Introduction to Probability Theory and Stochastic Processes

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Module - 02
Random Variable
Lecture - 06

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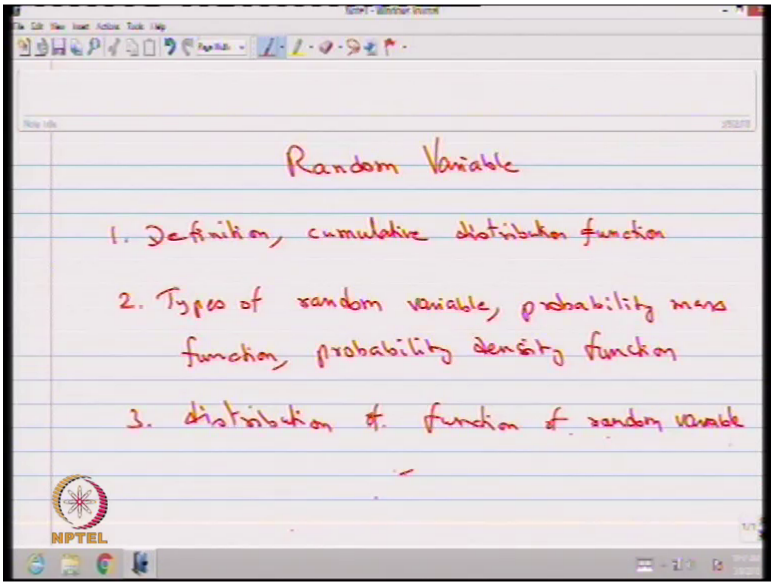

In this topic we are going to discuss three things in three different lectures. In the first lecture which we are going to discuss the definition of a random variable and then we are going to discuss cumulative distribution function of a random variable. So, this is going to be the lecture 1 of a week 2, and the second lecture which we are going to discuss that is a types of a random variable followed by we are going to discuss probability mass function, we are going to discuss probability mass function. Then we are going to discuss probability density function.

In the third lecture which we are going to discuss that is a distribution of a function of random variable. So, this is going to be the week 2 lecture 3 lectures. The first lecture is definition of a random variable and then a cumulative distribution function. And the second lecture is types of random variable probability mass function and probability density function and the third lecture is a distribution of function of random variable.

Let us start with the first lecture definition of random variable. This plays a important role of a describing a random data therefore, the random variable is very important concept in the probability theory.

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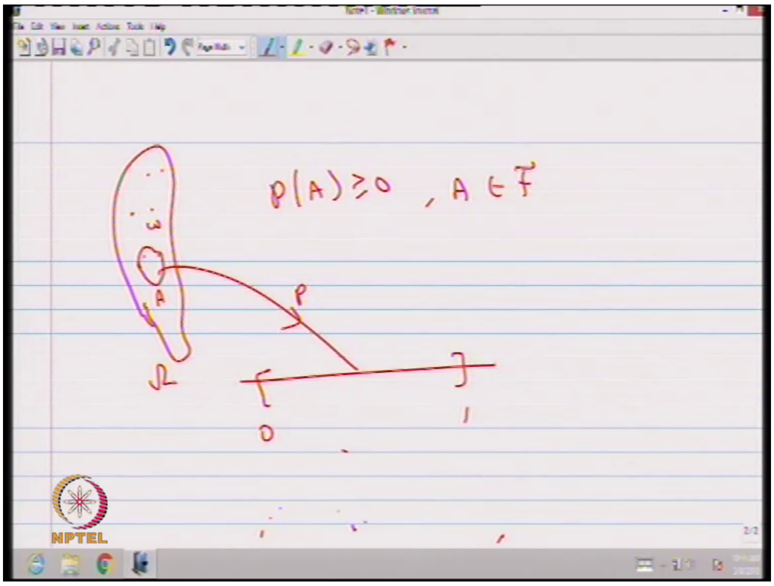

As you know in the random experiment you have a collection of a possible outcomes which you denoted by ws, and the collection of possible outcome is a sample space omega, which consist of all possible random sample or sample which you denoted by ws. Whenever we defined the probability of a event, the event is a collection of a elements of omega that is nothing but the event.

So, the probability of event is greater than or equal to 0 where A is belonging to the sigma field, which we discuss in the first week. Suppose, you know the probability of event you can find out the probability of a compliment of event probability of a collection of a few events as a union of or so many form of a probability of event you can find, but the problem with event is event with the probability definition you need a all the set operations. The set operation means union compliment intersection. But this is going to be very tedious job when you have a very complicated example in which the collection of a possible outcomes are huge. Therefore, the sigma field is going to be very big, it is a tedious to find out the probability of a any event where event is belonging to F where F is sigma filter.

So, to avoid the set operation, we transform by real valued function the probability of event can be computed through the set operations by nicely defining a real valued function. Now, we are going to define the random variable that is nothing but the real valued function, which will useful to find out the probability of a some events not only probability many more information you can get it for the random experiment instead of the set operation, which we have discussed in the first week.

So, let me just recall how the earlier definition goes. So, you take few possible outcomes that you make it as a event A. So, you define a probability over event A; therefore, you are getting the values from 0 to 1. So, this is the way one can define the probability of a event A, where event is nothing but the collection of fewer elements of omega. So, this values is going to be 0 to 1.

Now, what we are going to do in a little left different way that is, we are going to make I will go to the next slide. Let me give the definition of a random variable through that definition I am going to explain how it goes the definition of a random variable.

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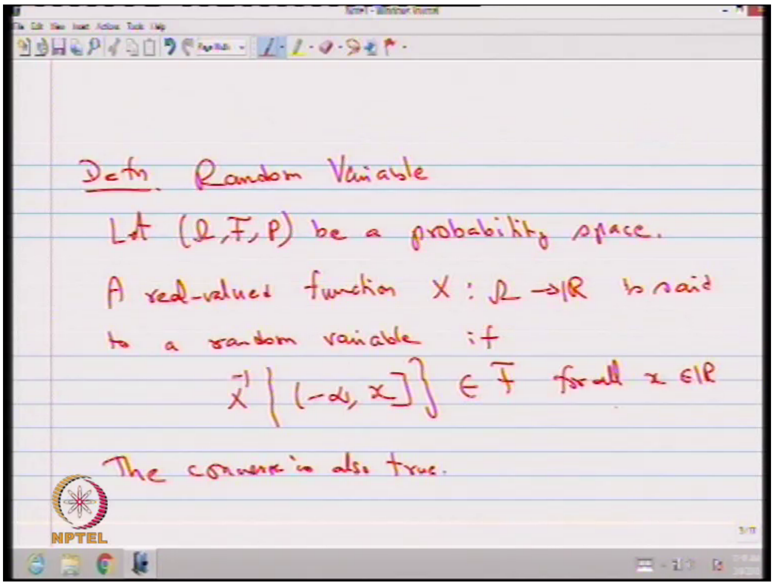

Let omega e F capital P be a probability space, omega is a collection of possible outcomes, F is the sigma field on omega and P is the set function defined on F such that satisfying the 3 conditions that is kolmogorov asymmetric conditions of probability that is p of a is greater or equal to 0, p of omega is equal to 1, p of union of ais when ais are mutually exclusive events. That is a must sum of p of ais then p is going to be call it as a probability function. Therefore, omega ef and p be a probability space.

Now, we are defining here real valued function x defined on omega is said to be a random variable a real valued function x define on omega is said to be a random variable. If X inverse X inverse of minus infinity to X it is a semi closed interval that is belonging to F for all X belonging to real line. Whenever a real valued function defined on omega satisfying the inverse image of minus infinity to X. That is belonging to F for all X belonging to r then that real valued function is set to be a random variable.

Let me give the pictorial representation or pictorial way of explaining the same thing.

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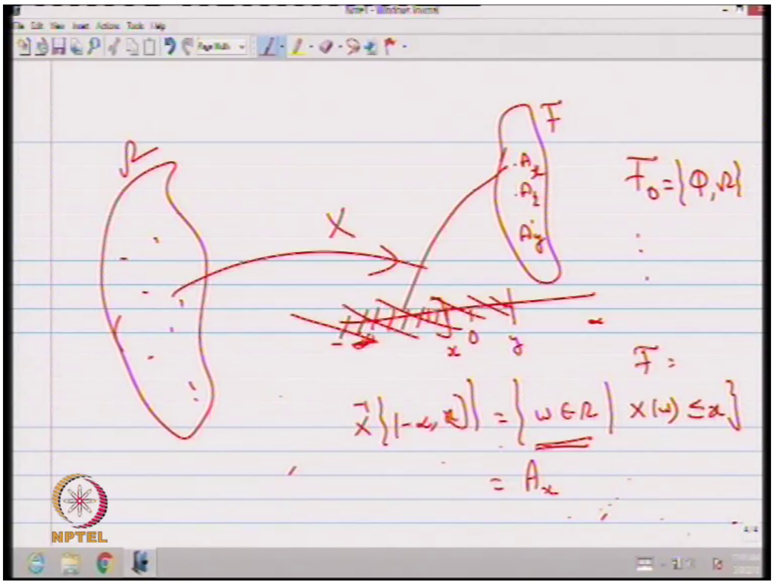

That means, you have a omega have a many elements in it, it could be finite or it could be countable infinite or it could be unaccountably many elements in it. You make a mapping it is a real valued function, from omega to r. So, it is a real line, if you go for inverse image from minus infinity to any point X. Suppose I take this point as X, the inverse image of minus infinity to small x collect all the possible outcomes which is going to give the value from minus infinity to x that is nothing but the few elements of omega satisfying the condition that that.

Under the mapping X it gives the values from minus infinity to small x, if I collect those possible outcomes then that should be one of the elements in the F. So, the F may have many elements A 1 A 2 and so on. So, there is a possibility this minus infinity to x will give one of the element here.

So, if when you change the small x from minus infinity to infinity, whatever the inverse image of under the mapping x minus infinity to that x if that is belonging to F or that is one of the element in the capital F, we call this real valued function as a random variable. That means, there is a possibility some real valued function may not be a random variable for that particular F you recall the sigma field. The smallest sigma field is empty set, and the whole set that denoted by F naught empty set and the whole set like that you can create many more you can create the many more sigma field the largest sigma field is nothing but the power set collection of all subsets of omega, And the number of elements in the largest sigma field is going to be 2 power n by n where n is a number of elements in the omega.

So, some real valued function may not be random variable with respect to the given F. So, I can expand the x inverse of minus infinity to small x that is nothing but a collection of ws belonging to omega, such that under the mapping x it is a real valued function that is going to give the value less than or equal to small x that is equivalent of the x of w is lies between minus infinity to till small x.

So, since you are getting a collection of possible outcomes, this is going to be the event this is denoted by A suffix x. If I define A suffix x, x as a event; that means, it is nothing but collection of possible outcomes such that under the operation x capital X real valued function it gives the values from minus infinity to small x; that means, this I can rewrite A 1 as the a suffix x. Suppose I change the value instead of x into some other y, then minus infinity to till small y under operation x the inverse image will be belonging to A suffix y like that if. Whatever be the x between minus infinity to infinity, the inverse image from minus infinity to till that point is belonging to one of the elements in the capital F then this capital x is said to be a random variable.

Now, I will go for few remarks, then I will move into the next topic remark number 1.

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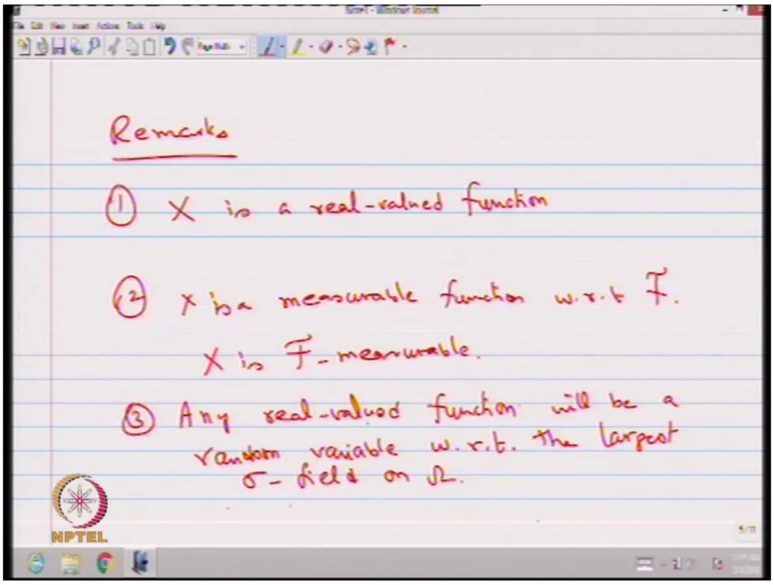

Remark number 1 x is a real valued function that is a x is a real valued function, then why it is called a random variable? Because it is defined from omega to r and the omega comes from the all possible outcomes of the random experiment therefore, this real valued function is called it as a random variable. So, it is a real valued function define it from omega to r, omega is the collection of all possible outcomes of the random experiment therefore, it is called a random variable. By default random variable means it is a real valued function.

Someone may ask can I have a complex valued function. It is possible to create complex valued random variable, in the form of a one real valued function plus square root of minus one times another real valued function. That means, with the help of 2 real valued functions, you can make a complex valued random variable that is possible. But as far as this course is concern, we keep x as a real valued function and the random variable is formed with the real valued function that is it.

Second one x is a measurable function, with respect to the sigma field F what is the meaning of measurable function? The inverse image of a semi closed interval that is belonging to one of the elements in the capital F. Therefore, x is going to be call it as a measurable function with respect to sigma field F. So, there is another name for the random variable that is a measurable function.

So, if you do the advanced level probability course probability theory course. So, we call a random variable as the measurable function with respect to F. So, there are some books they use called a X is a F measurable; that means, X is a random variable with respect to the probability space omega F p. Kindly note that, there is notion of probability is not coming to the picture when you define the random, variable to define a random variable you need omega and F there is no need of p; p is nothing but the probability.

So, through probability one can study the random variable x that is called studying the distribution of the random variable x. So, there are few more remarks I want to make it so that I will do some more remarks after giving some examples. So, we will go to the example of random variable.

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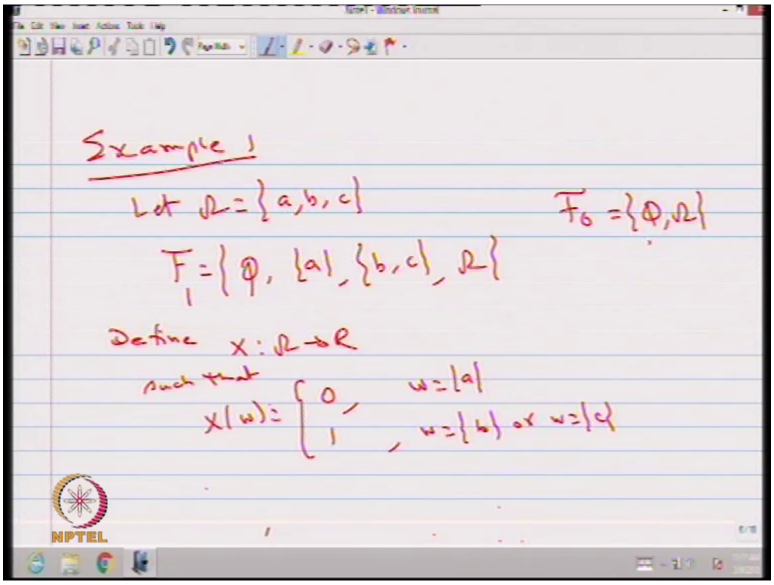

So, the first easiest example 1: let omega consist of three elements with the three elements, you know that we can frame the sigma field the smallest one which consist of empty set and the whole set then you can go for the largest one, but since I want to explain the random variable. So, I am going to introduce I am going to define the sigma field, which is in between the smallest and the largest.

So, the F which I am going to consider here that has a four elements empty set, singleton element a and both the elements b and c and the whole set. So, this is not the smallest and this is also not largest sigma field. Now, I am going to define a real valued function from omega to R such that X of w that is going to take the value 0, when w is going to be singleton element a. It is going to take value 1 if the w is going to be b or w is going to be c. So, this a real valued function mapping from omega to r omega consist of three elements a b c fine.

Now, we will check whether this real valued function is a random variable or not let us go for it.

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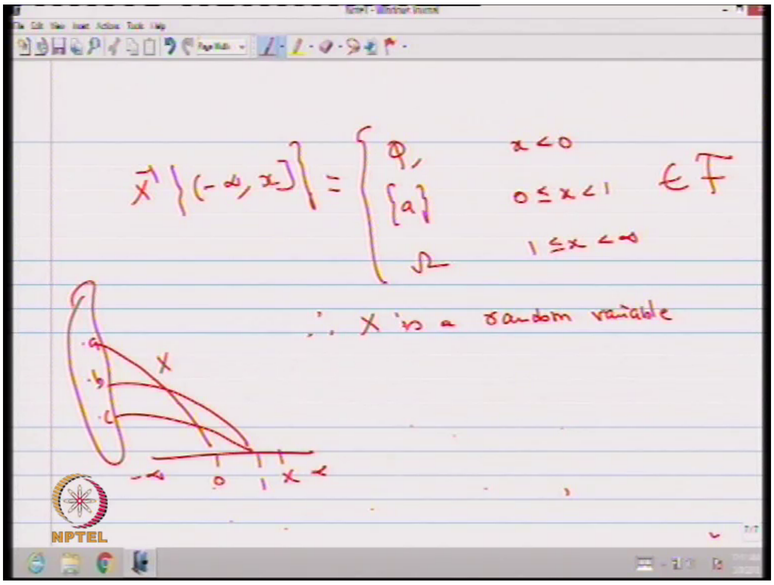

We will go for finding x inverse of minus infinity to small x closed to interval that is going to be empty set, if x is going to be less than 0, you recall x is the mapping from omega to r it takes a value. So, a is connected with the 0 and b and c both are connected with 1. Therefore, I can redraw the diagram. So, there is no this line fine.

So, x is a mapping from omega to R, the x inverse of minus infinity to small x. So, when small x is less than 0 suppose you treat x is going to be somewhere here, between minus infinity to 0 then you are looking for what is the inverse image of minus infinity to x under the operation capital X. You are not getting any possible outcomes because one possible outcome a is mapped with 0 b and c is mapped with 1. Therefore, when small x is less than 0 you are not getting any possible outcomes. Therefore, it is going to be empty set.

Now, we will go for when small x is lies between 0 to 1 including 0 excluding 1; that means, somewhere now the treatment is x is not there, x lies between 0 to 1 it can include 0 also. Now, the inverse image of minus infinity to x, where small x is lies between 0 to 1. Now you find out what is the inverse image; that means, you look for what is the possible outcomes which gives the values from minus infinity to small x where small x is lies between 0 to 1. Since 0 is included therefore, the inverse image is going to be the singleton element a correct.

Now, we will go for when small x is lies between 1 to infinity; that means, now the x is lies between 1 to infinity somewhere here what is the inverse of each? That means, whatever the possible outcomes which is going to give the value from minus infinity to small x where small x is one to infinity, you can include all the possible outcomes.

So, since a is mapped with 0, b and c are mapped with 1 therefore, it is going to be singleton element a singleton element b singleton element c. So, when you include all these three elements, then it is nothing but omega. Now we will check whether these inverse image of the values are belonging to F or not. So, for us the F is empty set singleton element a and b and c together and the omega, whereas the here we land up empty set, singleton element a and omega all are belonging to F. There is a possibility fewer elements of the F may not come into the picture here that does not matter, but whatever the element comes here that should be belonging to F or not that is the question.

So, when x moves from minus infinity to infinity whatever be the possible x all possible x the inverse image of semi closed interval that is belonging to the F. Hence, x is a random variable, it is a real valued function in a probability space omega F p this real valued function satisfying this condition. Therefore, this real valued function is a random variable, it is a measurable function with respect to F. In the hidden thing is random variable with respect to F that is hidden, but we usually wont write we just say that x is a random variable, but actually it is a random variable with respect to F. Why I am saying this now we are going to give a another example, in which that real valued function is not going to be a random variable for the same F. So, then we will go for concluding some result then we will make the remark or the remarks.

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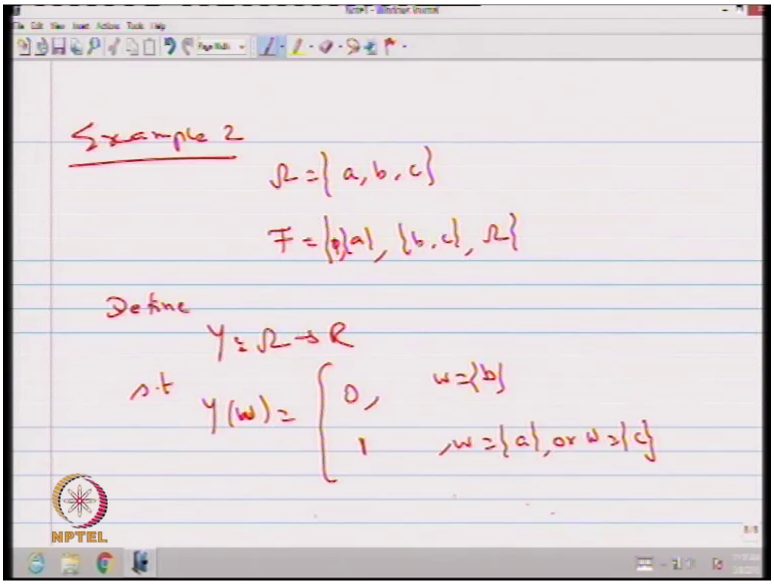

We will go for the second example, second example we take the same omega we take the same F singleton element sorry empty set singleton element b and c and the whole set. Now, we define the new real valued function Y from omega to R such that y of w is going to give the value 0, when w is going to be a singleton element b or it is going to give the value 1, when w is going to be a or w is going to be c. So, this is a change. So, this is also real valued function omega to R and Y of w takes a value 0, when w is equal to b it takes a value one or w is equal to a or w is equal to c let check whether this is going to be a random variable.

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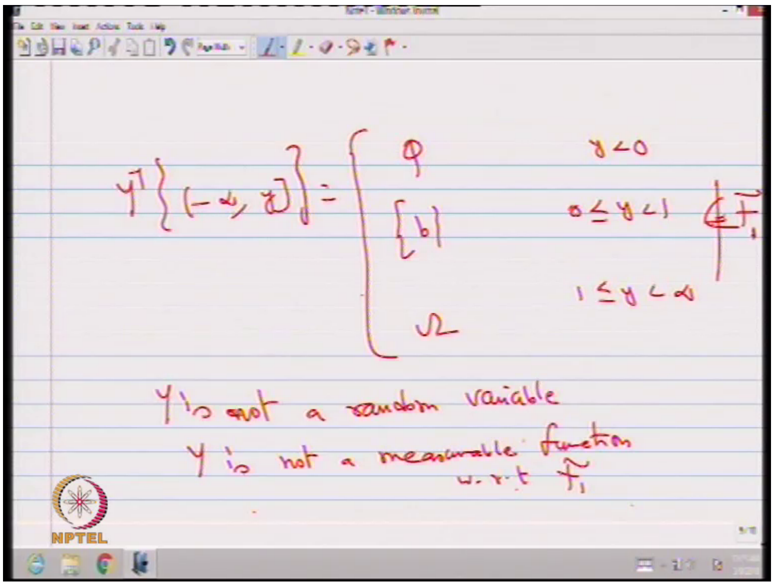

Y inverse minus infinity to small y close to interval, that is going to give the value, when y is between minus infinity till 0 you are not going to get anything any possible outcomes therefore, it is a empty set. When y is from 0 to excluding 1, the inverse image is going to give only the pass singleton element b, because y of w is equal to 0 when w is equal to singleton element b. When y is lies between 1 to infinity similar to the earlier exercise example, I will end up with collecting all possible outcomes therefore, it is a or b or c union everything therefore, it is going to be the whole set.

Now, let me check whether these three elements are belonging to F or not. So, the here the F is the empty set singleton element a b and c and omega. Empty set is belonging to [FL], whole set is belonging to F whereas, the singleton element b is not belonging to F therefore, all these elements are not belonging to F. That means, it is a real valued function, but it is not going to be a random variable with respect to the [FL] which we have discussed; that means, y is not a random variable with respect to this F. You can raise the question when this real valued function is going to be the random variable. That means, by changing or by having a new probability space same omega with the different F may give a this real valued function going to be a random variable. Can you guess what will be the F in which this real valued function is going to be the random variable? Yes you would have guessed it basically.

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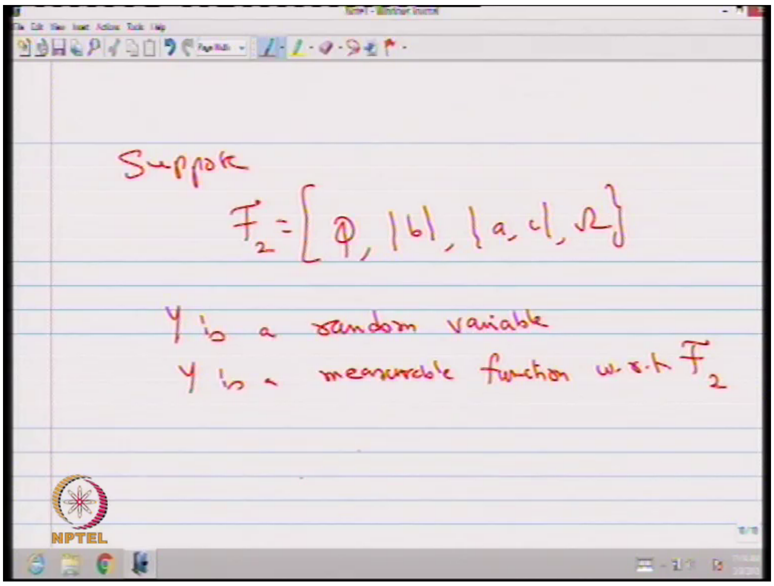

Suppose omega sorry suppose F is going to be empty set singleton element b. So, a and c and the whole set in this F the y is a random variable or y is a measurable function with respect to this m. Since, we have more than one sigma field, I am going to note down the notation as a this I treat it as I will go to the example 1, this I treat it as the F 1. F naught I will keep it as empty set and the whole set. So, I denote this as the F 1, therefore this is belonging to F 1, therefore x is a random variable. Come to the second example, same F we are using F 1. So, y is real valued function it is not belonging to F 1. Therefore, y is not a random variable y is not a measurable function y is not a measurable function with respect to F 1 ok.

Suppose we keep another sigma field that I denote it as a F 2. Y is a random variable or y is measurable function with respect to that is very important, with respect to F 2 not F1. Now, we can make a another constraint based on the 2 examples what is the F in which both the real valued function going to be a random variable. I will repeat the question in the example 1 x is a real valued function that is going to be a random variable with respect to F 1.

Second example y is a real valued function, but y is not a random variable with respect to F 1 whereas, y is a random variable with respect to F 2 another sigma field. Now I am going for the third example same real valued function that is x and y what is the F in which both the real valued functions are going to be random variable that is a third example.

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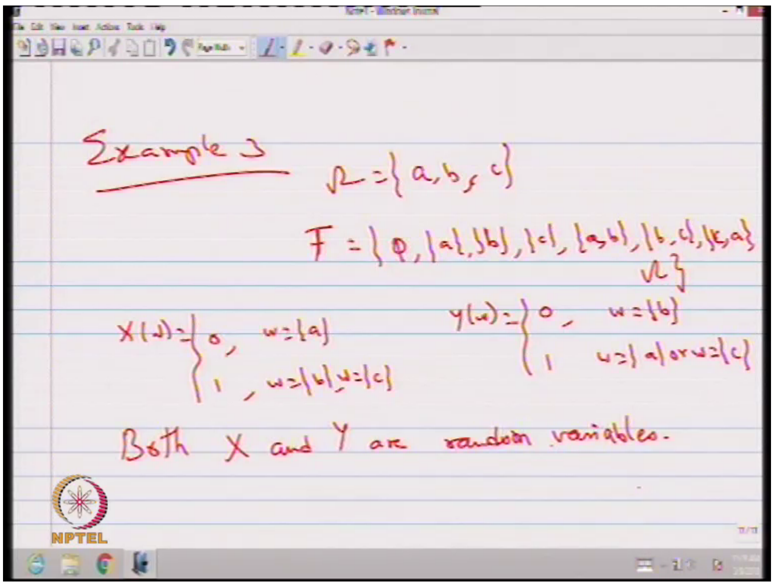

That is I have a three elements, what is the F in which the real valued function omega is equal to a it takes a value 1, when omega is equal w is equal to b or w is equal to c and another real valued function that is also takes a value 0, when w is equal to b and takes a value 1 w is equal to a or w is equal to c. These 2 real valued function are going to be a random variable under one sigma field.

So, what is the sigma field? Suppose I keep the largest sigma field suppose I keep the largest sigma field that is nothing but the power set. You can count 1 2 3 4 5 6 7 8 elements because omega has three elements. So, 2 power 3 8. So, the number of elements in the largest sigma field that has the 8 elements that is nothing but the power set. Under this sigma field [FL], both the real valued function going to be the random variable you can prove it the same way whatever I have done the derivation, you can go for it its already there, only you have to check whether that is belonging to F. But if you go for x inverse of semi closed interval similarly y inverse of semi closed interval both are belonging to F. Therefore, one can conclude both x and y or a random variables that is both x and y or measurable functions with respect to this F that is a largest sigma field.

From these three examples one can conclude one can conclude whenever you have a largest sigma field any real valued function is going to be the random variable. So, that is the next remark, which I have given earlier. Yes we have given 2 remarks now I am going to make a third remark, that is any real valued function will be a random variable with respect to with respect to the largest sigma field or sigma algebra on omega.

Therefore, many of the probability course if it is a elementary, they would not discuss the random variable in the form of a through the measurable function concept that is x inverse of semi closed interval is belonging to F. Whenever you have a largest sigma field or you keep the probability space with the largest sigma field, then any real valued function will be a random variable. That means, it is a mapping from omega to R that real values function will be the random variable.

The way I have given 2 examples, you can think of many more examples instead of three elements you can go for easy example of tossing a coin which has a 2 possible outcomes head and tail, then you can have a F, since it has a 2 elements in the omega. Therefore, the you will end up with the F is going to be the largest sigma field unless otherwise you take the smallest one

So, you will have a F with the empty set h d omega. So, it has four elements. So, whatever the way you define the real valued function on the random experiment of tossing a coin, any real valued function is going to be a random variable because the F is largest sigma field. So, having a example with the 2 elements, it would not serve the purpose of a defining a random variable in a nice way that is what I have taken three elements.

Now, the next question can I go for 4 and 5 elements or finite number of elements. So, if you have a finite number of elements for example, you can think of throwing a dice, which has omega has 6 elements 1, 2, 3, 4, 5, 6 here it has 3 elements. So, when you throw a dice you have consist of 6 elements. Therefore, other than the smallest sigma field other than the largest sigma field which is the power set you will have a many sigma fields in between. In that case again you can have a any real valued function for example, getting even number that real valued function is 0 getting a odd number that values is going to be 1. So, you can map x is getting even number becomes 0 and odd number becomes 1. Need not be 0 and 1 also you can go for any values as long as it is a finite value, you can have a any value mapping from omega consisting of 6 elements to the any real numbers that is a real valued function.

Then the F which you have sigma field accordingly some real valued function may be a random variable, some real valued function may not be a random variable. But in that example also if you keep the power set as the sigma field the largest one, then any real valued function you define it over the omega consisting of six elements also that is going to be a random variable. You can go for some books they use the indicator function whether that is going to be a random variable or not. So, you can go for the inverse image whether that is belonging to the F or not; if you have a F has all the elements including this elements then it is going to be a random variable if not it is not going to be a random variable.

So, I have given three remarks. So, I want to introduce one more remark that is the one which I have given as a if condition, the real valued function is said to be a random variable if inverse image of this is belonging to F, this condition is if and only if condition; that means, the converse also true that has the fourth remark.

Basically the minus infinity to x a close semi closed interval that is nothing but the Borel set. So, since I made the Borel set of the form minus infinity to x therefore, many of the other events can be written in terms of minus infinity to x, this semi closed set form. Therefore, this Borel set will make if and only if condition; that means, if this condition is satisfied then that is going to be a random variable, if you have a random variable always this condition will be satisfied that is the meaning of if, and only if that is a meaning of converse.

So, since I am going for a very particular type of a semi closed interval of minus infinity to x, this makes this condition is going to be a if and only if condition. That means, any real valued function satisfying this condition will be a random variable as well as any random variable having this condition sorry; any random variable will always have this condition for all x belonging to R, x inverse of minus infinity to small x that is belonging to F.

So, this is also one of the very important remarks whatever I have said it in earlier three remarks, this is going to be the fourth remark.