Point Set Topology

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Week 03

Lecture 14

Okay, so welcome to the next lecture. So, in this lecture we shall introduce the notion of metric spaces. So, metric spaces provides perhaps the most important class of topological spaces. So, after we introduce metric spaces we will also see I mean we have talked about open subsets, closed subsets, continuous maps, we will give descriptions of these in terms of the metric. So, let us begin by recalling the definition of a metric space. So, we recall the definition of a metric space.

Let X be a set and let d from X×X to the real numbers >= 0, be a function, be a map which satisfies the following three conditions: d(x, y) = 0 if and only if x is equal to y, d(x, y) is equal to d(y, x), and d(x, z) is less than equal to d(x, y) + d(y, z) for all x,y,z in X. So, this is called the triangle inequality. So, the d is for distance, and we know that if we have a triangle, This is x, this is z, and this is y. Then the distance between x and z is less than equal to the distance between x and y and y and z.

So, if these three properties are satisfied, then we say that then the pair (X, d) is called the metric space, and the map d is called the metric on X or the distance function. So, we just need any function d which satisfies these three conditions. So, given any set X, there could be a lot of, lots of metrics on X, but the most common example of a metric space we encounter of metric space we encounter is ℝ^n with the Euclidean metric. So, that is if we have two vectors in ℝ^n. So, x is (x_1, .

.., ,x_n) and similarly y is (y_1, ..., y_n). So, then d(x,y) is defined to be the square root of the sum of squares of the difference of x_i and y_i.

Let us check that this defines a metric. So, let us check that this defines a metric. Note that the first two conditions are trivial to check, the only nontrivial condition is the triangle inequality. So, which we will prove now here. So, this d(x,y) we shall write this as norm of x - y, the two norm.

We will use this notation instead of writing d(x,y). What we need to prove is that, in this notation, that norm of x-y is less than equal to, or alternatively, we can say, we can define. So, if for a vector x in ℝ^n, we define the two norm of x as summation x_i^2 and square root of this. Clearly I do not need to write this as a definition. Then clearly d(x,y) is equal to the two norm of x-y and to check that this defines a metric, we need the following proposition.

Proposition: Norm of $x-y \le$ Norm of $x-z$ and Norm of $z-y$, for all x,y,z in $\mathbb{R}^n$. So, let us prove this. It is enough to check that, and this left as an easy exercise, that it is enough to check this: Norm of $x+y \le$ Norm of x + Norm of y, For all x,y in $\mathbb{R}^n$. In order to do this, Consider the map The inner product $\langle , \rangle$ from $\mathbb{R}^n \times \mathbb{R}^n$ to $\mathbb{R}$ We first claim that this absolute value of this quantity is less than equal to norm of x into norm of y. So, let us first prove this claim and to see this.

Let us define $w = y - tx$ where t is a real number. and now let us compute $\langle w,w\rangle$, and this is equal to, this is bilinear in both both. So, what that means is if we fix x, then $\langle x,\lambda y_1+y_2\rangle=\lambda\langle x,y_1\rangle+\langle x,y_2\rangle$ (fixing x, it is linear in y) and similarly, if we fix y, then it is linear in x right. Using the bilinearity, we get that this is equal to $\langle y,y\rangle+t^2\langle x,x\rangle-2t\langle x,y\rangle$, and it is also symmetric bilinear and symmetric, because clearly for symmetry $d(x,y)=d(y,x)$. So, using the fact that this is bilinear and symmetric we get this, this is precisely equal to $\|y\|^2 + t^2\|x\|^2 - 2t\langle x,y\rangle$.

For a fixed x and y, we view this as a quadratic in t, there is a degree 2 polynomial in t, and as it only takes positive value greater than or equal to 0, it follows that the discriminant is less than equal to 0. But what is a discriminant. So, from the discriminant, $4|\langle x,y\rangle| \le 4\|x\|.\|y\|$, which proves our claim. This is the claim which we want to prove.

So, using this claim we will now prove the proposition. Note that, we have $\|x-y\|^2 = \langle x-y,x-y\rangle = \|x\|^2 + \|y\|^2 -2t|\langle x,y\rangle|$. and $\le\|x\|^2 + \|y\|^2 -2t|\langle x,y\rangle|$. So, over here we have used the claim. Sorry, we have not used the claim yet.

This is less than, we are just taking absolute value, this is less than equal to, now, here we have used the claim, $\|x\|^2 + \|y\|^2 +2\|x\|.\|y\| = (\|x\|+\|y\|)^2$. So, this implies that $\|x-y\|\le\|x\|+\|y\|$. and note that $\|y\|=\|-y\|$. Therefore, we can just replace y by -y and we get $\|x+y\|\le\|x\|+\|y\|$, thus, this shows that $(\mathbb{R}^n,d)$ is a metric space.

So, this proves triangle inequality holds and therefore this is a metric space. So, this is the main example of metric space. So, now given a metric space, how can we use the metric to define a topology on it. So, let us see that. So, given a metric space X, We can use the metric to define a topology on X.

For x in X and an $\varepsilon>0$, define the open ball of radius ε around x as follows: So, $B_\varepsilon(x)$ is defined to be those y in X such that the distance of x from y is strictly less than ε, and we consider the collection there is something which we are familiar with, $\boldsymbol{\mathcal{B}}$ contained in $P(X)$ defined as: So, $\boldsymbol{\mathcal{B}}$ is the collection of all these open balls around various points of x, x is in X and $\varepsilon>0$. So, it is easily checked that $\mathcal{B}$ satisfies the two conditions for being a basis. Thus $\mathcal{B}$ defines a topology $\mathcal{T}$ on X, which has $\mathcal{B}$ as basis. And so therefore, in other words, thus a subset U containing x is open in this topology, if and only if it satisfies the following property: for every x in U, there exists an $\varepsilon>0$ such that this open ball of radius ε around x is completely contained inside. So, this is very similar to the examples that we saw in the beginning of the course that is $\mathbb{R}^n$.

If we have a topological space X over here. So, a set is open if and only if given any point x, we can find a ball of radius ε around x which is completely contained inside. So, the ε of course depends on x. So, it is trivial to check and this is left as an exercise that the topology defined on $\mathbb{R}^n$ using the Euclidean metric is the standard topology. And this check is left as an exercise.

So now we want to understand what it means to be in the closure, in terms of a metric space. So, in terms of the metric. Definition: let X be a metric space and let $\{x_n\}$ for $n>=1$ be a collection of points of X. So, we say x_n converges to x if for every $\varepsilon>0$, there is an n sufficiently large, such that for all, let us say n_0, and this n_0 depends on ε. So, is that for all $n>=n_0$, the x_n's are contained in the ε-ball around x.

So, this x_n, I should write, x_n converges x_n converges to x, and it is often written as x_n converges to x. So, roughly, if we have x, and x is here, and we have sequence of x_n's. So, we want the, the sequence to get closer and closer to x as our n tends to infinity. Or precisely, what we want is, no matter how small we take ε, when we take the ε-ball around x, after finitely many terms, all the x_n should be in that ε-ball.

Let us prove this lemma. Let X be a metric space, and let A contained in X. Then x is in A closure if and only if there is a sequence x_n converging to x and x_n belongs to A. x is in the closure of A if and only if there is a sequence in A which converges to x. Let us take a simple example. If we take, let us say this region.

Then given any point on the boundary, we can find a sequence inside this red region, $x^2+y^2<1$, which converges to this point in the boundary. But if we take some point outside this boundary, then we would not be able to find a sequence. So, then we can find a small neighborhood around this point outside which does not meet the set A. Therefore, no matter which sequence we take inside A, it will not satisfy the definition of converging to x. If we take a point outside this circle $x^2+y^2=1$.

That is a small example of what is happening. Let us see a proof. Let us assume that Suppose x belongs to A closure. Let us say x is over here. So, then By definition of closure, every open subset U containing x meets A.

So, take U_n to be $B_{(1/n)}(x)$. We take U_n to be this collection, and we choose. So, each U_n meets A because x is in the closure. So, choose any x_n in $B_{(1/n)}(x)$ So, we are taking smaller and smaller neighborhoods, and inside each neighborhood we are choosing from each of these neighborhoods, it meets this red region, and we choose one point in the intersection with A. Then we claim that x_n converges to x right.

To prove this what we need to show. We need to show that, given any $\varepsilon>0$, there exists some n_0 very large. That such that for $n >= n_0$ naught we have x_n belongs to $B_\varepsilon(x)$. Let us show this. Note that we can first choose n_0 sufficiently large. So, that $1/n_0$ is strictly less than ε.

So, then we claim that for all $n>=n_0$, we have $B_{(1/n)}(x)$ is obviously contained in $B_{(1/n_0)}(x)$ which is contained in $B_\varepsilon(x)$, and moreover these balls are nested and so on. So, from this and x_n is here $x_{(n+1)}$ is here and so on. All this is contained in $B_{(1/n_0)}(x)$. So, therefore this shows that, this implies for all $n>=n_0$, x_n belongs to $B_\varepsilon(x)$, which implies that x_n converges to x. Conversely, suppose, So, in the first part of the proof, we showed that if x is in A closure, then we can find a sequence of x_n's in A, so that x_n's converge to x.

And conversely, Suppose x_n's is a sequence in A, such that x_n's converge to x, then we need to show that x is in A closure. So, once again we will use a definition. Let U containing x be an open subset. By the definition of the topology on X which is given by the metric right. Then there exists an $\varepsilon>0$ such that $B_\varepsilon(x)$ is completely contained inside U.

U is some open set which contains x and let us say x is here. So, we can find an ε. $B_\varepsilon(x)$ is completely contained inside U and as x_n converge to x, this implies that for all n, there exist n_0 such that for all $n >= n_0$, x_n belong to $B_\varepsilon(x)$. This implies that x_n belongs to $B_\varepsilon(x)$ intersected with A which is contained in U intersection A.

So, thus U intersection A is nonempty. Thus x belongs to A closure. So, in case of metric spaces we have given a nice and intuitive criterion for what it means for a point to be in the closure of a set right. So, once again we return to our earlier example. So, the closure of this set we are taking this open region A is equal to $x^2+y^2<1$. So, the closure of this is exactly the set A closure is equal to $x^2+y^2<=1$. So, we will end this lecture here.