral on the RHS of (4.20) namely

$$\int_R^\infty \frac{\xi(s)(\sin s)ds}{(x + 3\xi^2)^3}$$

converges absolutely. Now we need to show that the integral

$$y(x) = \int_{\mathbb{R}} \cos(x\xi + \xi^3)d\xi \tag{4.18}$$

does indeed solve the Airy's equation. As observed earlier differentiating under the integral sign is not easy to justify. We resort to another technique that is useful in many other contexts. This is the idea of *shifting the contour of integration into the complex domain*. For this purpose we consider the entire function

$$f(z) = \exp(ixz + iz^3)$$

We integrate this along a rectangle with vertices R, $R + i\eta$, $-R + i\eta$ and $-R$.

The integral along the base L_1 of the rectangle converges to (4.18) as $R \to \infty$. Exercise: Show that the integrals along the vertical sides V_1 and V_2 tend to zero as $R \to \infty$. Let us see what happens to the integral along the top edge L_2 of the rectangle. Along the top edge we have

$$z = \xi + i\eta, \quad -R < \xi < R \;\text{ and }\; \eta > 0 \;\text{ (fixed) }.$$

and

$$|\exp(ixz + iz^3)| = \exp(-x\eta + \eta^3 - 3\xi^2\eta)$$

so we estimate

$$\int_{L_2} |\exp(ixz + iz^3)dz| \le \exp(-x\eta + \eta^3)\int_{-R}^{R} e^{-3\xi^2\eta}d\xi$$

which shows that the intgeral along L_2 converges absolutely. Now by Cauchy's theorem,

$$\int_{L_2} f(z)dz + \int_{L_1} f(z)dz + \int_{V_1} f(z)dz + \int_{V_2} f(z)dz = 0.$$

Letting $R \to \infty$ we get

$$\int_{-\infty}^{\infty} f(\xi)d\xi = \int_{-\infty}^{\infty} f(\xi + i\eta)d\xi$$

In orther words,

$$y(x) = \int_{-\infty}^{\infty} \exp(ix\xi - x\eta + i(\xi + i\eta)^3)d\xi$$

Now we can differentiate under the integral sign and denoting $\xi + i\eta = z$,

$$\begin{aligned} y''(x) - \frac{xy(x)}{3} &= \frac{-1}{3i}\int_{-\infty}^{\infty}(3iz^2 + ix)\exp(ixz + iz^3)d\xi \\ &= \int_{-\infty}^{\infty}\frac{d}{d\xi}\exp(ixz + iz^3)d\xi = 0. \end{aligned}$$

It is elementary to obtain a solution of Airy's equation as a power series. While power series lend themselves to algebraic manipulations, it is difficult to obtain from them information such as zeros, asymptotic behaviour and estimates.

From integral representation it is easier to obtain asymptotic behaviour and such information. Asymptotic beahviour of Airy's function is important in optics.

Theorem (Riemann Lebesgue Lemma revisited - An exercise): If $f \in L^1(\mathbb{R})$ then the function $\xi \mapsto \widehat{f}(\xi)$ is uniformly continuous and bounded.

This will be a guided exercise. First select $A > 0$ such that

$$\int_{\mathbb{R}-[-A,A]}|f(x)|dx < \frac{\epsilon}{4}$$

Now we have

$$(\widehat{f}(\xi) - \widehat{f}(\eta)) = \int_{|x|\leq A} f(x)(e^{-ix\xi} - e^{-ix\eta})dx + \int_{|x|\geq A} f(x)(e^{-ix\xi} - e^{-ix\eta})dx$$

It is clear how to estimate the second piece and secure it less than $\epsilon/2$ in absolute value. Can you think of using the mean value theorem in the first piece?

1. Use Parseval formula to compute the integral

$$\int_{-\infty}^{\infty}\frac{4\sin^2\xi d\xi}{\xi^2}$$

   Use the fact that $\|f\|^2 = (2\pi)^{-1}\|\widehat{f}\|^2$ where, the norms are all L^2 norms.

2. Use the convolution theorem to determine the convolution $f_s * f_t$ where $f_s(\xi)$ is the Cauchy distribution:

$$f_s(\xi) = \frac{1}{\pi}\frac{s}{\xi^2 + s^2}$$

   The Cauchy distribution $f_s(\xi)$ is the Fourier transform of some known function $F_s(x)$. By the inversion theorem can you easily write $\widehat{f_s}$?

3. Compute the Fourier transform of $(\cosh ax)^{-1}$ where a is real positive.

Well, the problem amounts to computing the integral

$$I(\xi) = \int_{-\infty}^{\infty}\frac{\cos\xi x\; dx}{\cosh ax}$$

You could use complex analysis to do this. Usually when you have hyperbolic functions in the denominator it is convenient to use rectangular contours. Select a rectangular contour with vertices $-R, R, R + it$ and $-R + it$ where t has to be selected appropriately so that the integral over the top edge is a multiple of the integral you want. You need to show that the contributions from the vertical sides go to zero.

There is a simple pole inside the contour at the point $z = i\pi/2a$.