Fourier Analysis and its Applications

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25 A formula of Srinivasa Ramanujan

A formula of Srinivasa Ramanujan As a last item in this chapter we shall discuss a remarkable formula discovered by the great Indian Mathematician *Srinivasa Ramanujan.*

(1) *Robert Kanigal, The Man who knew infinity: Life and Genius of Ramanujan, Scribner, 1991.*

(2) Gamma function featured prominently in the works of Ramanujan.

(3) The formula:

$$\int_{-\infty}^{\infty} |\Gamma(a+it)|^2 \exp(-it\xi)dt = \sqrt{\pi}\Gamma(a)\Gamma\Big(a+\frac{1}{2}\Big)\cosh^{-2a}(\xi/2), \quad a>0.$$

Ramanujan, Messenger of Mathematics, 1915. Cited in G. H. Hardy, Collected papers, Volume 7, p. 98ff. In the next few slides we shall prove this remarkable Fourier transform formula of Ramanujan.

We first show that the function

$$t \mapsto |\Gamma(a+it)|^2$$

decays very rapidly. For this we need to recall the *Stirling's approximation formula*:

$$n! \sim n^n e^{-n}\sqrt{2\pi n}, \qquad \text{as } n\to\infty.$$

This formula given by *James Stirling* in his *Methodus Differentialis* in 1730 is unarguably the most remarkable formulas in classical analysis. The corresponding version for the gamma function reads:

$$\Gamma(x+1) \sim x^x e^{-x}\sqrt{2\pi x}, \qquad \text{as } x\to\infty.$$

We need to look at the behaviour of the gamma function $\Gamma(z)$ for values of z lying in the region R_δ in the complex plane given by $|\text{Arg } z| < \pi-\delta$ and $|z| >> 1$. In this region the $z^{z+\frac{1}{2}}$ is defined using the principal branch of the logarithm namely

$$z^{z+\frac{1}{2}} = \exp\Big((z+\frac{1}{2})\log z\Big), \quad \log z = \ln|z| + i\text{Arg } z.$$

The Stirling's formula in the complex domain now states that

$$\lim_{z\to\infty,\, z\in R_\delta} \Gamma(z+1)\Big(z^{z+\frac{1}{2}}e^{-z}\sqrt{2\pi}\Big)^{-1} = 1.$$

Reference for this is *B. C. Carlson, Special functions of applied mathematics, Academic Press 1977, pp. 45-47.* Since

$$|\Gamma(a+it)| = |\Gamma(a+it+1)|(a^2+t^2)^{-1/2},$$

it suffices to show that $|\Gamma(a+it+1)|$ *is in the Schwartz class and we are ready to use the Stirling's formula.* The vertical line $\{a+it \,:\, t\in\mathbb{R}\}$ certainly lies in the region R_δ for any choice of δ. Let us look at the approximation to $\sqrt{2\pi}\Gamma(a+it+1)$ given by Stirling's formula:

$$\exp((a+it+\frac{1}{2})\log(a+it)-(a+it))$$

To estimate the absolute value of this we need to look at *exp of* the real part of

$$(a+it+\frac{1}{2})\log(a+it)-(a+it)$$

Ignoring the multiplicative constant $\exp(-a)$ we are left with estimating

$$\exp\Big((a+\frac{1}{2})\ln|a+it| - t\,\text{Arg}(a+it)\Big)$$

Clearly, as $t \longrightarrow +\infty$,

$$\exp\Big((a+\frac{1}{2})\ln|a+it|\Big) = t^{a+\frac{1}{2}}O(1).$$

while the other factor

$$\exp\Big(-t\text{Arg}(a+it)\Big) = \exp(-t\pi/2)O(1)$$

since $\text{Arg}(a+it)$ tends to $\pi/2$ as $t \to \infty$. This proves that $|\Gamma(a+it)|$ decays exponentially fast whereby we can try to compute the Fourier transform of $|\Gamma(a+it)|^2$ directly using the definition. The behaviour as $t \longrightarrow -\infty$ is similar and is left as an exercise. Next, observe that since a and t are real,

$$|\Gamma(a+it)|^2 = \Gamma(a+it)\Gamma(a-it) = \Gamma(2a)B(a+it, a-it).$$

$B(p,q)$ denotes the *beta function* and we have used the *beta-gamma relation*. Recall that

$$B(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1}dt, \quad \text{Re}\, p > 0, \text{Re}\, q > 0.$$

Setting $t = (1+e^u)^{-1}$ in the integral gives after a little algebra,

$$B(p,q) = \int_{-\infty}^{\infty} \frac{\exp(\frac{qu}{2} - \frac{pu}{2})\,du}{(e^{u/2}+e^{-u/2})^{p+q}}$$

Exchanging the roles of p and q and using the symmetry of the beta function we get

$$\begin{aligned} B(p,q) &= \int_{-\infty}^{\infty} \frac{\exp(\frac{qu}{2} - \frac{pu}{2}) + \exp(\frac{pu}{2} - \frac{qu}{2})}{(e^{u/2}+e^{-u/2})^{p+q}}\frac{du}{2} \\ &= \int_{-\infty}^{\infty} \frac{\exp(qu-pu)+\exp(pu-qu)}{(e^u+e^{-u})^{p+q}}\,du \end{aligned}$$

we use this with $p = a+it$ and $q = a-it$ and we get that

$$|\Gamma(a+it)|^2 = \Gamma(2a)\int_{-\infty}^{\infty} \frac{e^{2iut}+e^{-2iut}}{(e^u+e^{-u})^{2a}}du$$

We now multiply this by $\exp(-it\xi)$ and integrate with respect to t to compute the Fourier transform of $|\Gamma(a+it)|^2$ as an iterated integral:

$$I(\xi) = \Gamma(2a)\int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} \frac{e^{it(2u-\xi)}+e^{-it(2u+\xi)}}{(e^u+e^{-u})^{2a}}du$$

It is very tempting to switch the order of integrals:

$$\Gamma(2a)\int_{-\infty}^{\infty} \frac{du}{(e^u+e^{-u})^{2a}} \int_{-\infty}^{\infty} (e^{it(2u-\xi)}+e^{-it(2u+\xi)})dt$$

We see the emergence of the problem of coping with oscillatory integrals, as was the case with the Fourier inversion theorem. To get around the difficulty we must resort to *the* $\exp(-\epsilon t^2)$ *trick!* namely,

$$I(\xi) = \lim_{\epsilon \to 0} \Gamma(2a) \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} \frac{e^{it(2u-\xi)-\epsilon t^2} + e^{-it(2u+\xi)-\epsilon t^2}}{(e^u + e^{-u})^{2a}} du$$

We may now safely switch the order of integrals and

$$\begin{aligned} I(\xi) &= \lim_{\epsilon \to 0} \Gamma(2a) \int_{-\infty}^{\infty} \frac{du}{(e^u + e^{-u})^{2a}} \int_{-\infty}^{\infty} (e^{it(2u-\xi)-\epsilon t^2} + e^{-it(2u+\xi)-\epsilon t^2}) dt \\ &= \lim_{\epsilon \to 0} \Gamma(2a) \sqrt{\frac{\pi}{\epsilon}} \int_{-\infty}^{\infty} \frac{\exp -((2u-\xi)^2/4\epsilon)}{(e^u + e^{-u})^{2a}} + \lim_{\epsilon \to 0} \Gamma(2a) \sqrt{\frac{\pi}{\epsilon}} \int_{-\infty}^{\infty} \frac{\exp -((2u+\xi)^2/4\epsilon)}{(e^u + e^{-u})^{2a}} \end{aligned}$$

The change of variables $2u \mp \xi = \sqrt{4\epsilon} v$ in the integrals now gives the closed form expression for the Fourier transform:

$$I(\xi) = 2\pi \Gamma(2a)(e^{\xi/2} + e^{-\xi/2})^{-2a} = 2^{1-2a} \pi \Gamma(2a) \cosh^{-2a}(\xi/2)$$

To see how this is the same as Ramanujan's formula we need the *Duplication formula of Legendre.*

$$\sqrt{\pi}\Gamma(2a) = 2^{2a-1}\Gamma(a)\Gamma(a + \frac{1}{2}).$$

We have completed the proof of Ramanujan's formula. As a reference for this material:

(1) D. Chakrabarty and G. K. Srinivasan, On a remarkable formula of Ramanujan, Archiv der Mathematik, **99**, 125–135 (2012).

(2) G. K. Srinivasan, A unified approach to the integrals of Mellin-Barnes-Hecke type, Expositiones Math., **31**, 151-168 (2013).

In the second reference you will find many other integrals of a similar kind evaluated using a generalization of the $\exp(-\epsilon t^2)$ trick.

Some thoughts of the Duplication formula For completeness let us sketch a proof of the duplication formula. Start with the formula

$$B(p,p) = \int_0^1 t^{p-1}(1-t)^{p-1} dt = 2 \int_0^{\pi/2} \sin^{2p-1}\theta \cos^{2p-1}\theta d\theta$$

To use the double angle formula we rewrite this as

$$B(p,p) = 2^{1-2p} \int_0^{\pi/2} (2\sin\theta\cos\theta)^{2p-1}(2d\theta)$$

Setting $2\theta = \phi$ we get

$$B(p,p) = 2^{1-2p} \int_0^{\pi} \sin^{2p-1}\phi d\phi = 2^{2-2p} \int_0^{\pi/2} \sin^{2p-1}\phi \cos^{2(1/2)-1}\phi d\phi = 2^{1-2p} B(p, 1/2).$$

Now assume that p is real and positive. Using the beta-gamma relation:

$$\frac{\Gamma(p)}{\Gamma(2p)} = 2^{1-2p}\frac{\Gamma(p)\Gamma(1/2)}{\Gamma(p+\frac{1}{2})}$$

where we have used the self-evident fact that $\Gamma(a) \neq 0$ if a is real positive. Cancelling $\Gamma(p)$ and rearranging gives for for p real positive,

$$\sqrt{\pi}\Gamma(2p) = 2^{2p-1}\Gamma(p)\Gamma(p+\frac{1}{2}).$$

Using the identity theorem from complex analysis we conclude that the result evidently extends to complex p, whenever both sides are defined. The Duplication formula proved by Legendre in 1809 and generalized by Gauss in 1812 may seem mysterious but to de-mystify it let us recall the *Reflection formula of Euler:*

$$\Gamma(z)\Gamma(1-z) = \pi/\sin(\pi z)$$

The formula says that *The gamma function is "one half of the sine function"* in a multiplicative sense. So any factorization formula for sine is likely to have a gamma analogue. Well, the sine function $f(x) = \sin \pi x$ has the factorization

$$f(2x) = 2f(x)f(x+\frac{1}{2})$$

with a striking similarity with the duplication formula:

$$\Gamma(2x) = 2^{2x-1}\Gamma(x)\Gamma(x+\frac{1}{2})$$

For more on these matters see *R. Goenka and G. K. Srinivasan, Gamma function and its functional equations, Resonance,* **26**, *367-386 (2021).*