Matrix Computation and its applications

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Lecture - 56
Pseudo-inverse/Moore-Penrose inverse

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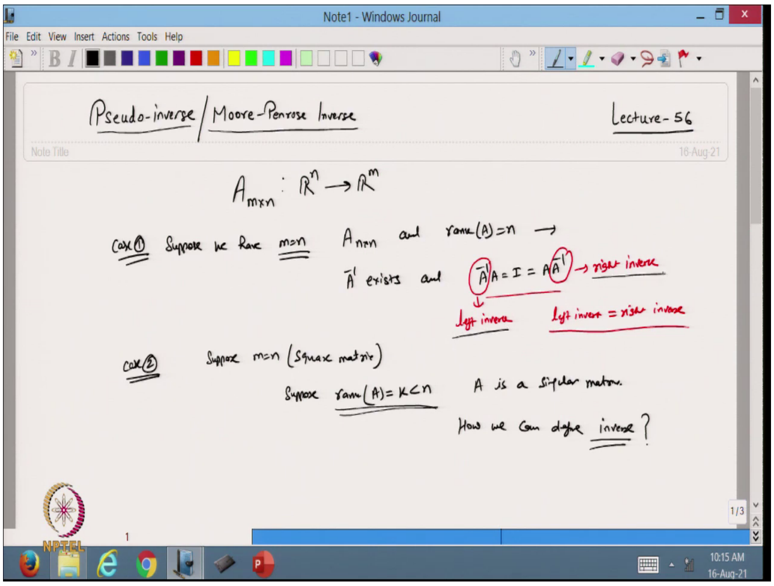

Hello viewers. So, welcome back to the course on Matrix Computation and its application. So, today we are going to discuss a very important concept that is Pseudo inverse : how we can define the inverse of a matrix which is in the maybe n* n matrix, but its rank is less than n or we have a rectangular matrix. So let us discuss that one.

So, today we are going to discuss the pseudo inverse and this is also called Moore–Penrose inverse. So, in this case what we are going to discuss is that suppose I have a matrix A that is of order m*n. And this matrix I know represents the linear transformation from R^n to R^m. Now, I will define the three different cases.

So, suppose that case 1 suppose we have m = n. So, in that case we have a square matrix, so that A will be n* n and let rank of A is n. So, in that case we know that the A^{-1} exists and if I take $A^{-1}A$ that will be equal to I and this is also equal to $A A^{-1}$.

So, in this case we are putting the inverse on the left-hand side of A. So, this is called the left inverse and this is my right inverse. And in this case when we are able to take the inverse the classical inverse or whatever the inverse we have defined. So, in this case my left inverse is also equal to the right inverse.

So, we have already seen that whenever the matrix is a nonsingular matrix then we can define the left inverse and the right inverse and then we can define it like this one. So, this is the case we have already discussed.

So, I will take case 2. Now we have a matrix that is equal to n, again the square matrix. Then suppose the rank of A is K < n. So, in that the matrix will be, so the matrix A is a singular matrix. So, in that case how can we define the inverse? We can define inverse, so in this case definitely we can now define the regular inverse as we have discussed here. But we have to define the inverse of this type of matrix.

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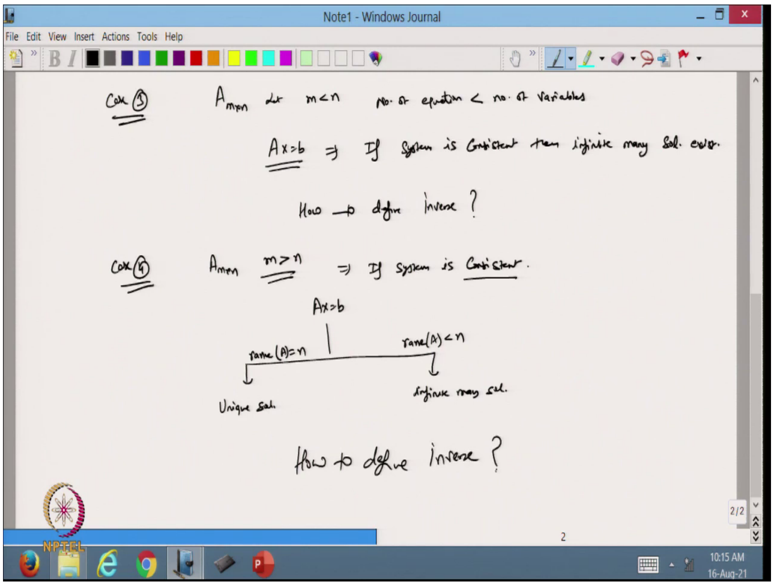

Maybe you can take the case 3: I have a matrix A m*n and let I just take m < n, it means the number of equations < the number of variables. So, in this case whenever we have a system of equations like Ax = b then if the system is consistent then always.

So, if the system is consistent then infinitely many solutions exist; then infinitely many solutions exist. Because in this case the number of equations is less than the number of variables. And so, in this case also how to define the inverse, that is also one of the questions.

Similarly, I can take the case 4; when I have A matrix m*n matrix and my m > n. So, in that case the number of equations is greater than the number of variables and in this case also the system is consistent then the system. So, the same system I am talking about, Ax = b. So, if the system is consistent then it may have. So, in that case I can say, so this system has I can make a condition here.

So, in this case m > n now what will happen if I take the rank of matrix A is. So, the number of equations > the number of variables. So, what would happen if it had a full rank? It means the columns are linearly independent. So, if the columns are linearly independent then in this form then we are going to have a unique solution.

And when the rank of A < n the number of variables and the system is consistent then is going to have infinitely many solutions. So, in this case also, how to define the inverse? So, all these systems we will discuss here how we can define the inverse. So, now, definitely for such types of systems we are unable to define the regular inverse. So, this type of inverse is called the regular inverse.

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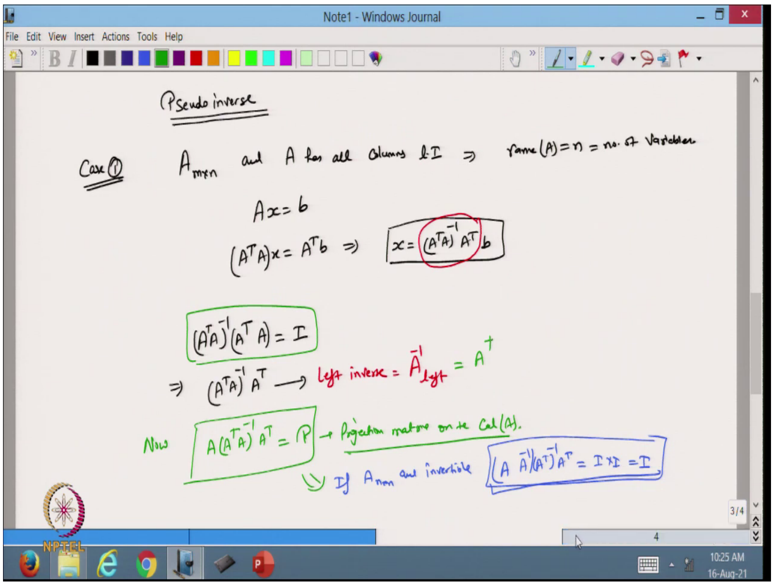

So, let us define another term that is we call it pseudo inverse. So, in the pseudo inverse let us take all the cases. So, I will take case 1, suppose I have a matrix A that is of order m cross n. And in this case suppose and A has all columns linearly independent. It means that it has full rank in terms of the number of variables that implies that the rank of the matrix A is n that is the number of variables.

So now in this case, suppose I have a system Ax = b then I know that I can pre multiply by the transform; $A^T A x = A^T b$. And from here I can write now, because this matrix is a full rank matrix ; all its columns are linearly independent. So, I know that the $A^T A$ is invertible and from here I can define my $x = (A^T A)^{-1} A^T b$.

So, these things we have already seen in the terms of least squares. Now, so from here I define; so this term I will define as now let us see what will go and what is going to happen. See if I take $(A^T A)^{-1} (A^T A) = I$ then in this case if you see I will get an identity matrix.

So, from here I can say that this$(A^T A)^{-1} A^T$, so this is A transpose if I can call this Left inverse or I can represent by A^{-1} left. So, this is my left inverse because I am multiplying on the left of the matrix A and I am going to get the identity matrix.

Now what will happen let us see; so, this is going to be in the case of left inverse, now let us see what will happen if I take on the right-hand side. Suppose, now I take A and put it on the right-hand side $A(A^T A)^{-1} A^T = P$. So, that is the projection matrix on the column space of A that we already know. So, in this case I will get my P the projection matrix and that which is the projection matrix on the column space of the matrix A, this one.

Now, somehow suppose from here let us see what is going to happen. Now I take if A is n*n and invertible if A is n*n and invertible. Then I can write from here it will become $(AA^{-1})(A^T)^{-1} A^T = I * I = I$.

So, in this case it will also become the right inverse. So, that is the condition for the projection matrix that if the matrix is an invertible matrix then this will be the identity matrix. So, this is my left inverse we have defined.

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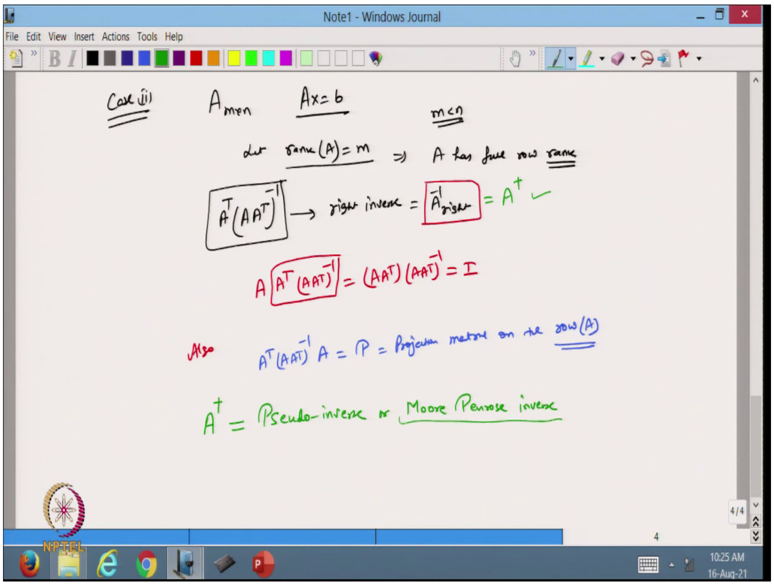

Now, I will take the second case; so I have taken case 1, now I am taking case 2. So, A is m*n and in this case now suppose, so I have the Ax = b this is my system and I am taking let the rank of the matrix A is m. So, suppose we have a m < n, the number of equations < the number of variables. So, in that case suppose I have a rank equal to m, so from here I can say that the matrix A has full row rank.

So, in this case what we are going to define is now let I have the system. So now, I define a matrix that is AA^T and then I am taking its inverse and then this one. So, let us see what is going to happen in this case. So, I define the term this one and I call it right inverse and I represent it by A^{-1} right.

So, in this case let us see, so why is it right? If I take A and I multiply by this inverse then this will be equal to AA^T and $A^T(AA^T)^{-1}$ and that is equal to y. So, from here I can say that this is in the right inverse of the matrix A and from here then I can write like this one. Also, if I try to put it on the left side, let us see what is going to happen.

Now, also if I take $A^T(AA^T)^{-1}$ I am putting on the left-hand side because it is the right inverse. Then if you see, then this is also a P. It is a projection matrix on the row space of A, because instead of A^T you just put A and then you will get the same value as the previous one.

So, that was the column space, so now I take just on the A^T. So, I get this value and that is the projection matrix on the row space. And the same way if you take this matrix is invertible and then this becomes the identity matrix. So, in this case we are able to write the left inverse and the right inverse. So now, we can define the terminology.

So, now we define the terminology A^+ with this sign the plus sign. So, this is we call it Pseudo inverse or we also call Moore Penrose inverse, because this is given by the mathematician Moore Penrose. So now, from here I can call that this left-hand inverse also can be written as a dragon. So, this is like a.

And this also can be written as. So, it is a plus sign like a cross, so this is the pseudo inverse. So now, from here we are able to write the pseudo inverse that is the we call it right-hand inverse and the left-hand inverse. Now we are going to take the next case.

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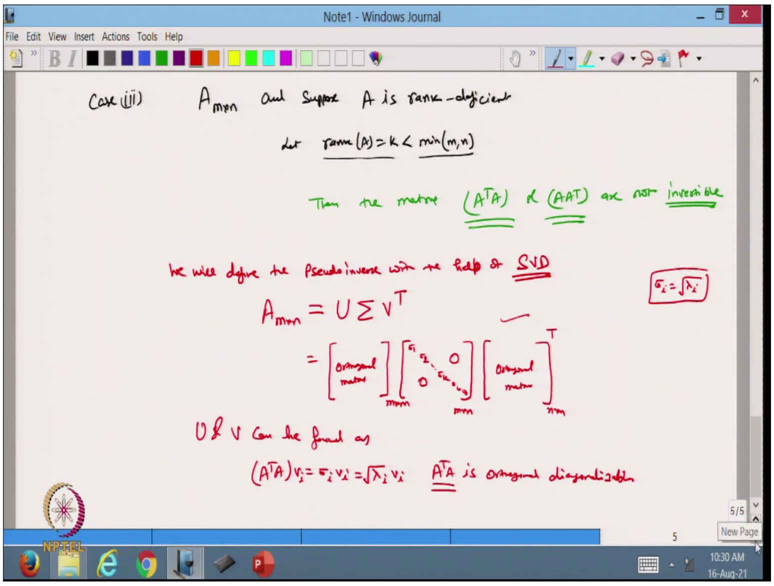

So, let us take the third case. Now we have the matrix A that is m* n. And suppose A is rank deficient, so rank deficient means it is not a full rank. So, let the rank of matrix A is k and that is less than whatever the minimum value I am taking this. Because if a number of rows is more than the number of variables then it can be n, minimum value can be n otherwise minimum value can be m. So, the rank of the matrix is even less than this one.

So, in that case, so now from here now if this is the case then the matrix AA^T and AA^T are not invertible, this thing also we have seen. So, if it is not invertible then if you see from here then I cannot define the left inverse and the right inverse. So, in this case how are we going to define? So, now, from here I will take the help of SVD.

So, we will define the pseudo inverse with the help of SVD, singular value decomposition. So, in this case what we are going to do is now I have matrix A, so this is my matrix A that is m*n. So, I know that I can write its singular value decomposition, so I will take the $A = U\sum V^T$.

So, these things we already know where this U is a square matrix of order m* n having the; so here it is an orthogonal matrix.

Now, summation sigma is the diagonal matrix with singular values $\begin{bmatrix} \sigma_1 & 0 & 0 & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ \vdots & \vdots & \sigma_k & \vdots \\ 0 & 0 & 0 & \ddots \end{bmatrix}$

and then (0 0 0). And this is, so this is of order m*n and this is again the n* n matrix taking its transpose. So, this is a V^T, so this is also an orthogonal matrix. And we know that in this case we can find out the orthogonal matrix. So, U and V we know that we can find U and V as; so U and V can be found as. So,

$$\left(A^T A\right)v_i = \sigma_i v_i = \sqrt{\lambda_i} v_i$$

So, now, from here because I know that the $\sigma_i = \sqrt{\lambda_i}$. So, maybe I just write this one lambda i, this value. Now from here, it means that this A^T A is orthogonally diagonalizable. So, from here I can find the value of V, so this is the V I can find.

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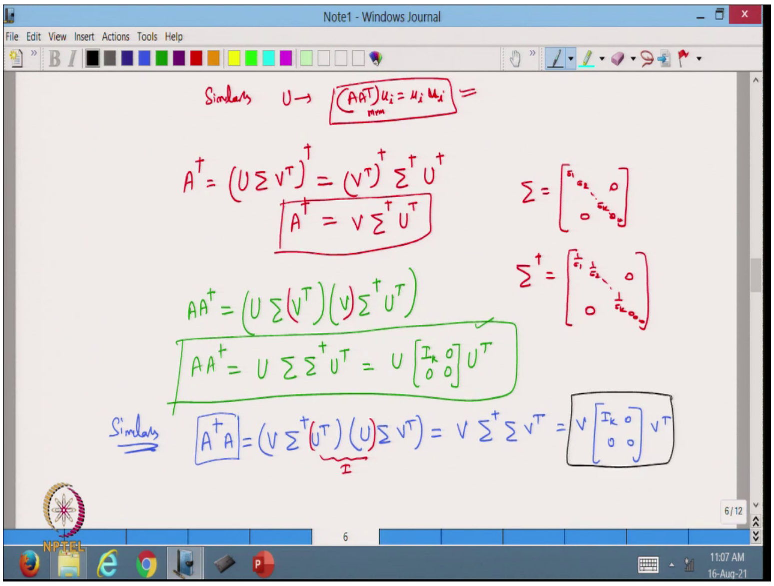

Similarly, we can define U, so U we can define from AA^T and then putting this ui's and that is also equal to whatever the values we are taking, so maybe I can write them $\mu_i u_i$. So, this is also how we can define the diagonalized form of the matrix AA^T, because this is going to be the m*m matrix. And this is going to the n* n matrix. And this is a symmetric matrix, so we know that it is always diagonalizable, so we can write this form and then we can substitute here and we get this value.

Now, let us see, so what is going to happen now. Let us define a pseudo inverse. So, this one I will define, so it is U V^T pseudo inverse. Now I can define this as

$$A^{\dagger} = \left(U\Sigma V^{T}\right)^{+} = \left(V^{T}\right)^{+}\Sigma^{+}U^{+}A^{\dagger} = V\Sigma^{+}U^{T}$$

So, it is a pseudo inverse, so it becomes the transpose because U is an orthogonal matrix. So from here I will get this form.

Now, let us see what is going to happen here. Now we know that the sigma is basically a diagonal matrix with singular values $\sum = \begin{bmatrix} \sigma_1 & 0 & 0 & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ \vdots & \vdots & \sigma_k & \vdots \\ 0 & 0 & 0 & \ddots \end{bmatrix}$

See if I take the sigma inverse the pseudo inverse, so this will be

$$\Sigma^{+}=\begin{bmatrix} 1/\sigma_1 & 0 & 0 & 0 \\ 0 & 1/\sigma_2 & \cdots & 0 \\ \vdots & \vdots & 1/\sigma_k & \vdots \\ 0 & 0 & 0 & \ddots \end{bmatrix}$$

Now let us see I define AA inverse pseudo inverse. So, this is

$$AA^{\dagger} = \left(U\Sigma V^{\top}\right)\left(V\Sigma^{+}U^{\top}\right)$$

$$AA^{\dagger} = U\Sigma\Sigma^{+}U^{\top}$$
$$= U\left[I_k\ 0\ 0\ 0\right]U^{\top}$$

Now, in this case this and this will be an identity matrix because it is an orthogonal matrix. So, I can write from here then this will be equal to U^T.

And if you see from here then I am multiplying this by this matrix. So, it will be a U and then I will get, here I will get the identity matrix of order k. So, if I put the pseudo inverse on the right-hand side I will get this value; so AA pseudo inverse.

So, I am not going to get the value I here because we have this matrix and if this matrix has a full rank and then it is complete I then it will be I and then U U^T will be I otherwise this will be on this form. Similarly, what we are going to define is, now I take A inverse A. So, in this case I will get V summation and U transpose and put the value here A.

So, A is U Σ V^T and from here you will get V Σ. Now in this case also this is an identity matrix because it is an orthogonal form. And from here this will be equal to sigma V^T. So, in this case also we are getting V then I_K and then V^T. So, that is how we are going to get when we take on the left-hand side of A.

So, this is the way we can define the pseudo inverse for any type of matrix. If it is diagonalizable then I can define its left inverse and the right inverse. And means if the matrix is a full rank, then we can define the left inverse and the right inverse. And if the matrix has the rank deficient form, so in that case we can use the help of this SVD to find out the pseudo inverse.

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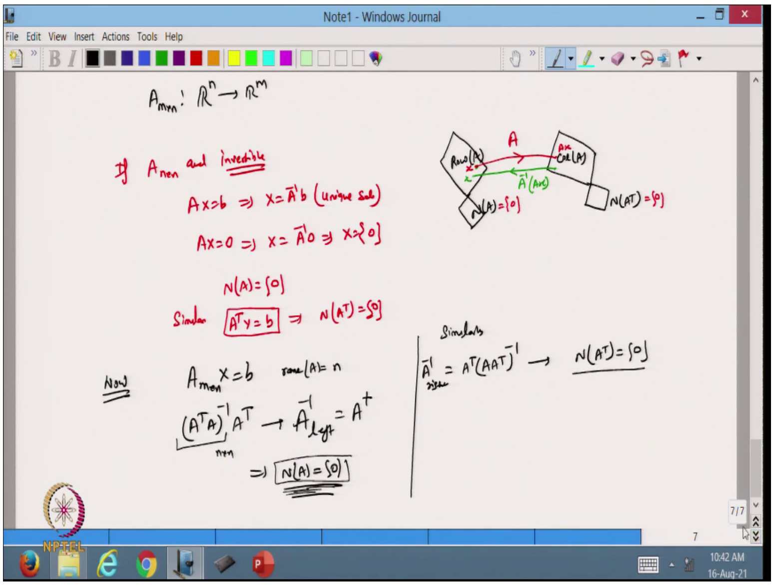

Another important thing we just want to discuss here is that. Now how this, because we know that, if I have a matrix A m* n then this is a transformation. And we all also know that these are the four subspaces and this is orthogonal to each other. So, I know that this is a row space of A, this is a column space of A, this is a null space of A and this is the null space of A^T.

Now, from here if you see that if A is n* n and invertible then I know that Ax = b and from here my x will be A^{-1} b. So, in that case I have a unique solution. Now suppose I take Ax = 0. And from here you know that this will be $A^{-1}0$ and that gives you that X= {0} element because it will be 0, so x will be 0 element only here.

So now, if the matrix A, so now, this is my transformation A. So, in this case if the matrix is invertible then I can say that the null space of A has only 0 element. Similarly, I can take the A^T and then we can define some y = b. So, in that case also you will see that if A is invertible then A^T is also going to have the rank full rank and in that case it is also going to give the unique solution.

So, from here you can see that the same way I can define that the null space of A^T is also 0 element. So, whenever we are going to have an inverse of the matrix A then in that form always the null space of N (A) and the null space of A^T is 0 when this inverse exists. So, this is the A we are going to take and my A inverse is going back from here to here. So, that is my A inverse.

So, suppose this is my some let us take this my x. So, this will be Ax and this is going to give back Ax. So, this is my Ax, if I take this x and this is going to give you the x the back. So, that is the meaning of inverse in this case.

Now, similarly, the left inverse is basically this is the left inverse. So, left inverse what it is doing is killing the null space on the left-hand side and right inverse killing the null space on the right-hand side. So, similarly we can define that we have a system A that is m*n x = b. So, in this case if we define $(A^T A)^{-1} A^T$, so this is my left inverse.

So, I know that the dimension of $A^T A$ is n*n because we have taken that it has the full rank the rank of A we have defined is equal to n the number of variables. So, in this case also if you see the number of variables are n, it means I can say that the null space of A will contain only 0 element because this matrix has a rank that is n.

So, we can see that this is the whole of the dimension n then at its dimension will be 0 by the rank nullity theorem. So, in this it means that the null space of A is 0. So, it means that the left inverse pseudo inverse is killing the null space on the left-hand side.

And similarly, $(AA^T)^{-1}$ on the right-hand side in this case you can say that the null space of A^T is 0. And it means that the right, so that is I can write as A^{-1} right-hand side. So, it will kill the null space on the right-hand side, so that is my right-hand side. So, that is the meaning of the inverse, so finding the inverse means it kills the null space related to that vector space from where it is defined. So, this is just the way we can define it.

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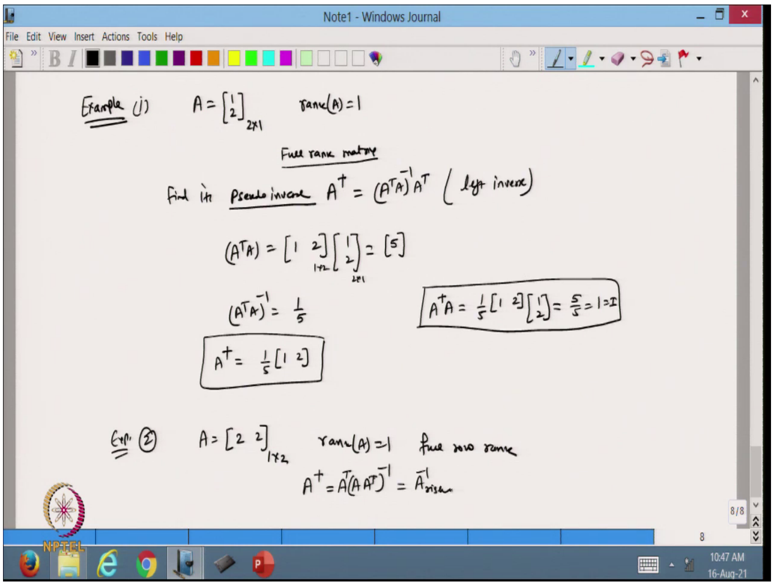

So now, let us take one example of how we can define the pseudo inverse for the corresponding given matrix, so let us take one example. Suppose, I have a matrix A that is a column matrix. So, let us take one very simple matrix. I just take the column matrix $A = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

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So, this is my matrix of dimension 2*1 and I know that the rank in this case is 1. So, it is full rank; full rank matrix. So now, I want to define its pseudo inverse.

So, it means that it is a full rank, so I want to define its pseudo inverse. So find its pseudo inverse. So, pseudo inverse means I need to define this value and in this case this value will be equal to because we know that it is the full rank matrix. So, I can define my $(A^T A)^{-1} A^T$. So, this is basically what I am going to define here as the left inverse. So, let us see.

Now I define my A^T A. So, A transpose A will be [1 2] and this is [1; 2], and from here you will see that it is 5 because it is 1* 2 it is 2 *1, so it is 1* 1. So, I will get the matrix 1* 1 matrix with the element 5 and A^T A inverse because in this case I know that this is invertible. So, it will be 1/5. So that is my inverse in this case.

Now, I can define my pseudo inverse, so that will be 1/5 and A^T, so this is [1 2]. So, that is my left inverse. Because you can check from here, then if I write like this one then it will be 1/5[1 2] and A is [1 2]. So, it is 5/ 5 that is 1, so my A^{-1} A will be I that is 1. So, basically it is an identity matrix of dimension 1 *1. So, that is the way we can define my pseudo inverse for the matrix of this form.

So, let us take another example, the second example. I can take the matrix A now I just take this matrix row matrix 2 cross, so it is 1* 2 matrix. Now, I can say that it has a rank(A)is 1, that is the number of rows. So, it has a full row full row rank, so has full row rank. So now, I can define its pseudo inverse that will be $(AA^T)^{-1}$, because I am putting on the right side so that is my A^T and this is I know that this is A^{-1} right. Now let us see what is going to happen in this case.

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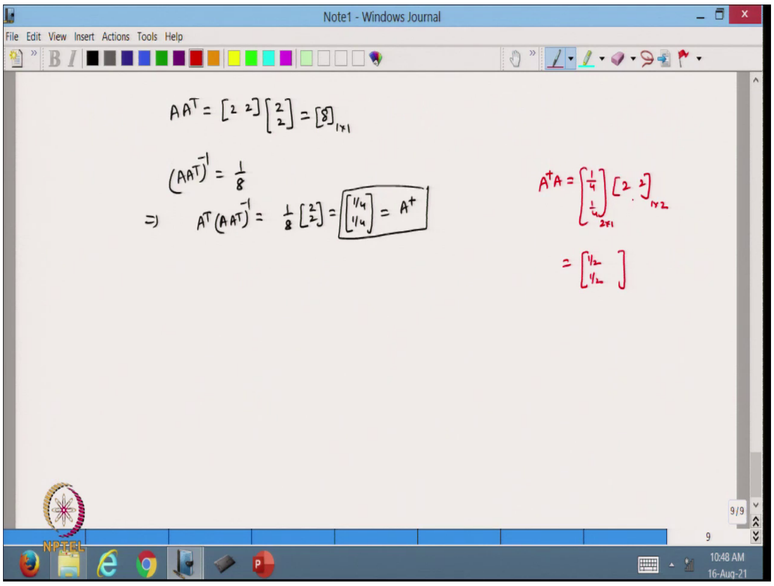

Now, I define $$AA^T = \begin{bmatrix}2 & 2\end{bmatrix}\begin{bmatrix}2\\2\end{bmatrix} = [8]$$

So, I will get the matrix that is 1* 1 matrix. And $(AA^T)^{-1}$ will be 1/8. And now, from here I can define my $A^T(AA^T)^{-1}$, so it is 1/8 and A^T is [2 ;2]. So, I can assume that I just multiply here, so it will be [1/4 ; ¼].

So, that is my pseudo inverse in this case. And I can check this one that my A, so this is [1/4 ;1/4] and then my A is [2 2]. So, it is 2*1 it is 1* 2, so it will be a 2* 2 matrix. Now, multiply here, so it is 1/ 2. This is also multiplied here, so it is 1/2; it is a right inverse not on the left side.

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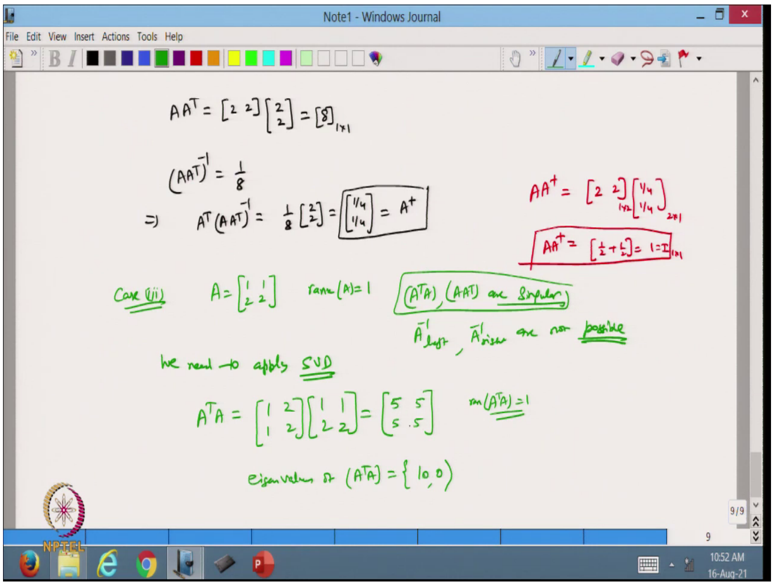

So, I have to write AA^{-1} , so my A is [2 2] and this is 1/4 and 1/ 4. So, it is 1* 2 and this is 2*1. So, from here this will be equal to, so 2/4, so it is[1/ 2 + ½] and that is 1. So, my right inverse is when I apply on the A I get the value I this is my identity matrix of dimension 1*1. So, from here we are able to verify that this is the right inverse.

Now I take case 3. I take the matrix A. Let us take it may be 2* 2 matrix I just take $A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$ So, in this case you know that the rank of the matrix A is 1. So, it is a rank deficient matrix and from here I know that $A^T A$ and AA^T are singular. So, I cannot define the left inverse and the right inverse then I have to take the help of SVD theorem.

So, since these are singular matrices then we can now define A^{-1} left and A^{-1} right are not possible in this case. So, I have to take the help of SVD, so we need to apply for the SVD form, so let us take this one. So, I just want to write the SVD form.

So, I will take $A^T A$ in this case. So,

$$A^T A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 5 \\ 5 & 5 \end{bmatrix}$$

And I know the rank of this matrix $A^T A$ is always the same as the rank of A, so from here I can say that its rank is 1.

Now, $A^T A$ the rank is 1, so it means that this is a singular matrix, so it is going to have one eigenvalue 0. So now, from here I can find out I can say that the eigenvalues of $A^T A$. So, this is 10 and 0 because I can write directly, because the sum of eigenvalues should be equal to the trace of the matrix. So, the trace is 10, so I can take this value 10 and 0.

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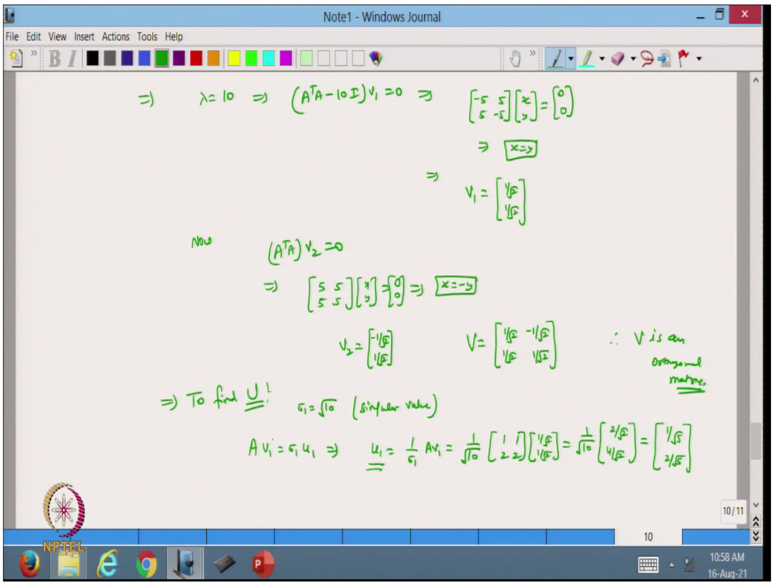

Now, from here I can find out the eigenvector corresponding to, so lambda is 10 basically.

Now, from here I can define $(A^T A - 10 I)\, v_1 = 0$ $\begin{bmatrix} -5 & 5 \\ 5 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

So, $x = y$. And from here I can take my first eigenvector $v_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

because I need to find its magnitude 1; the v1 should be of magnitude 1. So, that is my first eigenvector we have taken.

Now the second eigenvector is corresponding to the 0 eigenvalue. So, it should be equal to $(A^T A)v_2 = 0$ and from here you will see that this is $\begin{bmatrix} 5 & 5 \\ 5 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

and that gives me x =-y. So, if x = -y I can choose my v2. Maybe I can take $v_2 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

Because from here now I can define my matrix V as $V = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$

And you can check that V is an orthogonal matrix. So, we are able to find this orthogonal matrix. Now from here I need to find the U, so to find U this one we need to find.

So, in this case it is also of 2 dimensions, so I will define with; so first we have a non zero singular value. So, I will define for $\sigma_1 = \sqrt{10}$ that is my singular value, so this is my singular value. So, in this case I will take $Av_1 = \sigma_1 u_1$. I already know how we can find u1.

So, from here this value is, so I can find my $u_1 = \frac{1}{\sigma_1} A v_1$. So, that is equal to

$$= \frac{1}{\sqrt{10}}\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \frac{1}{\sqrt{10}}\begin{bmatrix} 2/\sqrt{2} \\ 4/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$$

So, this will become like this one So, I am able to find u1 because u1 is also a unit vector. Now I can define my u2. So now, I cannot define these things because my other value is 0, so then what we need to do is that.

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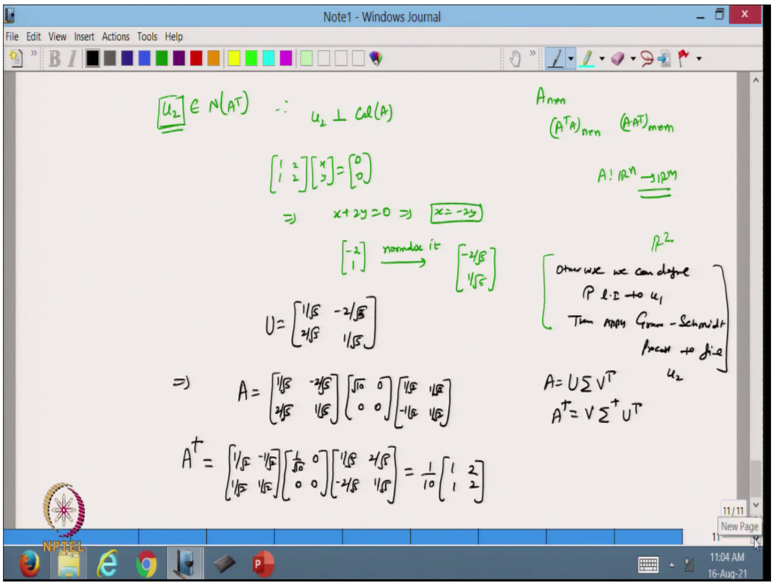

Now, u 2 is I can define from, so if you see from here this u is generally of order n cross m. So, suppose we have a A n cross m then A transpose A is always n cross n and AA transpose is always m cross m and my A is from R n to R m. So now, from here you can see that my u 2 will belongs to null space of A transpose, because my u 2 will be perpendicular to the subspace of column space of A.

$$U = \left[1/\sqrt{5} \;\; -2/\sqrt{3}\; 2/\sqrt{5}\; 1/\sqrt{5}\right]$$

$$\Rightarrow A = \left[1/\sqrt{5} \;\; -2/\sqrt{5}\; 2/\sqrt{5}\; 1/\sqrt{5}\right]\left[\sqrt{10}\; 0\; 0\; 0\right]\left[1/\sqrt{2}\; 1/\sqrt{2} \;\; -1/\sqrt{2}\; 1/\sqrt{2}\right] A^{\dagger} = \left[1/\sqrt{2} \;\; -1/\sqrt{2}\; 1/\sqrt{2}\; 1/\sqrt{2}\right]\Big[$$

Because, here it will be the column space and that is of dimension m cross m, so this is basically m cross m. So, in that case the rank is 1 only here and it is of 2 dimension R 2 in this case. So, 1 is lying in the column space that is u 1. So, another will lie perpendicular to this one because we need the orthogonal matrix. So, that will definitely belong to the null space of A transpose.

So, I can write that u 2 belongs to the null space of A transpose. So, from here I can write that A transpose is 1 1 2 2, so it will be 1 1 2 2 x y and that equals 0 0. And from here I will get x plus 2 y is equal to 0 and that gives me that my x is equal to minus 2 y.

So, from here I can take my vector as I suppose y is equal to 1. So, maybe I will just take x as equal to 1 or maybe y is equal to 1 and x will be minus 2, and now I normalize it. So, if I normalize this will become, so 2 raise to power 2 4 plus 1; 5. So, it will be minus 2 by root 4 and 1 by root 5, so this way I can define root 5.

So, this is my u 2 and you can check that this is orthogonal to u 1. Another way is that we can define. Otherwise, we can define a vector some vector I can define some vector maybe P that is linearly independent to u 1 and then apply Gram Schmidt. So, then I apply the Gram Schmidt process to find u 2. That is also one of the ways.

So now, from here we are able to get the values. So, my u is now, so it is 1 by root 5 2 by root 5 and this is minus 2 by and 1 by root 5. So, 1 by root 5 2 by root 5 and if you take the dot product, this is going to be the 0 value and from here you can see that, which implies that my matrix A is now U. So, this is 1 by root 5 and this summation will be root 10 0, 0 0, because 0 is the other eigenvalue of A transpose A.

And here I will take V transpose, so my V value is basically 1 by root 2. So, I will take the transpose of this, so the minus will come here. So, it is equal to 0 over root 2 minus 1 over root 2 1 over root 2 and 1 over root 2. And if you do this one you will get the value, so that you can verify yourself.

Now, from here I want to define my pseudo inverse. So, pseudo inverse in this case will be because my A is U summation V transpose. So, this one I want to define, so this will be equal to V U transpose. So, in this case my this will be equal to; so now, I need to find the V. So, this will be equal to 1 by root 2 minus 1 by root 2 1 by root 2 and 1 by root 2.

And this is the inverse of this, so it will be 1 by root 10 0, 0 0 and U transpose, so this is my U minus 2 by root 5 2 by root 5 and 1 by root 5. So, that is the value we are going to get and if you solve this one, maybe I can do the calculation for this. So, if you do the calculation I will get the value 1 by 10 and this is 1 2, 1 2. So, that is the pseudo inverse we are going to get in this case.

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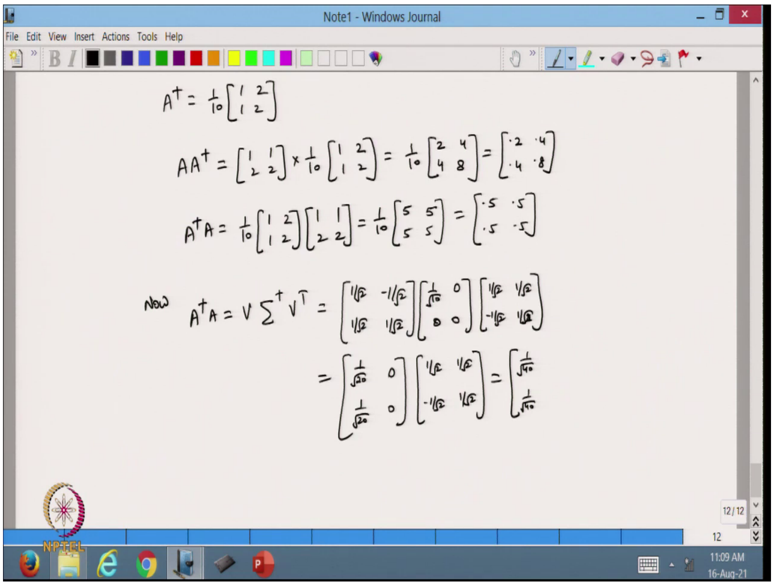

So, my pseudo inverse is 1 2, 1 2. And now you can check from here that A A pseudo inverse, so my A is in this case is 1 1, 2 2 1 by 10 it is 1 2, 1 2. So, in this if I take the calculation, so I will get the value this is 1 and 1 2, this is 4; it is 4 and this is 4 plus 4; 8. So, I will get the value here 2 by 10; 0.2 0.4, 0.4 0.8

And if you take a pseudo inverse A, so this is 1 by 10 1 1 2 2, so this is 1 by 10. And this is I am taking it will be 5 here and then it is 5 here 5 5. So, this is equal to 0.5 0.5, 0.5 0.5. So, that is my pseudo inverse and you can also verify this with the help of.

Now, after getting this value you can see that we are unable to find here the identity matrix. That we already know that we are not going to get the identity matrix. And these things we can also verify by taking this form. So, A pseudo inverse if you see from here A this one.

So, A pseudo inverse, yeah A pseudo inverse is this form V I K V T. So, it is equal to V summation V T. So, V is basically we already know, so my V is 1 by root 2 1 by root 2, so I can just take from here also. So, 1 by root 2 and minus 1 by root 2; this is 1 by root 2 and 1 by root 2 and this is 10 by root 10 0 0 0. And V transpose is just and if you do the calculation here, so you will see from here I will get this value. So, let us calculate this first.

So, I will get this, so I will get 1 by root 20 1 by root 20 and then, so this is going to be this is going to be this now this is 0 0. So, I will get this value; into 1 by root 2 1 by root 2 minus 1 by root 2 and 1 by root 2. And if you do the calculation here, so 1 by root 2 is here now from here I will get 1 by root 40 and this is 1 by root 40 this is the value. Now this should be I basically, if you see from here this is I.

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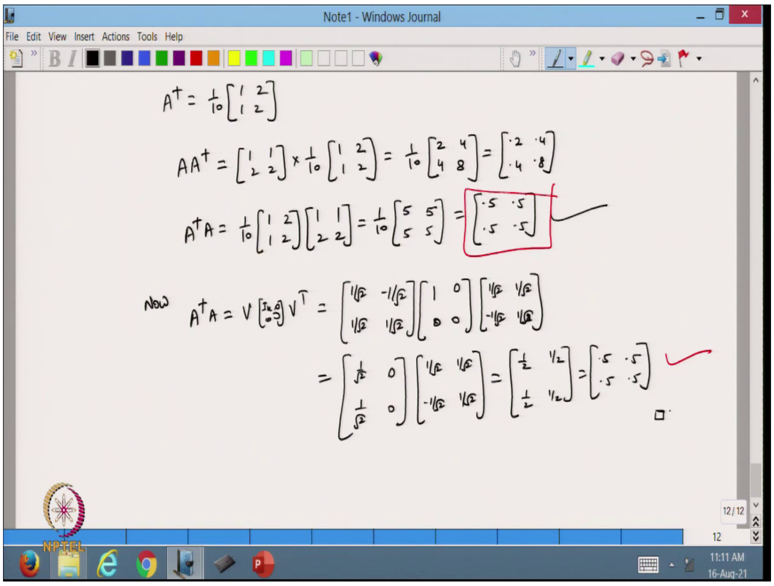

So, it should be 1. So now, from here if I do this one it should be 1 by root 2 and this should be 1 by root 2 and now from here I can check. So, 1 by root 2 and 1 by root 2 1 by 2, now this is also 1 by 2, this is also 1 by 2 and this is 1 by 2. And that is 0.5 0.5, 0.5 0.5 and this is also how we are getting this way. So, this is either we can verify from here and we can verify from here also.

$$A \ = \frac{1}{10}[1\ 2\ 1\ 2\ ]$$

$$AA^{+} = [1\ 1\ 2\ 2\ ] \times \frac{1}{10}[1\ 2\ 1\ 2\ ] = \frac{1}{10}[2\ 4\ 4\ 8\ ] = [2\ 4\ 4\ 8\ ]A^{+}A = \frac{1}{10}[1\ 2\ 1\ 2\ ][1\ 1\ 2\ 2\ ] = \frac{1}{10}[5\ 5\ 5\ 5\ ] = [.5\ 5\ 5$$

So, and this is not it is basically I K 0 0 0 because I have taken this form this way. Similarly, we can define from the other way and we can verify. So, this is the way we can take the pseudo inverse for any type of matrix: it is either as a full rank or in the matrix or square matrix with the lower which has the row deficiency. So, this way we can define the pseudo inverse of the given matrix. So, we will stop here.

So in today's lecture we discussed the pseudo inverse of a given matrix. So, we have shown that if the matrix is a rectangular matrix and if it has the full rank then we can define its left inverse and the right inverse. And if the matrix is having the rank deficiency then we can take the help of SVD to find out the pseudo inverse of the given matrix. And this pseudo inverse is also known as a Moore Penrose inverse or the pseudo inverse. So, I hope that you have enjoyed this lecture.

Thanks for watching.