Introduction to Probability Theory and Stochastic Processes
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Module – 01
Basics of Probability
Lecture – 01

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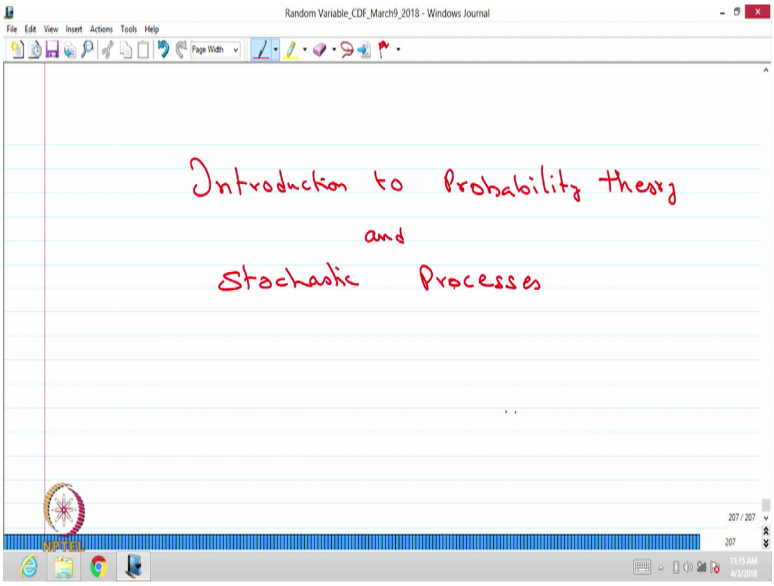

This is a course on Introduction to Probability Theory and Stochastic Processes is a course on Introduction to Probability Theory and Stochastic Processes. This is a undergraduate level course. In this course, we are going to discuss the two topics; one is a probability and the second topic is a stochastic processes.

Since, it is undergraduate course, we are going to discuss probability and the stochastic processes in the introductory level; that means, we can always have a post graduate course level on probability separately and there will be a separate course on stochastic processes where as this is undergraduate course on Introduction to Probability Theory and Stochastic Processes. The word theory is important in this course because we are going to cover some of the theoretical aspects of probability as well as the some problems on the probability as well as the stochastic processes.

So, we are going to discuss the theory part of probability as well as a some theory part of a stochastic processes also in this course. The whole course is a divided into 12 models, each models is roughly of a 3 hours lectures. So, this 12 models; out of out of 12 models, 8 models- we are covering for probability theory and the remaining 4 models for the stochastic processes.

Out of 8 models in probability theory, we started with the first model of basics of probability, second model we discuss the random variable, third model we have a movements and inequalities, fourth model standard distributions. So, basically the first 4 models; we discuss about the one dimensional random variable that is basically sort of very elementary level of probability, then the fifth model; we started with the random vectors; that means, multi dimensional random variables.

Then in the sixth model, we discuss the functions of several random variables in the seventh model, we discuss the cross movements and eighth model we discuss the limiting distributions that is a very important topic in the probability theory limiting distributions. So, with that 8 models we cover the probability theory then from ninth model onwards we have stochastic process.

So, there also, we disuses only the introductory level; that means, we started ninth model with introduction to stochastic processes in which we define definition properties some common random processes or stochastic processes and some important properties, then in the tenth model. We discuss the discrete timer coaching in detail and the eleventh model, we discuss the continuous timer coaching in the twelfth model, we discuss the simple Markovian queuing models simple means the underline stochastic process is going to be a birth death process.

So, we discuss stochastic processes in 4 models and probability theory in 8 models, let me start with the first model that is basics of probability.

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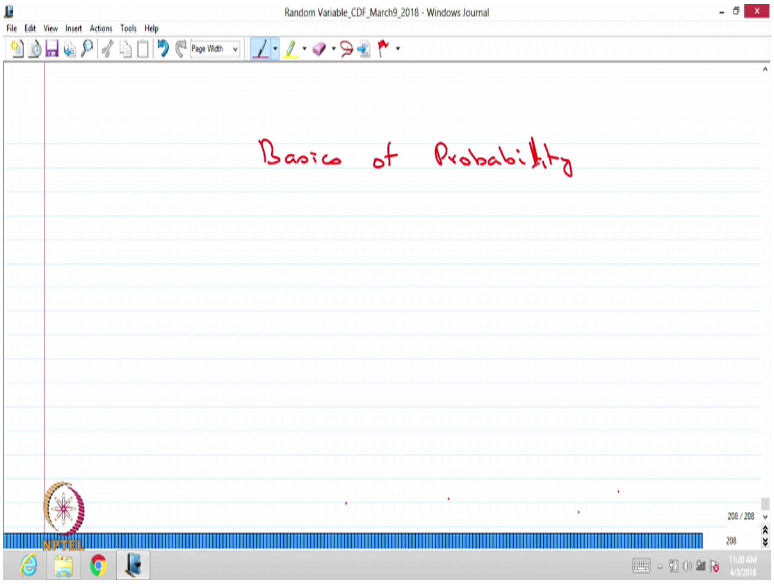

In this model, we are going to discuss about the random experiment, then sample space, then axiomatic definition of probability and probability space in the first lecture and the second lecture, we are going to discuss the conditioning probability and the independent of events. And in the third lecture, we are going to discuss the total probability rule, multiplication rule and base theory with that we are going to complete the first model that is called basics of probability.

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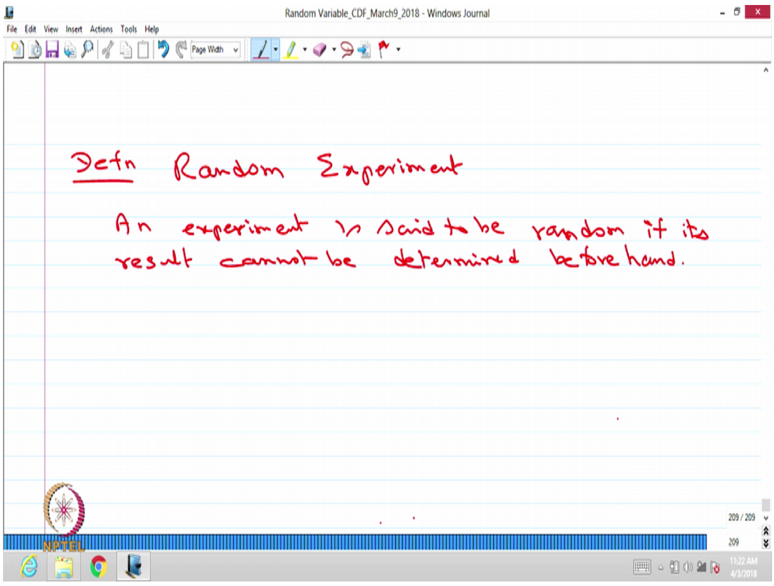

Let me start with the definition of a random experiment, the first definition that is called random experiment an experiment. An experiment is said to be random if its result cannot be determined beforehand whenever in any experimental result cannot be determined beforehand, then we will say that experiment is a random experiment in the whole probability theory.

We started with the random experiment; that means, we have a experiment in which the results cannot be determined beforehand, we can think of a many examples over the course, we are going to discuss one by one examples, therefore, I am going to give the examples little later.

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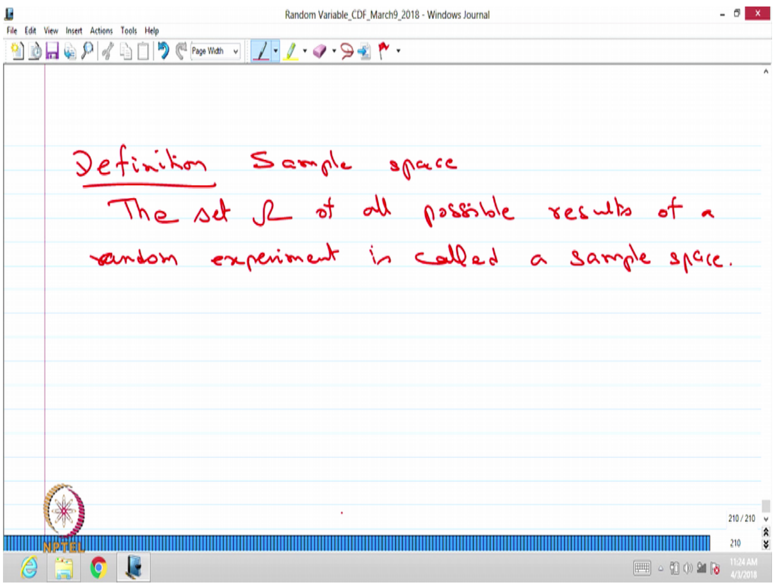

The next definition; sample space, the set omega of all possible results of a random experiment is called a sample space.

The sample space is nothing, but the collection of all results or outcome of a random experiment; that means, we have a random experiment in which the results cannot be determined beforehand. If you able to collect the all possible outcomes of a random experiment that collection are that set of all possible results that is going to be call it as a sample space; each possible outcome that is called the sample either possible results or possible outcomes one and the same.

So, each possible outcome or each possible result that is going to be called it as a sample and collection of all possible results or all possible outcomes that going to call it as a sample space since we have a random experiment of a many kind the set could be consisting of a finite number of elements or it could be countably infinite number of elements in it or it could be uncountably many elements the set omega may consisting of finite elements or countably infinite number of elements or uncountably many elements not only that; those elements could be numerals or non numerals, you may have some random experiment in your mind all possible outcomes or results of a random experiment, if you make a collection that collection could be numbers or it could be non numbers.

Similarly, it could be finite or uncountably infinite or uncountably many elements.

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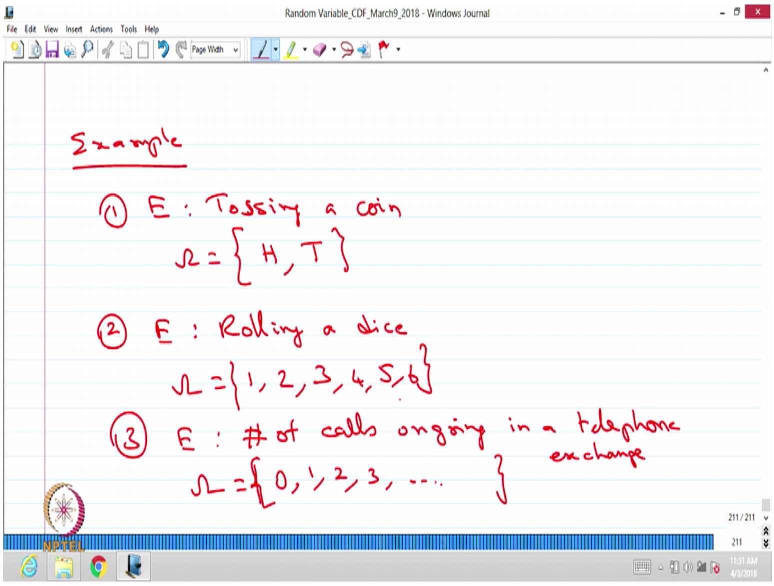

So, I will go for few examples for the sample space. Example one; the easiest example in the random experiment, I use a notation E for random experiment, random experiment is the suppose you have a very simplest example tossing a coin, tossing a coin that is a random experiment in the set of all possible outcomes.

In these random experiment is going to be I use a notation H for if I get the head I use a notation H for obtaining head. So, only I use the notation capital T for getting a tail; that means, the only two possible outcomes are head or tail.

So, this is the collection of all possible outcomes in these random experiment lets go for another example, the random experiment is rolling a dice here the all possible outcomes, I show the each side that number, it is a number which I am going to get either the number 1 or I am going to get the number 2 or 3 or 4 or 5 or 6, these are all the possible outcomes in the dice whenever you roll, the possible outcomes going to be either 1 or 2 or 3, 4 or 5 or 6. These are all the possible outcomes. So, the omega consisting of all possible outcomes that all is very important; you should collect you should make a set consisting of all possible outcomes.

So, first one has a only 2 elements, whereas, a second one has 6 elements, third example; random experiment is a number of a calls telephone calls ongoing in a particular telephone exchange, number of a ongoing calls, calls ongoing or ongoing calls in a telephone exchange the omega all possible outcomes. There is a possibility no calls or there is a possibility only one call or there is a possibility two calls and so on; even though it may a finite number of calls.

So, you can just putting a dot dot dot; that means, it is a huge number. So, it could be 1000 or it could be 2000 or whatever be the number. So, I am just writing 0, 1, 2, 3 and so on. So, so many calls are going this is going to be a all possible outcomes of a these random experiment it could be finite also. So, this is going to be the collection; collection or set, you see the first example; it is not a numbers whereas, the second example we have a numerals third example we have a numerals.

So, the way you have a random experiment, the collection of all the possible outcomes could be numerals or non-numerals, and the collection could be finite number of elements or countably infinite number of elements or uncountably many elements also, suppose I give a fourth example in which the random experiment is temperature of a particular city.

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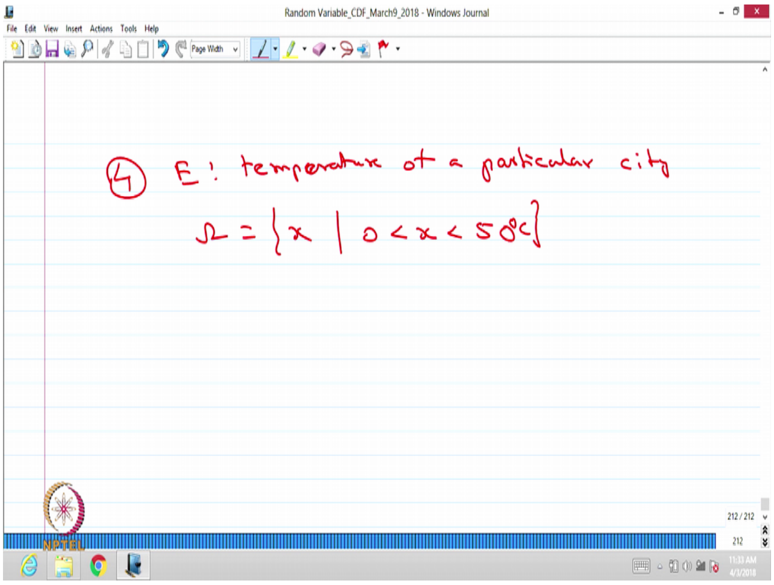

In these random experiment, the omega is going to be the temperature can vary from some range for exampled may range from collection of a X such that in that particular city the temperature is a most of the time say lies between 0 to 50 degree centigrade. Suppose, I make the assumption, the temperature is always goes from 0 to 50.

So, the omega is going to be a collection of values lies between 0 to 50. So, it is a real number, therefore, a the elements of omega is a countably many between the interval 0 to 50 degree, 50 degree centigrade, I have given the example in which it is going to be numerals or non numerals, similarly, it could be finite or countably infinite or uncountably many elements also.

Now, we move into the next definition that is called sigma field.

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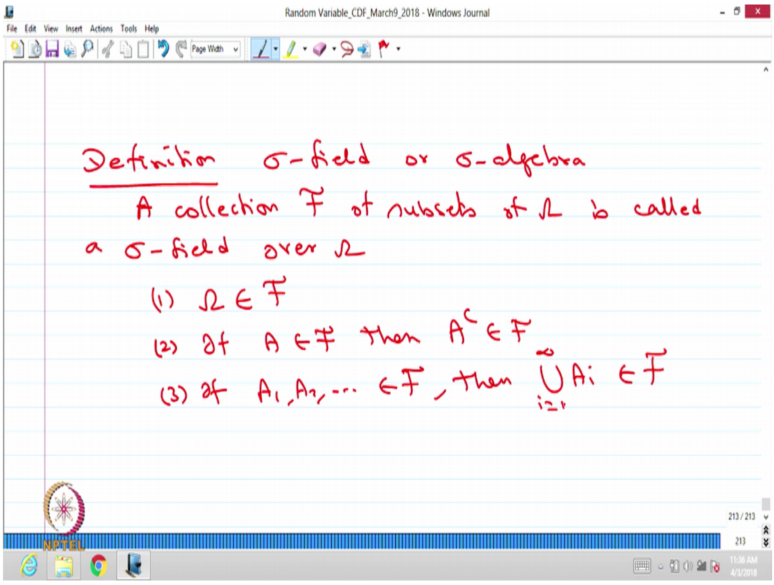

Or there is another name called sigma algebra, a collection a collection that is denoted by F of subsets of omega; that is called a sigma field over omega, either, you can use over omega or an omega; the omega is a nonempty set such that the omega is belonging to F second one if a is belonging to F then a compliment that is also belonging to F a collection F of subsets of omega is called sigma field over omega or on omega, whenever it satisfies these 3 condition, the first condition is omega belonging to F.

If a belonging to F, then A cap A compliment is also belonging to F the third condition, if A 1, A 2 and so on; that is belonging into F, then the countable union i belonging into one to infinity that is also belonging to F whether you take a finite elements in the F or countably infinite elements in the F the union of those elements also belonging to F.

If a any collection F of subsets of omega satisfying these three conditions, then we call that collection is called sigma field on omega. So, omega has to be a nonempty set then satisfies these 3, these 3 conditions of a collection of a subsets of omega, then it is called the sigma field or sigma algebra. This is going to be a very important to define a probability, let me give a examples for the sigma felid, then we will go to the definition of probability.

Now onwards, I want discuss the random experiment, I will start with the sample space.

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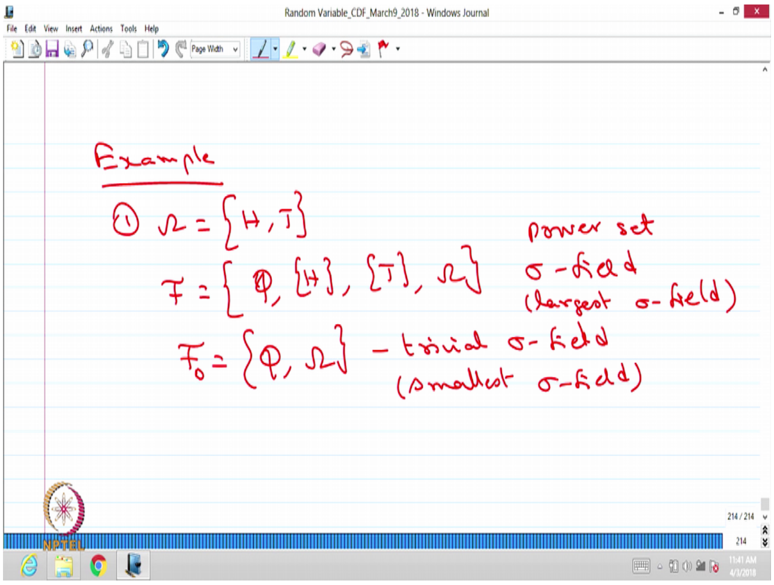

Suppose the sample space is the same, example head, tail, we can create the sigma field or sigma algebra on omega that is F empty set, this is empty set single term element H tail and omega, you can verify whether all three conditions of the sigma field is satisfied empty set is A 1 element compliment is the whole set both are belong in to F, the third condition, if I take a few elements union is also belonging to the same F. Suppose, if I take H as well as empty set union is going to be again H that is also belonging if I take tail and the omega the union is going to be omega and if I take a head and tail union, then that union is going to be omega that is also belonging that is a third condition second condition, if I take any one element the compliment is also belonging to the F.

So, if I take empty set complement is omega that is also belonging and the H compliment is tail that is also belonging to F and tail compliment is a H that is also belonging to F omega complement is empty sectors are also belonging to F. Therefore, all those 3 conditions are satisfied by this collection which collection is a subsets of omega, therefore, this is going to be a sigma field or sigma algebra, this is not the only sigma field which satisfies the three conditions, we can have A 1 more sigma field that is empty set and the whole set.

This is also satisfies all the three conditions of a sigma field because omega is one of the elements compliment of empty set and the whole set is there and the union is also belonging to the collection therefore, this is also sigma field. So, this is called a trivial sigma field for any omega, you can always make a sigma field that is a consisting of empty set and the whole set that is a trivial sigma field, even some books, they use the notation that is F suffix naught if it is x suffix naught that is a trivial sigma field.

So, not only this trivial sigma field, this is a smallest sigma felid also, since I use the word smallest sigma field the other one which has a singleton element starting from empty set a singleton element and the whole set.

So, this is the largest sigma field, since it has A 2 elements omega consisting of two elements, if you go for 2 power 2 that is 4. So, number of elements in the this sigma field F as a exactly 4 elements, therefore, this is the power set also this is the power set which is the largest sigma field. In this example, we have a trivial sigma fields one is the smallest one and other one is the largest one that is a power set in between we are not getting any other sigma field, we are getting only 2 sigma fields because it has only 2 elements; one is if the smallest, other one is the largest. Now, I am going to discuss the some example in that we will land up and non trivial sigma fields other than smallest and the largest second example.

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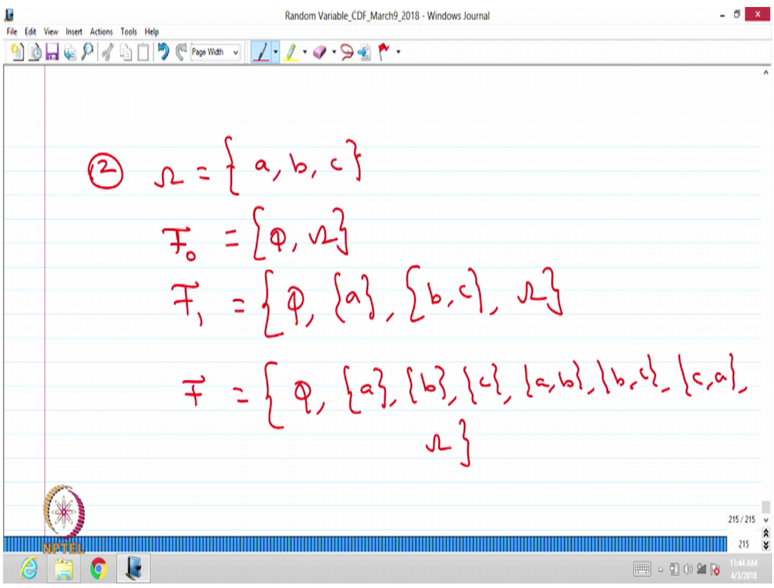

I would not bother about what is the random experiment, directly I give the sample space.

So, the sample space consisting of three elements, sample space consisting of 3 elements. Now, you will go for creating the sigma field, the smallest one always empty set and the whole set the another sigma field which I can create empty set I take a singleton element a b and c together therefore, a compliment is band c, b and c compliment is a and union of a, b and c that becomes the whole set.

You can verify; you can verify F 1 that is a sigma field because it satisfies all the three conditions of the sigma field, I can go for similarly, I can go for another sigma field keeping b separately and a and c together. Similarly, I can go for another sigma field keeping c separately, a and b together; that means, I can create two more sigma field of similar kind, I am going for the largest sigma field satisfying all the three conditions that is empty set singleton element.

Any two element ab, bc and ca and all three elements together count 1, 2, 3, 4, 5, 6, 7, 8 elements number of elements in the sigma field is 3. So, 2 power 3 is 8. So, this is the largest sigma field nothing, but the power set. So, in this example we have created the trivial sigma felids like the smallest one and the largest one and in between, we have created some more sigma fields, I have created one, similarly, you can create two more sigma fields; that means, for a non empty set omega based on the number of elements.

You can always able to create many sigma fields including the trivial sigma fields and the sigma fields is going to be a play important role in the probability that we are going to discuss in the later part the first example and the second example, we have a finite elements though omega consisting of finite elements, we can go for creating the sigma field when omega is uncountably many elements.

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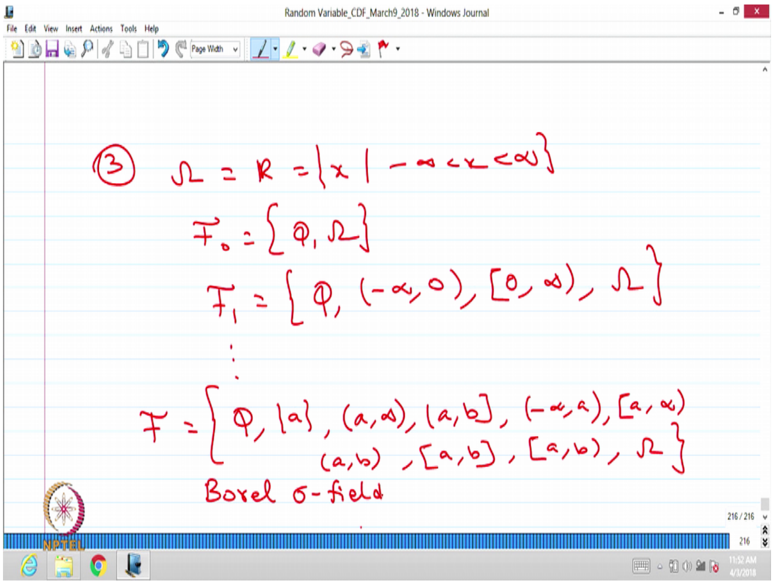

Let us go for third example that is omega is same as the real line; that means, the elements is from minus infinity to plus infinity that is a collection of x such that x lies between minus infinity to infinity real numbers uncountably many omega consisting of uncountably many elements in it.

Now, we are going to create the sigma field on omega the omega consisting of uncountably many elements; obviously, here also that trivial sigma field is going to be empty set and the whole set we can go for non trivial sigma field F 1 consisting of a empty set open interval minus infinite to 0 the other element is closed interval 0 open interval infinity.

Then the whole set; you can verify whole set is one of the element and the second condition is a complement of every element is also belonging to the same collection complement of empty set is whole set complement of minus infinity to 0; that is a 0 to infinity, the closed form and come complement of 0 to infinity; that is minus infinity to 0, complement of a whole set is empty set and if you make a union of any 2 elements, are any 3 elements, are all 4 elements that is also belonging to F; F 1 therefore, F 1 is also going to be a sigma field. The way I break minus infinity to infinity at 0 1 side open interval, other side 0, you can think of any real number between the interval minus infinity to 0, you can partition the whole interval into two pieces at any real point, then you can make A 2 elements.

Then you will have a many sigma fields this is a easiest sigma field one can create by partitiong the interval into 3 pieces, 3 subintervals, then it is going to be little difficult, you are to go for creating a union and the complement of everyone interval also belong into F and F 1 that is little TDS job, but still you can partition the interval minus infinity to infinity into finite number of subintervals then accordingly you put some more elements. So, that all the three conditions are satisfied like that you can create many sigma fields, but we are going to have a largest sigma field, you can just visualize what are all the forms of intervals are elements going to be the element of F or the collection of a subsets of omega.

So, that F is going to be sigma field in which it is going to be elements of it start the with empty set it is with the singleton element a and it is going to be having the element of a to infinity form, a can be a real number and it will be of the form a to b, b is also real number and it is going to be of the form minus infinitely to a open and it is going to be of the form closed interval a to infinity and it is going to be of the form a to b open interval, both closed interval a open interval b, here both a and b can take any real number. So, these are all the all the combinations of different real numbers we will land up.

Ok one more element that is omega also. So, these are all the different forms will form a collection that is going to be the largest sigma field on or over omega were omega is a real line special name for this sigma field that is called Borel sigma felid this is Borel sigma field on the real line, suppose, I have a omega which is not the whole real line. It is a in the sub interval 0 to 1 closed interval, then you can always a create a F which is Borel set on real line intersection with the 0 to 1you no need to have a whatever Borel sigma field, you created on the real line, suppose your omega is going to be a subinterval between minus infinity to infinity.

Then you can always make a sigma field on omega which is a subinterval of real line by intersecting that interval with the Borel sigma field, therefore, now we have discuss the how to create the sigma field. Whenever we have omega having a finite number of elements and uncountably many elements; obviously, when you have a countably infinite number of elements you can do the similar exercise for creating the sigma field.