Fourier Analysis and its Applications

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27 Vibrations of a circular membrane. Bessel expansion theorem

We now separate the radial and angular variables by setting

$$u(r, \theta) = v(r) \cos n\theta,$$

where n must be an integer owing to 2π periodicity.
Exercise:

3. The function $v(r)$ satisfies

$$r^2 v'' + rv' + (k^2 r^2 - n^2)v = 0 \tag{5.9}$$

   Check that this is the Bessel's equation after a rescaling of the variable r.

The indicial for the equation is $\rho^2 - n^2$ and only the positive index gives a solution which is finite at the origin. Since $r = 0$ corresponds to the center of the membrane which always remains at finite distance, the only physically tenable solution is $J_n(kr)$.

The Bessel functions of the first kind Thus we see that our special solution is (recalling $p = ck$),

$$z(x, y, t) = J_n(kr)(A \cos ckt + B \sin ckt) \cos n\theta.$$

Since the membrane is clamped along the rim we see that the solution vanishes along $r = 1$ for all values of θ and t. Thus the following boundary condition must be satisfied:

$$J_n(k) = 0. \tag{5.10}$$

Thus the frequency k must be a zero of the Bessel's function $J_n(x)$ and we shall see later that there is an infinite list of them. The frequencies therefore form a discrete set of values. The most general solutions are then obtained from superpositions whose coefficients are determined via initial conditions and Fourier Analysis. We illustrate this by means of an example where the oscillations are radial.

Radial vibrations of the circular membrane: Suppose the initial conditions, the value of $z(x, y, 0)$ as well as $z_t(x, y, 0)$, are radial functions (that is depends only on $\sqrt{x^2 + y^2}$ then so would the solutions. Thus the term $\cos n\theta$ would disappear (that is $n = 0$) and we have a sequence of solutions

$$J_0(kr)(A \cos ckt + B \sin ckt),$$

where k runs through the discrete set of zeros of zeros of $J_0(x)$ say $\zeta_1, \zeta_2, \zeta_3, \ldots$. The most general solution is then

$$z(r, t) = \sum_{j=1}^{\infty} J_0(\zeta_j r)(A_j \cos c\zeta_j t + B_j \sin c\zeta_j t) \tag{5.11}$$

Setting $t = 0$ in $z(x, y, t)$ as well as $z_t(x, y, t)$ we get the following pair of equations for determining the coefficients A_j and B_j:

$$z(r, 0) = \sum_{j=1}^{\infty} A_j J_0(\zeta_j r)$$

$$z_t(r, 0) = \sum_{j=1}^{\infty} C_j J_0(\zeta_j r)$$

where $C_j = B_j \zeta_j c$.

In order to proceed further we need a result from Analysis called the Bessel expansion theorem. We are obviously not equipped to prove this. The result is available for example in chapter 18 of the authoritative work *G. N. Watson, Treatise on the theory of Bessel functions, Second edition, Cambridge University Press, 1958.* See the historical introduction on pp. 577 - 579.

Theorem (Bessel Expansion Theorem): Suppose $f(r)$ is a smooth function on $[0, 1]$ then it admits a Fourier-Bessel expansion

$$f(r) = \sum_{j=1}^{\infty} A_j J_0(\zeta_j r)$$

The coefficients A_j are uniquely determined by the formula (due to *Lommel*)

$$A_j = \frac{2}{(J_1(\zeta_j))^2} \int_0^1 r f(r) J_0(\zeta_j r) dr. \tag{5.12}$$

So the coefficients A_j and B_j appearing in the solution of the vibration problem can be recovered from Lommel's formula applied to the initial conditions $z(r, 0)$ and $z_t(r, 0)$.

Orthogonality properties of Bessel's functions: Exercises:

4. Write the Bessel's ODE in self adjoint form. Check that the operator $x\frac{d}{dx}$ is scale invariant on $(0, \infty)$. Well, divide the Bessel's ODE by x and we get

$$xy'' + y' + (x - \frac{p^2}{x})y = 0$$

which can be written as

$$\frac{d}{dx}(xy') + (x - \frac{p^2}{x})y = 0$$

This is called the self-adjoint form of the equation.

5. Put $\phi_u(x) = J_p(xu)$ and check that

$$\left(x\frac{d}{dx}\right)\left(x\frac{d}{dx}\right)\phi_u(x) + (x^2u^2 - p^2)\phi_u(x) = 0. \tag{5.13}$$

6. Fix $p \geq 0$ and $\zeta_1, \zeta_2, \zeta_3, \ldots$ be the list of zeros of $J_p(x)$. Show that the family $\{J_p(\zeta_j x) \ : \ n = 1, 2, 3, \ldots\}$ is orthogonal over the interval $[0, 1]$ with respect to the weight function x. *Warning: The cases $p = 0$ and $p > 0$ have to be dealt with separately.*

Having discussed the orthogonality of the functions $J_p(\zeta_j x)$ ($j = 1, 2, 3, \ldots$), there remains the computation of

$$\int_0^1 x(J_p(\zeta_j x))^2 dx \tag{5.14}$$

since these are the Fourier coefficients in the Bessel expansion theorem.

7. Let ζ be a zero of $J_n(x)$. Multiply by $2x\zeta J_n'(\zeta x)$ the ODE satisfied by $J_n(\zeta x)$ and deduce that
$$2\int_0^1 (J_n(\zeta x))^2 x dx = (J_n'(\zeta))^2 = (J_{n+1}(\zeta))^2 \tag{5.15}$$
This is a bit tricky. We shall discuss this as a guided set of exercises.

8. Deduce the formula of Lommel. *We are not proving the Bessel expansion theorem. Only that if the expansion exists and the functions involved are smooth we are deriving the formula for the coefficients in a formal way.*

9. Determine the Bessel expansion for the constant function 1. Hint: Use $(x^p J_p(x))' = x^p J_{p-1}(x)$.

10. Show that
$$x^n = \sum_{j=0}^{\infty} \frac{2J_n(\zeta_j x)}{\zeta_j J_{n+1}(\zeta_j)}. \tag{5.16}$$

A very interesting proof of the orthogonality property of the Bessel functions suggested by physical considerations is available on pp 324-325 of *Lord Rayleigh, Theory of Sound, Vol - I, Dover, 1945.*

Solutions to some exercises. Orthogonality of $J_p(\zeta_j x)$ $(j = 1, 2, 3, \dots)$ Let $p \geq 0$ be fixed and $\zeta_1, \zeta_2, \zeta_3, \dots$ be the sequence of positive zeros of $J_p(x)$ arranged in ascending order. These are simple zeros (why?) and let us call
$$\phi_j(x) = J_p(\zeta_j x).$$
It is a routine calculation left to you to check that ϕ_j satisfies:
$$\Big(x\frac{d}{dx}\Big)^2 \phi_j(x) + (x^2\zeta_j^2 - p^2)\phi_j(x) = 0.$$
To begin with let us assume $p > 0$ and multiply this equation by $x^{-1}\phi_k(x)$ and integrate over $[0, 1]$. Note that since $p > 0$, the function $x^{-1}\phi_k(x)$ is integrable over $[0, 1]$ (why?). We get
$$\int_0^1 \phi_k(x)\frac{d}{dx}\Big(x\phi_j'(x)\Big)dx + \zeta_j^2\int_0^1 x\phi_j(x)\phi_k(x)dx - p^2\int_0^1 \phi_j(x)\phi_k(x)\frac{dx}{x} = 0$$
In the first of these integrals we perform an integration by parts:
$$-\int_0^1 x\phi_k'(x)\phi_j'(x)dx + \zeta_j^2\int_0^1 x\phi_j(x)\phi_k(x)dx - p^2\int_0^1 \phi_j(x)\phi_k(x)\frac{dx}{x} = 0$$
Explain why the boundary terms drop out. Now switch the roles of j, k and subtract the two equations and we get:
$$(\zeta_j^2 - \zeta_k^2)\int_0^1 x\phi_j(x)\phi_k(x)dx = 0$$
Since ζ_j, ζ_k are distinct zeros we get at once the orthogonality of ϕ_j and ϕ_k with respect to the weight xdx.

Now take up the case $p = 0$ where the equation for $\phi_j(x)$ simplifies to:
$$\Big(x\phi_j'(x)\Big)' + x\zeta_j^2\phi_j(x) = 0.$$
Multiply by $\phi_k(x)$ and integrate over $[0, 1]$. The rest of the argument is similar to the previous case.