Point Set Topology
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Lecture 28

Recall that in the previous lecture we had proved this nice result, we proved that $SO(n)$ is connected. So, before we proceed let me make the following remark. If we replace, so using the same methods or using similar ideas, we can also show that $SU(n)$ and $U(n)$ are connected. So, recall what is $SU(n)$, $SU(n)$ is all those complex matrices, complex nxn matrices such that A^*A is equal to identity. and determinant of A is equal to 1 and $U(n)$ is all those A in $M(n,\mathbb{C})$ which have already the first condition, the unitary matrices. So, just as in case of $SO(n)$ we define this map to S^{n-1}, for $U(n)$ we define this map to S^{2n-1}, so here a matrix A gets sent to the first column.

Now, $\langle A^*E1,A^*E1\rangle$ is equal to, so in the case of $U(n)$, we use the standard metric, sorry standard inner product on $\mathbb{C}^n$, which is given by $\langle V,W\rangle$ is defined to be W^*V and recall that W^* is equal to W transpose conjugate. So, we have a column vector and we take its transpose and conjugate all the entries, This with this definition, this inner product is going to become $(AE1)^* AE1$ which is equal to $E1^*A^*AE1$ and since A^*A is equal to identity, because A is in $U(n)$ there is going to be $E1^*E1$ which is equal to 1. But also notice that the columns of A, the first column consists of complex numbers z_1, z_2 upto z_n and if each of these z_i, if I write it as z_j, if I write it as x_j+iy_j, where x_j, y_j are real numbers, then this implies that, since the inner product is 1, this implies that summation of z_i^2 is equal to 1, which implies that summation of $x_j^2+y_j^2$ (1 to n) is equal to 1, which implies that we actually get an element of AE1, that can be viewed as an element of S^{2n-1}. Let us call this map, if we call this map ψ and we call this map ϕ here, over here we proved that in this situation, $\phi(A)=\phi(B)$ iff $B^{-1}A$ is a matrix of this type, this is something in S^{2n-1}.

So, here we will have to prove that $\psi(A)=\psi(B)$ iff $B^{-1}A$ is in $[[1,0][0,U_{n-1}]]$. And then the base case, so the base case for the induction will be U_1, which is, you can easily check this from S^1, which is connected. Similarly the base case, we can define a map from $SU(n)$, so we again, let me call this ψ_1 from $SU(n)$ to S^{2n-1}, once again this map A goes to AE1, and here the base case for the induction will be SU_1, which is simply 1, which is connected. So, the details are left as an exercise. Having said this, let us begin with today's lecture.

Today, we will start our discussion on compact metric spaces. Basically we are going to, so the main theorem of this lecture is the following: Let X be a metric space, X is compact, iff every sequence in X has a convergent subsequence. Let us prove this. First let us assume that X is compact. Suppose we are given a sequence x_n, $n \geq 1$.

So, we are given the sequence of points and we want to show that S has a convergent sub sequence. Let this be a sequence in X. If S is finite, so if the cardinality of S is finite, then there is some x in x_n in S, such that x is equal to x_{n_j} for infinitely many n_j's, for infinitely many indices. And then we can take this subsequence. So, in this sub sequence, every element, all the elements are the same and therefore obviously it converges.

So, let us assume, so let us assume that cardinality of S is infinite. We may replace S by a subsequence of S. Since we may replace S by a subsequence of S, we may assume that all elements of S are distinct. So, the cardinality of S is infinite, so therefore we can find a subset which is infinite and each member of that subsequence is, all the members of that subsequence are different. So, if you can find a subsequence of this subsequence which converges, then we would have found a subsequence of S which converges.

We can assume this. So, let Y be the closure of S in X. So S is a set of points like this and if $Y \setminus S$ is nonempty, that means there is some point x_0 in Y, in $Y \setminus S$, then there is a point x_0 in $Y \setminus S$. And as x_0 belongs to S closure, this implies when we take any neighborhood of x_0, $B(1/n,x_0)$ intersection S is nonempty, and this implies that there exists a sequence of x_n in S such that x_n converges to x_0. We are done in this case.

So, we are done. If $Y \setminus S$ is nonempty, then we have found, so let me say there is a sequence y_m such that y_m converges to x_0. So, then we have found a subsequence in S which converges. So, let us assume that $Y \setminus S$ is empty. So, note that S is always contained in S closure which is equal to Y, and $Y \setminus S$ is empty implies that Y is equal to S is equal to S closure.

So if x_n is any point in S and there is a subsequence in S, then there is a subsequence converging to x_n, then we are done. So, for instance we may have, we can take a sequence like this and this sequence may converge to this point x_n over here. Let us assume that, thus we may assume that for every x_n, there is no subsequence in S which converges to x_n. In other words, what this means is for for every x_n there is a small neighborhood, for every x_n there is a neighborhood $B(\delta_n,x_n)$ such that $B(\delta_n,x_n)$ intersected S is just this x_n. So now, as Y is S closure and this is equal to S, this implies S is a compact, is a closed subspace of a compact space.

Recall that we had assumed X is compact. So, thus S is compact, closed subspace of a compact space is compact. We may write S as the union over $n \geq 1$ $B(\delta_n,x_n)$ intersected S. This is x_n. This is simply equal to union $n \geq 1$ $\{x_n\}$'s.

This shows that S has an open cover which has no finite subcover which is a contradiction. So, this completes the proof of one part. This shows that S has a convergent subsequence. Now, let us prove the converse. Next we prove the converse.

Let us assume that, so suppose every sequence in x has a convergent subsequence. We shall show that x is compact. So, for this let us begin with an open cover, B and open cover of X. Our aim is to construct a finite subcover of this, So, we make a claim. There is a $\delta>0$ which works for all x, such that when we take this open set $B(\delta,x)$, so we are given, we are given a space X and this has an open cover.

So, no matter which x we choose, when we take this ball of radius δ around x, it will be contained in one of the U_i's. It is contained in U_i for some i. It is completely contained in one of the U_i's. Let us prove this. So, if claim one is not true, then for each, no matter how small we take δ, there will be an x such that the ball of radius δ around x is not going to be contained in any of the U_i's.

For each n greater than or equal to 1, there is an x_n such that we take $B(1/n,x_n)$ is not contained in U_i for any i. So, let us take S to be the sequence of x_n's. So, then there is by our assumption, there is a convergent sequence, a convergent subsequence x_{n_j} and let us say this converges to some x_0 in X. So, there is some U_0 such that x_0 is in U_0 in our open cover and for this U_0, there is an ε positive such that this ball of radius ε around x_0 is contained in U_0. So, our U_0 could be this and our x_0 could be somewhere here, this could be our U_0, and this ball of radius ε is completely contained inside U_0.

So, for j, for j very large, we will have, since our x_n, since our x_{n_j} is converged to x_0, x_{n_j} is going to go, belong to $B(\varepsilon/2,x_0)$. So, we can take the ball of radius $\varepsilon/2$ around x_0. So, we can take this to be x_{n_j}. So we can choose j large so that $1/n_j$ is strictly less than $\varepsilon/2$.

Then it is easily checked. In fact, if we take the ball, so x_{n_j} is here, our x_{n_j} is here, let us say this is x_{n_j}. So, if I take a ball of radius $\varepsilon/2$ around x_{n_j}, so then this ball is also going to be completely contained inside the ε-ball around x_0. Then it is easily checked, so before that, then the ball of radius $1/n_j$ around x_{n_j} is contained in the ball of radius $\varepsilon/2$ around x_{n_j} and this ball is going to be completely contained inside the ball of radius ε around x_0, which is contained in U_0. This contradicts the assumption that $B(1/n_j,x_{n_j})$ is not contained in U_i for any i. Recall that we had constructed this sequence x_n's by requiring that they satisfy this condition, but we have now contradicted this condition.

So, this proves claim 1. Now let us prove claim 2. In fact, claim 2 is the main assertion. So there is a finite subcover of x. So let us prove claim 2. By claim 1 there is a δ such that for every x $B(\delta,x)$ is contained in U_i for some i.

So thus we have $B(\delta/2,x)$ is contained in $B(\delta,x)$ is contained in U_i. So let us start a process of covering X as follows: choose any y_1 in X, and let X_1 be equal to $B(\delta/2,y_1)$. So our X is like this and we can just pick any y_1 and we take this ball of radius $\delta/2$ around y_1. So let me make this. We choose y_1 and we take this ball of radius $\delta/2$ around y_1.

Now choose y_2 from $X\setminus X_1$. So we can choose any y_2 over here, which is not in this ball, and we take the ball of radius $\delta/2$ around y_2. And let X_2 be equal to $B(\delta/2,y_1)$ union $B(\delta/2,y_2)$. Assume that we have defined x_n, and define x_{n+1} as follows: We choose any y_{n+1} in $X\setminus X_n$ and let X_{n+1} be equal to $B(\delta/2,y_{n+1})$ union X_n. So we have these balls and this is our y_{n+1} So then we have X_1 is contained in X_2, is contained in X_3 and so on. We claim that this process has to stop in finitely many steps.

Why? What happens if it does not stop? If not, then we get a sequence of points y_n's, such that the distance between y_i and y_j is greater than equal to $\delta/2$ for all i not equal to j. This is how the y_n's have been constructed. We take the open balls of radius $\delta/2$ around y_1 and we chose y_2 outside that ball which means that the distance of y_1 from y_2 is greater than equal to $\delta/2$. Now we take open balls of radius $\delta/2$ around y_1 and y_2 to their union and y_3 was selected outside this. That means that the distance of y_3 from y_1 and y_3 from y_2 is also greater than equal to $\delta/2$, And this sequence cannot have a convergence of sequence.

So thus this process stops. So this implies that our X is contained in the finite union of $B(\delta/2, y_i)$'s, i=1 to n. And each of these $B(\delta/2,y_i)$'s is contained in some U_{ij}, equal to 1 to n U_{ij}. So this implies that there is a finite subcover which implies X is compact. So this completes proof of the theorem. So we will end this lecture here.