Point Set Topology
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Lecture 34, Part II

Next as an application of quotient topology, we will define the Grassmannians. Let us begin. Let G be a group, such that the set G, the underlying set G has a topology. We say that G is a topological group if the two maps, since G is a group, we have the multiplication map, which is (x,y) maps to xy. The product in the group, and we also have the inverse map x goes to x^{-1}. If these two maps are continuous, this question makes sense because G is also, there is an underlying topology on G, and therefore G x G has the product topology.

So we can ask if both these maps are continuous, and if both these maps are continuous then we call G a topological group. So, let us begin by making a few observations about topological groups. First if G is a topological group, then left translations. So, these are the maps L_a from G to G, L_a(g) is defined to be ag and right translations, these are the maps R_a from G to G, R_a(g)=ga.

These are continuous. So, let us see why this happens. So, we can look at GxG to G, we have the multiplication map right and here we can look at {a}xG, inside GxG. So, this inclusion is given by (a,g) maps to (a,g), and when we multiply we get ag. So, this is homeomorphic to G.

So, this composite is L_a. Since this inclusion is continuous and multiplication is continuous, as G is a topological group, this implies that L_a is continuous. Similarly, we can take Gx{a} sitting inside GxG and this is homeomorphic to G, and to see that the right translation R_a is continuous, So, the inverse of L_a, the left translation is bijection and the set theoretic inverse is exactly L_a^{-1}. So, this implies that L_a and R_a are homeomorphisms. Similarly for R_a are homeomorphisms.

Similarly, the inverse of i, since the inverse map is a bijection, and i^2 is equal to identity. This implies that the inverse map is also a homeomorphism. So, left translations, right translations and inverses are all homeomorphisms. The next observation we want to make is: let {U} be the collection of open subsets containing the identity element. Then for a in G, the collection aU, here aU is equal to the left translate of this set U, is the collection of open sets containing a.

Clearly, since L_a is a homeomorphism, L_a(U) is an open subset and it contains a because the identity is in U. And conversely if we take an open subset W which contains a. So, then this will imply that $a^{-1}W$ contains e, and since this is equal to $L_a(W)$. Since L_a^{-1} is a homeomorphism, $L_a^{-1}(W)$ is going to be in this in this collection over here. Therefore, $a^{-1}W$, we can write as $a(a^{-1}W)$.

Therefore, even this W is obtained is in this collection. Similarly the collection Ua is the collection by the same region, containing the element a. So, let us make this third observation, which is a lemma, which is a very useful lemma. So, let V containing e be an open subset. Then there is an open set U which contains the identity, such that the set $U*U = \{ab$, with a and b in $U\}$.

So, this is equal to U^2. So, we have U^2 is going to be containing V. So, given any open set V, which contains the identity, we can find a smaller open set U such that U^2 is contained in V. So, for instance if we take the real line under multiplication, I am sorry under addition, and we take the neighborhood of 0. If we take an open subset which contains 0, let us say $B(0,\varepsilon)$, this V then we can take U to be $B(0,\varepsilon/2)$.

So, the multiplication over here is just the addition. So, if you take any (x,y) in U then clearly $x+y$ is in V. Let us prove this. Consider the multiplication map from GxU to G. This is the multiplication map which is continuous.

This implies that the inverse image of V is an open subset and it contains this element. In particular there is a basic open subset, which contains this element, and it is contained in $m^{-1}(V)$. This implies there exists U_1 containing e and U_2 containing e, such that $U_1 x U_2$ is contained in $m^{-1}(V)$. So, we just take U to be U_1 intersection U_2. Then UxU is containing $m^{-1}(V)$, which implies that $U*U$ is contained in V.

Let us make the fourth observation. Let V be an open set containing an element x in G. So, then there exists an open set U, which contains the identity, such that U is equal to U^{-1} and so U^{-1} is the set of u^{-1} where u belongs to U, or equivalently this is $i(U)$. So, U is equal to U^{-1} and UxU is contained in V. So, the proof is very similar to the previous lemma.

So, let us prove this. Proof so consider the map GxGxG to G. We can first project to the first two coordinates, sorry not this, this is (a,b,c) maps to (ab,c) which maps to abc. So, this map is continuous, this is easily checked. So, we restrict it to to the subspace $Gx\{x\}xG$.

So, this composite is then (a,x,b) maps to axb. So, this implies that. Let us call this composite map f, this is homeomorphic to GxG. So, since f is continuous, this implies that $f^{-1}(V)$ is open. So, again this implies that there exists $f^{-1}(V)$ is open and it contains the element exe.

So, this implies that there exists open sets U_1 containing e and U_2 containing e, such that U_1x{X}xU_2, this x is really playing no role over here. So, we could have simply considered the map GxG to G, given by (a,b) maps to axb, and once you prove that this map is continuous we can do this entire argument. So, is contained in f^{-1}(V). We take U_0 to be equal to U_1 intersection U_2.

So, we take U_0 first. So, this will imply that U_0x{x}xU_0 is contained in f^{-1}(V), which implies that U_0xU_0 is contained in V, but U_0 may not have the property that U_0^{-1} may not be equal to U_0. So, to rectify this we take U to be equal to U_0 intersected U_0^{-1} which is equal to U_0 intersected i(U_0). Since i is a homeomorphism i(U_0) is also an open subset. So both these contain the identity. So, the intersection is an open subset which contains identity and it is easily checked that U is equal to i(U), and obviously UxU is going to be contained in V.

Now that we have these lemmas in place, we will prove this interesting, we need one more lemma. Let H be contained in G be a closed subgroup. So, what this means is that H is a closed subspace of G, in the topology on G and also H is a subgroup. Let x be an element which is not in H. So, then there exists U containing identity and open, such that U is equal to U^{-1} and UxU intersected H is empty.

Let us prove this. Since H is closed and x does not belong to H, this implies there exists an open set V and we can simply take the complement of H containing x, such that V intersected H is empty. Therefore this previous lemma that we proved, this lemma 4 implies there exists U open which contains identity, such that UxU is contained inside V. So, this implies that UxU intersected H is empty. So, using these preliminary results we prove the following proposition: let H contained in G be a closed subgroup. Then G/H with the quotient topology is Hausdorff.

So, let us recall that G/H is equal to G mod equivalence, where x~y (every subgroup defines an equivalence relation) iff y^{-1}x belongs to H. So, we have this equivalence relation on G and G is a topological space. So, we can look at G mod equivalence and give it the quotient topology and our claim is that the space is Hausdorff. So, let us prove this. Let us take two points in G/H which are distinct.

So, be two cosets, then this y^{-1}x does not belong to H. Choose a neighborhood E of identity, open, such that U is equal to U^{-1} and Uy^{-1}xU intersected H is empty. We can do this using the previous lemma. Note that this happens: That Uy^{-1}xU intersected H is empty iff xUH intersected yUH is empty. That is an easy set theoretic check.

So, let us do this. So, suppose let us assume this is empty, and prove this that this intersection is empty. If not then there is xu_1h is equal to $yu_2h_1h_2$. This implies that $u_2^{-1}y^{-1}xu_1$ is equal to $h_2h_1^{-1}$. This implies that since u_2 is in U, u_2^{-1} is also in U, because U is equal to U^{-1} intersection H, is not empty, this is a contradiction. Similarly we can prove the other way, let us assume this is empty and let us prove this.

So, if this is not empty, then there is $u_1y^{-1}xu_2$ is equal to h. This implies that xu_2 is equal to $yu_1^{-1}h$, but this implies that xUH intersected yUH is not empty, which is a contradiction. Thus we see that this holds. Now if π from G to G/H denotes the natural map, then note that $\pi^{-1}(\pi(xUH))$ is equal to xUH. In fact one easily checks that $\pi^{-1}(\pi(A))=AH$.

So, this is equal to union of all h in H of Ah for every subset A contained in G. This is an easy check, which I will leave it to you, and using this check, what we will get is, this check implies $\pi^{-1}(\pi(xUH))$ is equal to xUH multiplied with H, but H times H is equal to H since H is a subgroup. So, this is just equal to xUH. So, similarly $\pi^{-1}(\pi(yUH))$ is equal to yUH. So, now as xUH is equal to union h in H of xUH, and these sets are open and right translations by elements of H are homeomorphisms.

So, each of these sets is open, and we are taking a union of open sets. So, this implies that this set is open, which implies by the definition of quotient topology that π(xUH) is an open subset in G/H which contains xH, this coset. Similarly, π(yUH) is an open subset which contains the coset yH. So thus, we have our xH here, we have our yH here, we have constructed two open subsets of these two, in the quotient topology and we claim that these are disjoint. If not, let us assume they are not disjoint.

So, what we have we have π(xUH) intersected π(yUH) is nonempty. Since π is surjective, that will mean that when we take π(xUH) intersected π(yUH) is nonempty. This implies that $\pi^{-1}(\pi(xUH))$ intersected $\pi^{-1}(\pi(yUH))$ is nonempty. So, this is nonempty as π is surjective, but this is equal to xUH and this is equal to yUH as we saw and this intersection is empty, which is a contradiction. Thus these two open neighborhoods are disjoint.

So, this shows that G/H is Hausdorff. So, this completes the proof of the proposition.