Fourier Analysis and its Applications

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58 Support of a distribution. Distributions with point support

Exercises:

16. Show that the Fourier transform of δ_0 is the constant function 1
17. Find the Fourier transform of the derivatives of the Dirac delta distribution in several variables.
18. Use the inversion theorem to find the Fourier transform of a monomial x^α where α is a multi-index.
19. Let D be the closed unit disc $\{(x,y) \in \mathbb{R}^2 \ : \ x^2+y^2 \leq 1\}$ and u be the characteristic function of D which is a tempered distribution. Find its distributional derivatives.
    Sketch: Apply the definition of derivative
    $$\langle \frac{\partial u}{\partial x}, g\rangle = -\langle u, \frac{\partial g}{\partial x}\rangle = -\iint_D \frac{\partial g}{\partial x} dxdy = -\int_{\partial D} n_1 g ds$$
    where the last expression arises upon applying the Gauss divergence theorem and (n_1, n_2) are the components of the outer unit normal to ∂D and ds is the arc-length measure along ∂D.

Recall Liouville's theorem from complex analysis that a bounded entire function is constant. There are analogues for harmonic functions we now state and prove

Theorem (Liouville's theorem): If u is a tempered distribution on $\mathbb{R}^n$ that satisfies $\Delta u = 0$ in the sense of distributions, then u is a polynomial. If in addition $u \in L^p(\mathbb{R}^n)$ then $u = 0$ if $1 \leq p < \infty$ and u is constant if $p = \infty$.

Proof: Well, let us take the Fourier transform of $\Delta u = 0$ and we get $|\xi|^2\widehat{u} = 0$ which means that the tempered distribution $\widehat{u}$ vanishes away from the origin or in other words $\widehat{u}$ has support contained in $\{0\}$. Thus
$$\widehat{u} = \sum_{|\alpha|\leq N} c_\alpha \delta_0^\alpha$$
We conclude that u is a polynomial since $c_\alpha\delta_0^\alpha$ is the Fourier transform of a monomial by the exercises in the last slide. The rest of the statements follow quickly. A polynomial in $L^p(\mathbb{R}^n)$ must be zero if $1 \leq p < \infty$ and a constant if $p = \infty$.

Restriction or localization Again we start with the one variable case since generalization to several variables is similar. Recall that if $f : \mathbb{R} \longrightarrow \mathbb{R}$ is a function and G is an open set in $\mathbb{R}$ we have the notion of restriction of f denoted by $f\Big|_G$. We discuss the analogous notion for distributions. How to restrict a distribution to an open subset of the real line?

Definition: Let u be a tempered distribution on $\mathbb{R}$. The *restriction of* u to an open subset $G \subset \mathbb{R}$ or *localization of* u *to* G is by definition the map
$$g \mapsto \langle u, g\rangle, \quad g \in C_c^\infty(G). \tag{10.17}$$
We use the notation $u\Big|_G$ to denote the map (10.17). Then the support of u can be described as the *complement of the largest open set* G *such that* $u\Big|_G$ *is the zero map.* It is clear that if we have two tempered distributions u, v such that
$$u\Big|_G = v\Big|_G$$

then $\text{supp}(u-v) \cap G = \emptyset$. We say that u and v agree on G. In particular if two distributions agree on $\mathbb{R} - \{0\}$ then they differ by a linear combination of Dirac delta and finitely many derivatives. *Show that $PV(\frac{1}{x})$ when localized to $\mathbb{R} - \{0\}$ is the smooth function $1/x$ on $\mathbb{R} - \{0\}$*. This suggests the following:

Definition: Suppose given a smooth function f on an open set $G \subset \mathbb{R}$, a distributional extension of f is a distribution u on the real line such that $u\Big|_G = f$.

Thus $1/x$ extends as a tempered distribution on $\mathbb{R}$.

20. Does the smooth function $1/\sinh x$ on $\mathbb{R} - \{0\}$ extend as a tempered distribution on the real line? Is the extension unique? Well,
$$\frac{1}{\sinh x} = \frac{1}{x}\Big(\frac{x}{\sinh x} - 1\Big) + \frac{1}{x}.$$

21. Does the smooth function $1/x^2$ on $\mathbb{R} - \{0\}$ extend as a tempered distribution on the real line? Is the extension unique?

22. Does the smooth function $|\Gamma(ix)|^2$ on $\mathbb{R} - \{0\}$ extend as a tempered distribution on the real line?

23. Optional Exercise: The the smooth function $\exp(1/x^2)$ on $\mathbb{R} - \{0\}$ DOES NOT extend as a distribution on the real line.

Problems of this kind become more interesting in several variables and they are highly non-trivial and call for sophisticated techniques from Algberaic Geometry. These are closely related to the so called *problem of division* in distributions.

Now that we have a notion of differentiation of a distribution and the operation of multiplication by polynomials we can ask whether the classical differential equations

$$\begin{aligned}
(1-x^2)y'' - 2xy' + p(p+1)y &= 0\\
(1-x^2)y'' - xy' + p^2y &= 0\\
x^2y'' + xy' + (x^2-p^2)y &= 0\\
xy'' + (1-x)y' + \lambda y &= 0\\
x(1-x)y'' + (c-(a+b+1)x)y' - aby &= 0
\end{aligned}$$

have solutions in spaces of distributions. The classical theory of ODEs such as the vector space of solutions being equal to the order of the differential equation FAILS across singular points namely the points ± 1 in the first two examples and the origin in the last three examples. In the last one both 1 and 0 are singular points.