Fourier Analysis and its Applications

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Fourier transform of a tempered distribution Before we take this up let us prove a simple lemma for the case of two functions in $\mathcal{S}(\mathbb{R})$

Theorem: Suppose $f, g \in \mathcal{S}(\mathbb{R})$. Then

$$\langle \widehat{f}, g \rangle = \langle f, \widehat{g} \rangle \tag{10.8}$$

Proof: Well,

$$\begin{aligned} \text{LHS} = \int_{\mathbb{R}} \widehat{f}(x) g(x) dx &= \int_{\mathbb{R}} g(x) dx \int_{\mathbb{R}} f(y) \exp(-ixy) dy \\ &= \int_{\mathbb{R}} f(y) dy \int_{\mathbb{R}} g(x) \exp(-ixy) dx \\ &= \int_{\mathbb{R}} f(y) \widehat{g}(y) dy = \text{RHS} \end{aligned}$$

Equation (10.8) suggests a definition of the Fourier transform of a tempered distribution. If f were a tempered distribution take the RHS of (10.8) as the definition of $\widehat{f}$. Formally,

Theorem: Suppose u is a tempered distribution then the map

$$g \mapsto \langle u, \widehat{g} \rangle, \quad g \in \mathcal{S}(\mathbb{R}) \tag{10.9}$$

defines again a tempered distribution. The tempered distribution given by (10.9) is denoted by $\widehat{u}$ and is called the *Fourier transform of u.*

Proof: This follows at once from the fact we proved that the Fourier transform is continuous as a map from $\mathcal{S}(\mathbb{R})$ to itself. In particular if $\{g_n\}$ is a sequence in $\mathcal{S}(\mathbb{R})$ converging to g in $\mathcal{S}(\mathbb{R})$ then $\widehat{g}_n \longrightarrow \widehat{g}$ and u being a continuous linear form on $\mathcal{S}(\mathbb{R})$ we get that

$$\langle u, \widehat{g}_n \rangle \longrightarrow \langle u, \widehat{g} \rangle$$

whereby we conclude that the RHS in (10.9) *does* define a tempered distribution.

Fourier transform computation:

11. As a first example let us compute the Fourier transform of an $L^1(\mathbb{R})$ function u in the above sense and see if it agrees with the classical definition given in chapter 4. To this end let $g \in \mathcal{S}(\mathbb{R})$ be arbitrary and the Fourier transform (in the sense of distributions) is given by

$$g \mapsto \langle u, \widehat{g} \rangle$$

The RHS is an integral pairing since $u \in L^1(\mathbb{R})$ and $\widehat{g} \in \mathcal{S}(\mathbb{R})$ so that

$$RHS = \int_{\mathbb{R}} u(y) \widehat{g}(y) dy = \int_{\mathbb{R}} u(y) dy \int_{\mathbb{R}} g(x) \exp(-ixy) dx.$$

Applying Fubini theorem we get

$$RHS = \int_{\mathbb{R}} g(x) dx \int_{\mathbb{R}} u(y) \exp(-ixy) dy = \langle U, g \rangle.$$

where U is the Fourier transform of u in the classical sense. We see that the distribution defined by the classical Fourier transform matches with the distributional Fourier transform showing the consistency of the two definitions.

12. Let us take up the Dirac delta next. For $g \in \mathcal{S}(\mathbb{R})$,

$$\langle \widehat{\delta_0}, g \rangle = \langle \delta_0, \widehat{g} \rangle = \widehat{g}(0) = \int_{\mathbb{R}} g(x)dx = \langle 1, g \rangle.$$

We see immediately that $\widehat{\delta_0} = 1$.

13. If the Fourier inversion theorem were to hold we would get

$$\widehat{1} = 2\pi\delta_0. \tag{10.10}$$

Let us verify the correctness of this. Well, for $g \in \mathcal{S}(\mathbb{R})$

$$\langle \widehat{1}, g \rangle = \langle 1, \widehat{g} \rangle = \int_{\mathbb{R}} \widehat{g}(y)dy.$$

Exercise: Use the $\exp(-\epsilon y^2)$ trick to compute the integral on the RHS and show that it equals $2\pi g(0)$. We have verified (10.10).

14. Use the $\exp(-\epsilon y^2)$ trick to compute the Fourier transform of $\exp(-ixa)$ (for real a) and hence compute the Fourier transform of $\sin x$ and $\cos x$ as tempered distributions.

15. Use the $\exp(-\epsilon y^2)$ trick to compute the Fourier transform of the *signum function* $H(x) - H(-x)$ as a tempered distribution.
Hints: Denoting the distribution by u and using the definition of the Fourier transform, we get that $\widehat{u}$ is given by

$$g \mapsto \int_0^\infty (\widehat{g}(\xi) - \widehat{g}(-\xi))d\xi$$

Introduce the $\exp(-\epsilon\xi^2)$ factor and the integral equals

$$\lim_{\epsilon \to 0+} \int_0^\infty (\widehat{g}(\xi) - \widehat{g}(-\xi)) \exp(-\epsilon\xi^2)d\xi.$$

Now swich the order of integrals.

We now see that the formulas obtained in chapter 4 carry over to the Fourier transforms of tempered distributions.

Theorem: Suppose $u \in \mathcal{S}'(\mathbb{R})$ then (as distributions)

$$\begin{aligned} \widehat{\Big(\frac{d}{idx}\Big)u} &= \xi\widehat{u} \\ \widehat{xu} &= \Big(-\frac{d}{id\xi}\Big)\Big(\widehat{u}\Big) \end{aligned}$$

Proof follows routinely from the definition of the Fourier transform of a distribution and the results proved for functions in the Schwartz class $\mathcal{S}(\mathbb{R})$. We shall only establish the first one. Well, for

$g \in \mathcal{S}(\mathbb{R})$ we have

$$\begin{aligned} \langle \widehat{\frac{d}{idx}u}, g\rangle &= \langle \frac{d}{idx}u, \widehat{g}\rangle \\ &= \langle u, -\frac{d}{idx}\widehat{g}\rangle \\ &= \langle u, \widehat{\xi g}\rangle \\ &= \langle \xi\widehat{u}, g\rangle \end{aligned}$$

It follows that

$$\widehat{\frac{d}{idx}u} = \xi\widehat{u} \tag{10.11}$$

The other equation is established along similar lines:

$$\widehat{xu} = \Big(-\frac{d}{id\xi}\Big)\Big(\widehat{u}\Big) \tag{10.12}$$

We now prove the version of the Fourier inversion theorem for tempered distributions:

Theorem (Fourier transform of a tempered distribution): Suppose $u \in \mathcal{S}'(\mathbb{R})$ and $g \in \mathcal{S}(\mathbb{R})$ then

$$\langle \widehat{\widehat{u}}, g(x)\rangle = 2\pi\langle u, g(-x)\rangle$$

Proof:

$$\text{LHS} = \langle u, \widehat{\widehat{g}}(x)\rangle$$

But we know from chapter 4 that $\widehat{\widehat{g}}(x) = 2\pi g(-x)$ whereby

$$\text{LHS} = 2\pi\langle u, g(-x)\rangle$$

as desired.

As a corollary we deduce immediately that *if* $u \in L^1(\mathbb{R})$ *such that* $\widehat{u} \in L^1(\mathbb{R})$ *then the inversion theorem holds namely,*

$$\widehat{\widehat{u}}(x) = 2\pi u(-x), \qquad \textit{as elements of } L^1(\mathbb{R}).$$

Tempered distributions in several variables So far we have confined ourselves to one variable for *notational simplicity*. The basic aspects of the theory readily generalize to several dimensions. We have already introduced in chapter 4 the Schwartz space $\mathcal{S}(\mathbb{R}^n)$. The topology on this space is defined in a similar manner. We shall not explicitly introduce the metric but just describe the convergent sequences:

Definition: A sequence $\{f_\nu\}$ in $\mathcal{S}(\mathbb{R}^n)$ converges to $f \in \mathcal{S}(\mathbb{R}^n)$ if for any multi-index $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ and any polynomial $Q(x)$,

$$\sup_{\mathbb{R}^n} |Q(x)D^\alpha(f_\nu(x) - f(x))| \longrightarrow 0.$$

as $\nu \to \infty$.

As in the one variable case the metric space $\mathcal{S}(\mathbb{R}^n)$ is complete and $\mathcal{S}(\mathbb{R}^n)$ is a locally convex TVS. The dual space $\mathcal{S}'(\mathbb{R}^n)$ is called the *space of tempered distributions* on $\mathbb{R}^n$. The Fourier transform of a tempered distribution is defined analogously. If $u \in \mathcal{S}'(\mathbb{R}^n)$ then its Fourier transform $\widehat{u}$ is defined as the tempered distribution

$$g \mapsto \langle u, \widehat{g} \rangle, \quad g \in \mathcal{S}(\mathbb{R}^n).$$

The support of a tempered distribution of several variable is defined in exactly the same way as in the one variable case. The theorem on distribution with point support now takes the form:

Theorem: If $u \in \mathcal{S}'(\mathbb{R}^n)$ has support $\{0\}$ then u is a linear combination of finitely many derivatives of the Dirac distribution:

$$u = \sum_{|\alpha| \leq N} c_\alpha \delta_0^\alpha$$

where δ_0^α denotes the derivative of δ_0 corresponding to multi-index α.