Introduction to Methods of Applied Mathematics
Prof. Mani Mehra & Prof. Vivek Aggarwal
Department of Mathematics
Indian Institute of Technology- Delhi

Module No # 06
Lecture No # 30
Daubechies wavelet

Welcome to all of you in the next class of this course so let us summarize what we were doing in last lecture.

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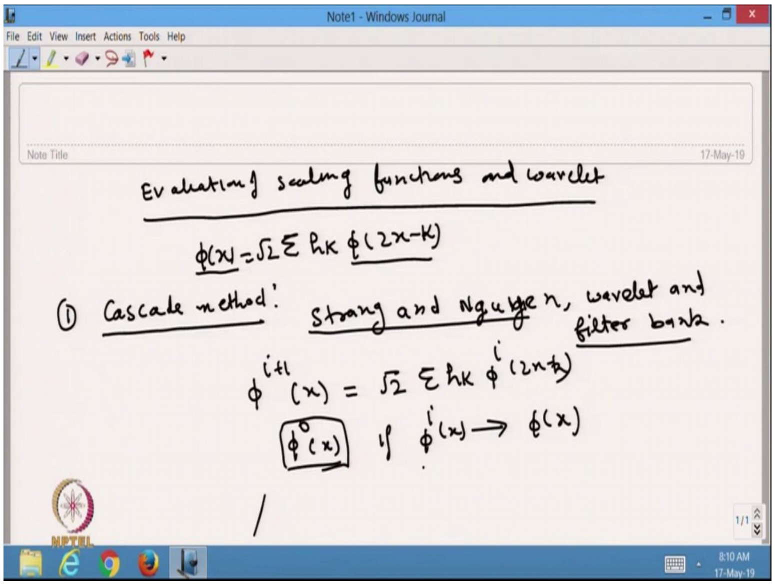

In the last lecture we were trying to evaluate evaluation of scaling function and wavelets of course it is in next lecture 32 ok. So as we have seen that with the help of MRA phi x can be written in the following way. So phi which is unknown which is that is available in the left hand side as well as in the right hand side so that is why we have to divide some specific algorithms to compute that.

And then in the last lecture I have stated 3 methods to compute this phi first one which I mentioned in the last lecture was cascade method. I will explain the basic idea behind this cascade method and for detail you could look at in this reference so for detail one could refer the reference Strang and Nguyen. Wavelet and filter brings this is the title of book wavelet and filter banks and this is published by springer.

This is the book for which you could refer the detail about this method the basic idea I am explaining here so what we are doing we are writing this in the following fashion. So we have to start with some phi naught x that is what we always do in iterative methods. So we have to start with some phi naught and then the if phi i x converges this to phi x it will solves the dilation equation.

So that is the basic idea behind cascade method that you start with some initial guess phi naught x and with that and then you keep iterating this procedures and if phi i x converges to phi x then that will solve the dilation equation. Because this is also called the dilation equation to equation 2 scale relation that we have already seen in the last lecture. So and again I am saying for detail about this method one could refer to the book but by Strang and Nguyen title of that book is wavelet and filter banks and it is published by Cambridge press ok.

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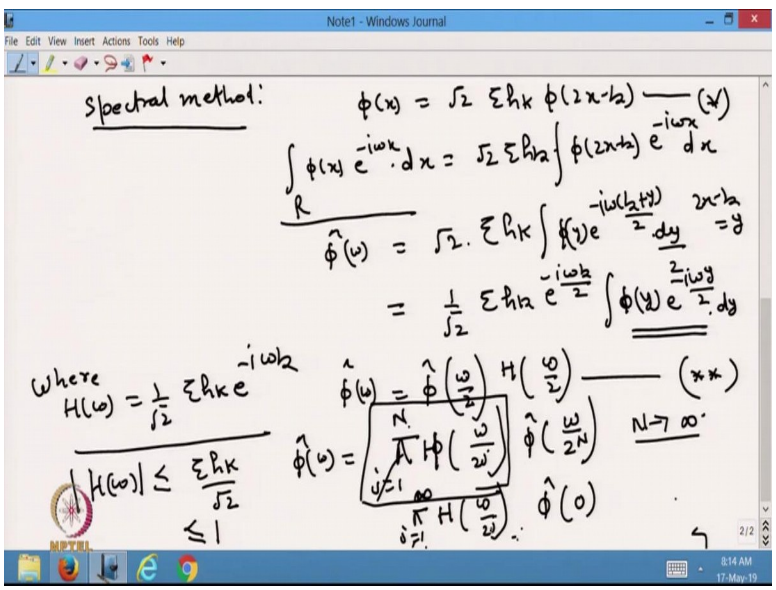

So this was the idea behind cascade method the next method which I stated in the last lecture was spectral method so in as the name is suggesting spectral means we are taking the into the Fourier domain. So for that purpose I am computing the Fourier transform of this relation both the sides ok I am taking the Fourier transform both the sides so for that purpose I am multiplying with this over real line ok.

So this left hand side will give me a phi hat omega under root 2 hk what I am doing I am putting 2x - k = y so this will be phi y e to the power –i omega k + y / 2 dx will be dy / 2 ok. So this will become whole expression will become hk e to the power –i omega k / 2 phi y e to the power –i omega y / 2 dy clear to everyone. So if I am denoting this with again phi hat omega / 2 ok and this functions I am denoting with H omega / 2 where H omega is given by the formula this ok.

So the same scaling relations which I have stated here I have taken that scaling relation in the Fourier domain with the help of this. So this is the same relation in the Fourier domain because I need the help of this relations to define spectral method that is why I am I have derived this in fact now I will keep iterating this way. So phi hat omega will become ok so this will sorry this should be H and then phi hat of omega / 2N ok.

So if now I am taking as N tends to infinity so for that reason I have to make sure that this product converges if this product converges what will this expression as N tends to infinity this will become 0. But I have to make sure that this converges only then I can write this as this way ok so that I can check from here whether it converges because this is a product so we know H omega if you look at summation of hk / under root 2 ok. And later on we will see that summation of hk = under root 2 that we will do later on so if this is true then this is less than 1. So under this condition we can say that this product will converge and if this product is and what is phi hat 0.

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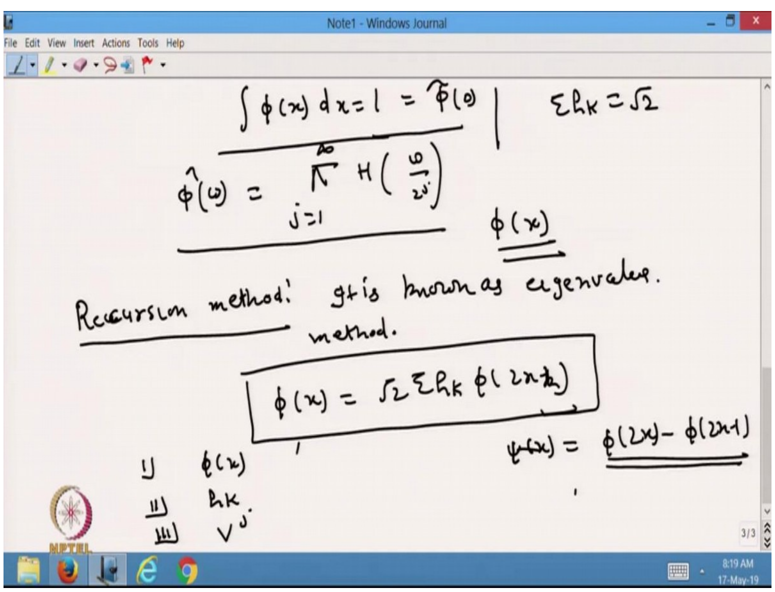

And we are making 1 more assumption that on phi that integral of phi x dx is 1. Because this is that phi hat 0 ok so these are 2 extra assumptions we have taken which we will see later on summation of hk is under root 2. After using this assumptions we can say that if we compute phi hat 0 is 1 so if we compute the inverse Fourier transform of this expression that will give me phi x ok. That will give me a inverse Fourier transform of this expression will give me phi x ok so this is the another method of computing phi x with the help of a Fourier transform that is why it is called spectral method.

So we have seen cascade method their basic assumption was using the iterative algorithm and then converging and but here also we are using iterations but in the Fourier domain. So that is the idea behind the spectral method the another method which I stated in the last lecture was recursion method. It is known as a eigenvalue method because at the end we have to compute the eigenvector so basically it is a eigenvalue problem ok.

So it is known as a eigenvalue method because in this method at the end we have to compute the eigenvector and that is known as a eigenvalue problem. So I will show you that detail of this method later on with respect to Daubechies compactly supported wavelet ok so that I will explain later on. So now what I wanted to say that if this is the relation I have been keep writing so this is the main relation so with the help of this relations I can compute phi ok with the help of any 1 of the method ofcourse recursion method I compute later on.

So now if so far I have given you a example of a only 1 wavelet because if you remember when I was discussing about this multi resolution analysis I said phi x is a box function ok. Box function which was 1 in 01 and rest were it is 0 and with the help of that box functions we also relate wrote this relations if you remember this was a graphical observation I could say or because we were knowing the analytic expression.

But as I am keep saying then in most of the wavelets we do not have analytic expression for phi. So the only way to compute phi is with the help of this relations. So there are 3 ways to start either all of them are equivalent either you choose hk filter coefficient first then you can compute phi and then you can compute approximation spaces vj or you choose phi then you can compute hk's or vj.

So means there are 3 ways to start the multiresolution analysis 1 is choosing scaling function another one is start with the filter coefficient and third one is you start with the multi-resolution approx scaling spaces vj itself. So what are the 3 ways 1 you start with phi x scaling function second you start with hk filter coefficient and third you start with scaling spaces Vj ok.

So that is how you could start with the multiresolution analysis ok so now as I am saying that analytic expression is not available to us. So how one could look at this whole construction and in fact I have said that recursion method I will explain later on.

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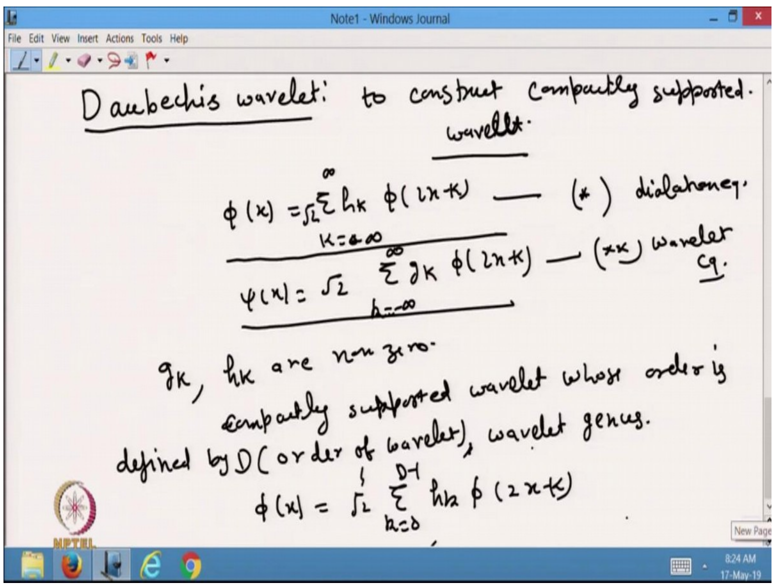

So for that reason now I am going to define a new category of a wavelet which is called Daubechies wavelet. I can say it is the most popular wavelet so far ofcourse it has couple of disadvantage as well and that is why we are also going now going into the domain of another wavelets but just to start for a beginners it is a good wavelet I could say. So Daubechies was the first one to use this MRA theory invented by Stephen Mallet Meyer to construct a compactly supported wavelet of some smoothness so ok.

So what Daubechies did she used this MRA theory to construct a compactly supported wavelet that was his not his that was her contribution ok. So how we were defining compactly supported wavelet with respect to scaling functions and wave ok so here 0 to infinity and psi x is also sorry

K is minus infinity to infinitive and this is because wavelet system is characterized by phi and psi.

That is why in the last topic when I was computing phi once you know how to compute phi you can also compute psi so that is why I have not written separately how to compute psi. So now this is the relation for phi this is the wavelet equation and this is dilation equation ok. So now what she did she said that if finitely many hk's are non-zero then phi and psi are finitely many hk's and gk's are non-zero then it is a compactly supported wavelet.

In fact I have stated this definition of a compactly supported wavelet in my last lecture as well so in case of a compactly supported wavelet whose order is defined by D which is also D is the D we call order of wavelet as well as wavelet genus ok. So we are defining phi x will be similarly in psi x also we can write down here k is minus infinity to infinity but for a compactly supported wavelet we can write down k is from 0 to D – 1 ok k so D is order of the wavelet.

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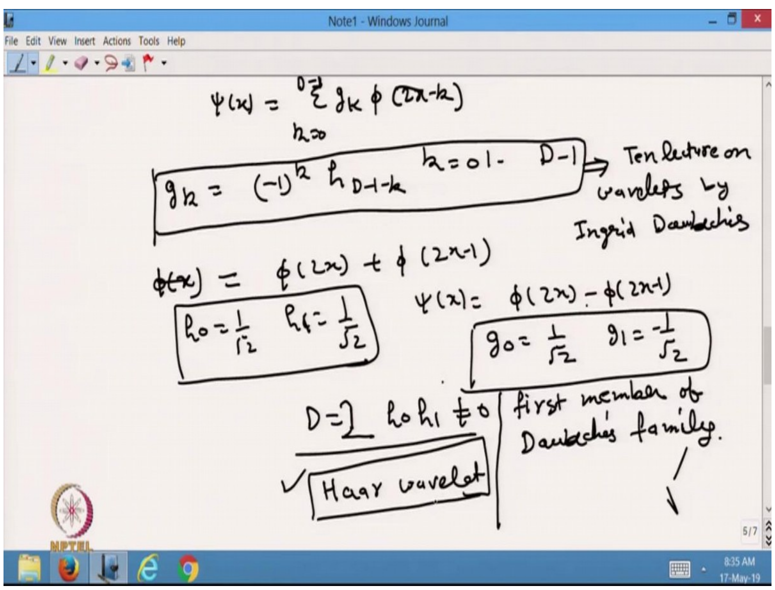

Ok ofcourse as psi is related and this one could prove that this gk and hk's are also related by the following relation ok. Gk's and hk's are related by the following relation where k is from 0 to D – 1 D is the order of the wavelet or you could also call it as a wavelet genus. For a proof of this how these are related one could look at the reference by book ten lectures on wavelets by Ingrid Daubechies ok.

I have also given you the reference where you could see this so now as I said that we have to look at how to compute first if we are starting with the filter coefficients. How to compute this hk's and gk's ok so for that reason I could say that ok when I was defining a phi x as a box function then I have stated this result if you remember from the last lecture. So basically h naught was 1 / under root 2 h1 was also 1 / under root 2 and when I was writing psi x with the this relation I was writing g0 is 1 / under root 2 and g1 is – 1 / under root 2.

So you can cross check this relation will be satisfied for these values of h and g that you could check that you can take it as a exercise and you could check that this relation should be satisfied. So and what will be the order of this wavelet because the order of this wavelet will be 1 ok. Because only h naught and h1 are non-zero ok so this is first family of a first member of the Daubechies family.

So the wavelet which I defined in my last lecture with the help of analytic expression is a first member of Daubechies family ok. It is also called Haar wavelet fortunately when we were constructing a wavelet of Daubechies family we ended up that when we are saying D = 1 it corresponds to scaling function corresponds to box function and wavelet function corresponds to this kind of a functions which I have stated in my last lecture.

So that was also a Haar wavelet why it is called why it is name is also Haar wavelet because Alfred Haar is the name of the scientist who separately discovered this as a basis function ok so that was the idea.

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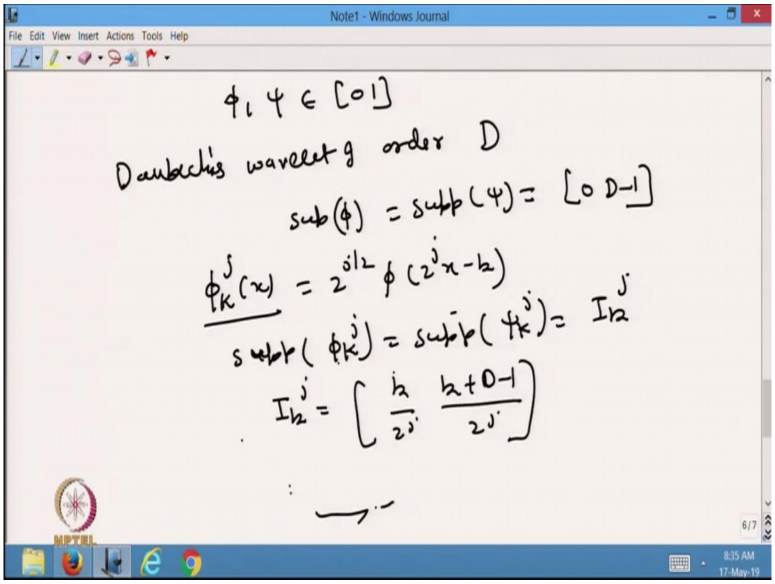

Now I am saying if the you could say that the box function was having a support in 01 and the wavelet Haar wavelet is phi and psi both are having a support in 0 to 1 ok. So basically if I wanted to generalize this idea I could say that for it Daubechies wavelet of order D phi and psi support of phi and psi will be this you could say ok or maybe what you could say that support of phi = support of psi = 0 to D – 1 ok clear.

Daubechies wavelet of ordered D will have support both scaling function and wavelet function because every wavelet is have or in fact I should say wavelet system is characterized by 2 function phi and psi. Phi is a scaling function and psi is wavelet which are constructed by MRA ok so support is 0 to D – 1 so if now if I am saying what with the support of this function which is dilated by 2j and translation by k.

So the support of this function you could also define this way support of will be Ikj where Ikj is basically k / 2j k + D – 1 2j ok. So this will be the support of phi kj and psi kj if support of phi and psi is given by this 0 to D – 1. Now I will tell you how to compute this filter coefficients for a general order D wavelet in the previous slide we have computed hk’s and gk not in a general way.

Because we were knowing the analytical expression for phi so that is why either from the function or for the graphical representation by observing we have computed hk’s and gk. Now I am telling you the method to compute hk’s and gk’s in a general way for any wavelet of whose order is D or and it comes under the Daubechies family so ofcourse you will be needing some specific set of equations.

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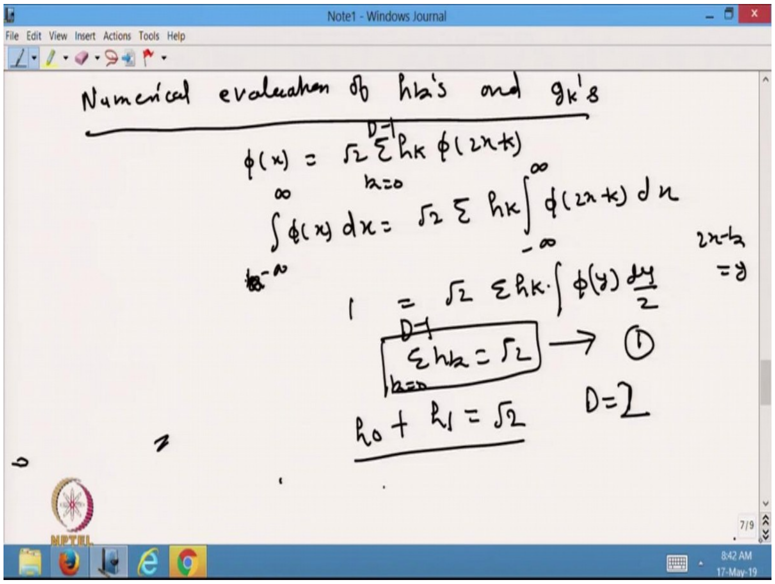

So for hk's to compute that so what I am doing for that numerical evaluations of hk's and gk's ok. So ofcourse again I have to write this results k 0 to D – 1 ok so this that is why you know MRA is the heart of the wavelet theory because you can see every time I am keep using the same results. So if I am integrating this in the both sides ok minus infinity to plus infinity ok.

So as I have said that I will be stating 1 condition that integral oh phi x dx this 1 so this is under root 2 summation of hk. So if now I am putting 2x – k = y this will become phi i and dy / 2 so summation of hk will be under root 2. So this is the results which I used earlier and I said that I will prove it later. In fact it is not specific to k = 0 to D – 1 only for compactly supported wavelet in fact if we would have written k is from minus infinity to infinity then also it is true.

That is why we were able to use it at that point of time where we were not defining just compact compactly supported wavelet we were driving a family of the wavelet in a general sense ok. So integral so this one of the results we are getting in case of a ofcourse here I have to make assumption that k 0 to D – 1. So now this is just if I wanted again ok whatever computations we have done in case of a Haar scaling function and wavelet earlier.

By observing the function I can show you here also so if D is 1 this will become h0 + h1 sorry D is 2 ok. So h naught + h1 will become under root 2 ok if this is the case here also it should be D = 2 ok then it will corresponds to Haar wavelet. Now but we cannot solve it for hk we need some additional conditions as well.

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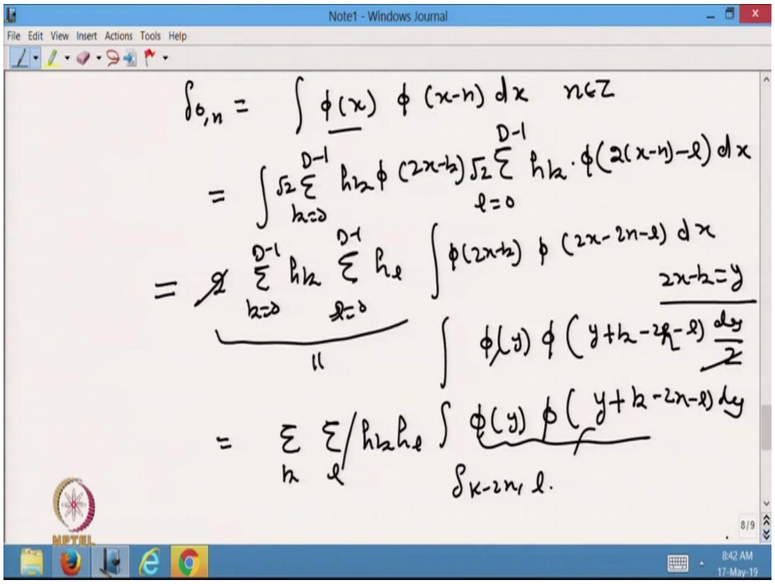

So for that reason I am stating another assumption which we have anyway we have already made that phi x – k forms a orthonormal basis for space V naught ok. So this will be phi x phi x – n dx when n belongs to Z this is the relation we know from MRA assumption. So now I am using again 2 scale relations here I am taking a index l ok so this will become 2 summation of k = 0 D – 1 hk ok and another summation will be on l hl ok.

And then this integral will become phi 2x – k phi 2x – 2n – l into dx clear to everyone. So now I am saying that this 2x - k = y so if this is true I can copy the same thing here what will be the value of this integral in that case phi y and phi will become y + k – 2n- l dy / 2 ok. So this 2 we can cancel from here so finally this is summation over k this is summation of l hk and hl and this function will become if you remember phi y and sorry this y + k and this should be 2n ok.

So this will become phi y + k – 2n – l dy so I can write down this in the form of a delta function k – 2n into l ok k – 2n into l. That is why because it is a assumption of MRA that it forms a orthonormal basis.

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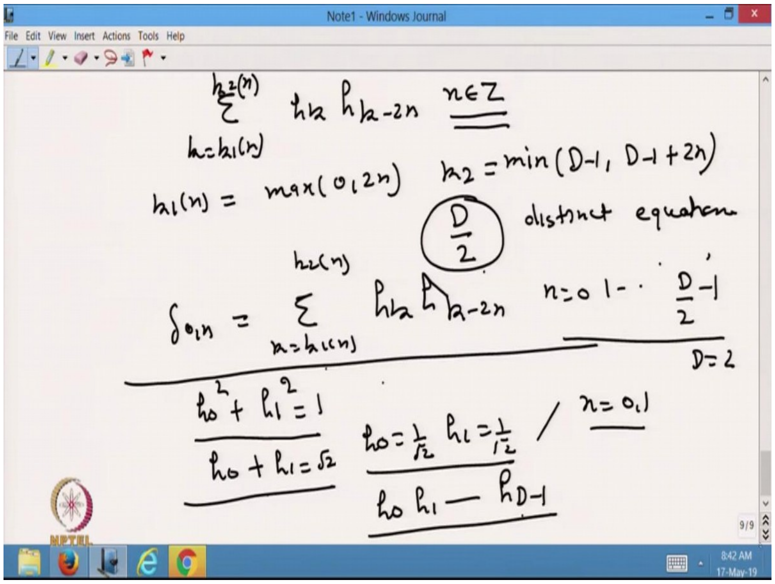

So if this is the idea and whatever non-zero terms I will get that I can write down in this form hk – 2n n belongs to Z where k1 n will become maximum of 0 to 2n and k2 will become minimum of D – 1 + 2n ok. So there are you could get as many question as n belongs to Z but there are you will you can observe that you will get only distinct D these many distinct equations ok.

So what I mean to say that hk – 2n where n is 0 1 to these many distinct equations you will get ok after using this MRA assumptions. So if for D = 2 Haar wavelet n is 0 to 1 so basically you will have h naught square + h1 square = 1. So which is a non linear equation you could observe and the other equations which I have written earlier was this. So after solving both of them you could say h naught is 1 / under root 2 and h1 is also 1 / under root 2.

So this was in the case of just specific D = 2 which corresponds to Haar wavelet generally from here how many equation distinct equation I have mentioned D / 2 and what is the non-zero value of the filter coefficients you have to compute total h naught h1to this h1 and h D – 1. So total unit D equations D / 2 distinct equations you got from here one equations you got from here if you remember so one equation from here D / 2 from here.

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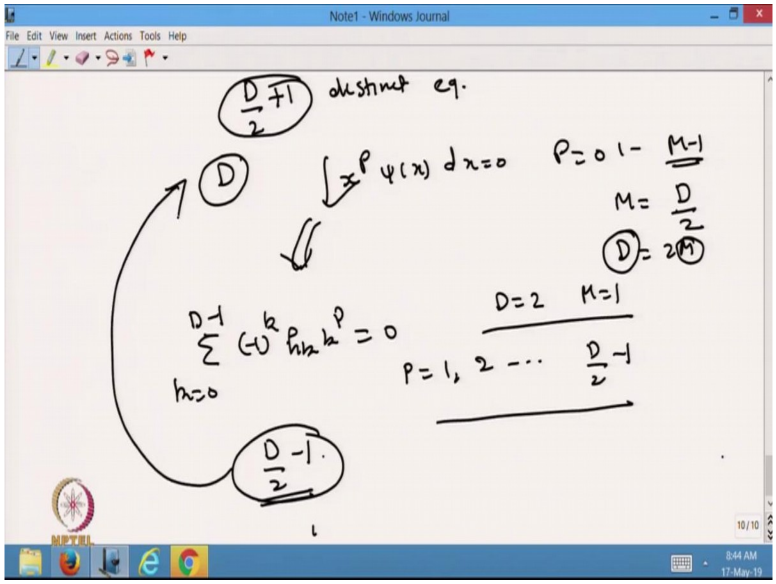

So total we have D / 2 + 1 distinct equation so far ok so and how many equations I need total equation should be this D ok. So I need remaining equations so from where I will bring this remaining equations I am bringing this remaining equations from one of the fundamental property of the wavelet which I have stated in previous lecture that was vanishing moment property.

Vanishing moment property was saying that x p psi x dx is 0 when p is 0 to M – 1 and this is the relation we have with M and vanishing moments M is D / 2 or basically D is 2M. D is called the order of the wavelet in case of a Daubechies wavelet and M is vanishing moment of the wavelet. So for a Haar wavelet D is 2 M is 1 so vanishing moment is 1 order is 2 and from this relations 1 can construct this ok.

For p is 1 to so from here I got these many distinct equation so if you had this term and you had this term you underwith these many equations that was the goal because if I to have a unique solvable system I should have a same number of a linear equation as same number of a unknowns that all of us already know from numerical analysis course or you could say linear algebra course. So that is how you know from the assumption of MRA one could see how to compute this filter coefficients.

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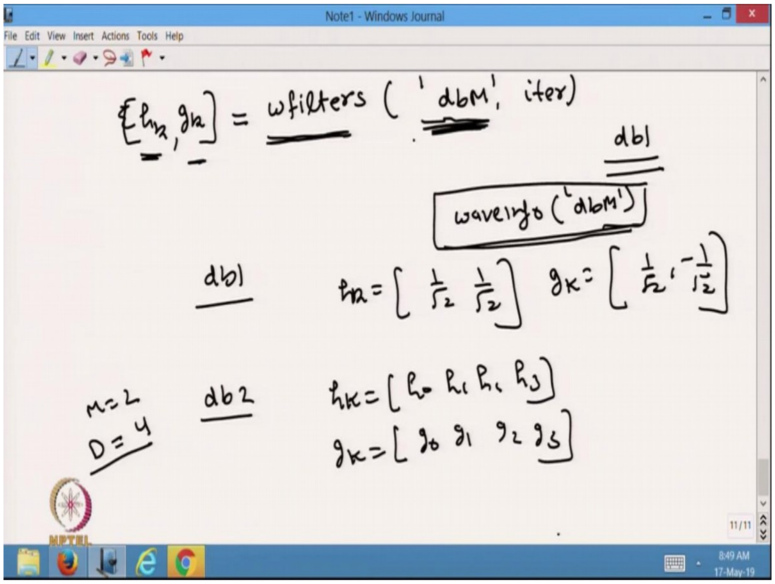

Infact you do not need to go through all this algorithm because there is a inbuilt Matlab function which is w filters in the matlab which gives you a filter coefficients for a wavelet which you write down here. So this is the syntax of a Matlab function which I am writing ok suppose all of you know Matlab and if some of you want to try to implement wavelet in the numerical method wavelet in any other field of applied mathematics.

First of all you have to how to start with it and this is the first step because the most simpler wavelet is Daubechies wavelet and this is the simple so if we should know how to start with the Daubechies wavelet. So it is a first what we have to do we have to compute the filter coefficient so for that there is a inbuilt Matlab function w filters which computes this low pass filter coefficient hk's and high pass filter coefficients and this is the name of the wavelet you have to give.

But right now we have just seen only one kind of wavelet Daubechies family of the wavelet so that is why we can write dbM but later on if you know any other wavelet also that name also you can write this in a single quote and iteration is the parameter which 1 has to use that in the syntax of the computation ok so dbM means Daubechies db stands for Daubechies and M is the vanishing moment.

So in case of a Haar moment what you will write? You will write db1 ok so if anyone wants to know the information about this function they could you could type in the matlab wave info ok dbM. It will give you a complete information about Daubechies wavelet ok the inbuilt function is wave info if later on also we study some another wavelets we can type waveinfo and that the name ofcourse you should know this short form of it before typing.

Otherwise how can you type it wave-info dbM it will so 1 can try if you want to know more about this category of the wavelet ok. So if I give db1 then you could see h naught h1 should be 1 / under root 2 and g naught g1 should be 1 / under root 2 and so what is the results of db 1 should come. Results of db1 should come this way ok so if you look at general order wavelet.

so in that case let us say db2 so it means M is 2 D is 4 so how many filter coefficients will be non-zero 4. So you will have here h naught h1, h2, h3, gk will also be g naught, g1, g2, g3 and the value of this hk's and gk's you could compute from this inbuilt matlab function w filters ok. So by this time all of us should feel very comfortable how to compute this filter coefficients once filter coefficients are known to you.

You can compute the value of phi and psi with the help of a dilation equation and wavelet equation ok because if you remember initially I have talked evaluation of scaling function and wavelets. That was with the help of 3 methods especially spectral method, cascade method and recursion method and recursion method I said I will explain later this is the time I can explain you the recursion method to compute phi and psi.

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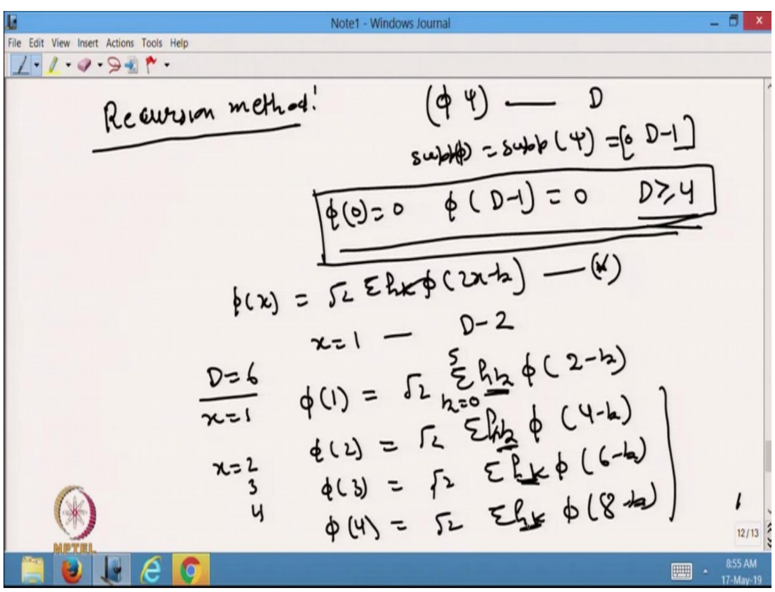

Recursion method so what this method does we know if phi and psi is a Daubechies family wavelet whose order is D then support of we know support of phi and psi will be D minus this ok. So this is also proved in the book of Ingrid Daubechies Ten lectures on wavelet if phi x suppose at the end point it is 0 for D is greater than equal to 4 ok this relation is in the book of ten lectures on wavelet by Ingrid Daubechies.

So we will be using this relation in recursion method ok so what we are doing we know this thing ok so this we will be putting the x 1 to D – 2. We will putting these value of x in the following relation so if we are taking let us start computing after taking this D = 6. So if x is 1 phi 1 because you know we are not interested in phi 0 because phi 0 is already 0 so phi 1 will be equal to under root 2 summation of hk phi 2 – k ok.

Similarly if so for D = 6 I will be keep substituting so this corresponds to x = 1, x = 2, 3, 4 ok so phi 2 will become 4 – k phi 3 will become 6 – k and phi 4 will become 8 – k ok. So now and k is varying from 0 to D – 1 k is varying from in each case k is varying from 0 to 5 so now this you can take it as a homework exercise.

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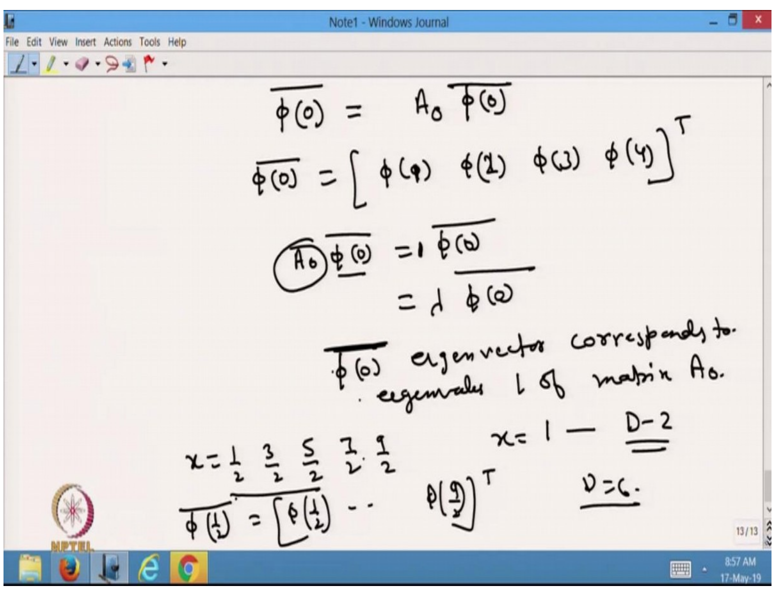

You expand this and then finally we will be able to write this as a phi 0 is basically A naught phi 0 so this is a vector we are phi naught is basically if you the value of phi at 0 phi sorry 1, 2, 3 and 4 ok. Because we have taken D = 6 we will be computing for this vector so what this problem has become if you look at this is this problem can you relate this problem to the problem of linear algebra. Ofcourse it is a eigenvalue problem as I have said that in recursion method it is known as a eigen value method.

Because at the end we have to compute the eigenvector so basically phi this vector is a eigenvector corresponds to which eigenvalue corresponds to eigenvalue 1 because if I can write down this way this is a eigenvector. I or a phi naught so this eigenvalue problem we know how to compute a eigenvector corresponds to any eigenvalue. So if you know that you know how to compute this vector ok.

Ofcourse for that reason lambda should be eigenvalue of this matrix that you can check and you can take it as exercise one is a eigenvalue of this matrix A naught and what is the entering the matrix A naught. A naught will be a matrix which will contain entries of a low pass filter coefficient because that is how you could see this is low pass filter coefficients ok. So phi naught is a eigenvector corresponds to eigenvalue 1 of matrix A naught ok phi naught eigenvector corresponds to eigenvalue 1 of matrix.

Once you know the value of phi at some integer points now you start substituting x = half 3 / half and 5 / half because if you remember we were trying to put earlier x 1 to D – 2 and D = 6. So we are putting this value then we can compute the value of phi at this grids 5 half 5 3/2 5/2 because 7/2 9/2 ok.

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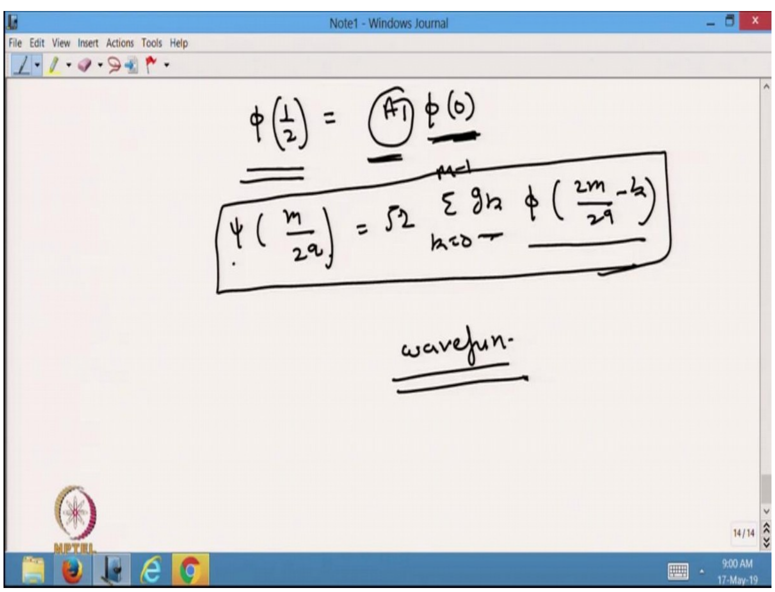

So then basically you could denote this as a this vector ok and this whole then this will become this matrix again A1 is a matrix which contains no low pass filter coefficients. This is an eigenvector we know so if we know this we can compute this so you can keep iterating this procedure to compute the value at any dyadic grid to compute the value of phi at any dyadic grid ok.

So now it is not dyadic let me say again we are taking for computational efficiency you could take any other grid also but here for computational efficiency we will be using only for a dyadic grid ok. So once you know phi you can also compute psi at the dyadic grid because that also can be given by this relation ok. So this you can compute at any grid or for computational efficiency we are saying dyadic grid.

So initially we will compute phi, phi is known to you gk's are known to you then you can compute psi at any m / 2q dyadic grid. So but I have explained you the algorithm how to compute phi and psi but if you want to implement it is upto you. You can implement it also and there is a inbuilt matlab function also what is that inbuilt matlab function says that I will explain in the next lecture.

But there is a inbuilt matlab function which is called wavefun ok which will compute the value of phi and psi at the dyadic grid for you people. You do not need to implement this algorithm that is the beauty of this inbuilt matlab function but really before using the inbuilt matlab function you should be knowing what it is doing or what is will be its computational cause. So for that reason you should know what is the algorithm they will be invoking.

So the recursion method basically if you look at use eigenvalue it you can consider it as a eigenvalue problem so thank you very much that wavefun function I will explain in the next lecture.