SCIENTIFIC COMPUTING USING PYTHON
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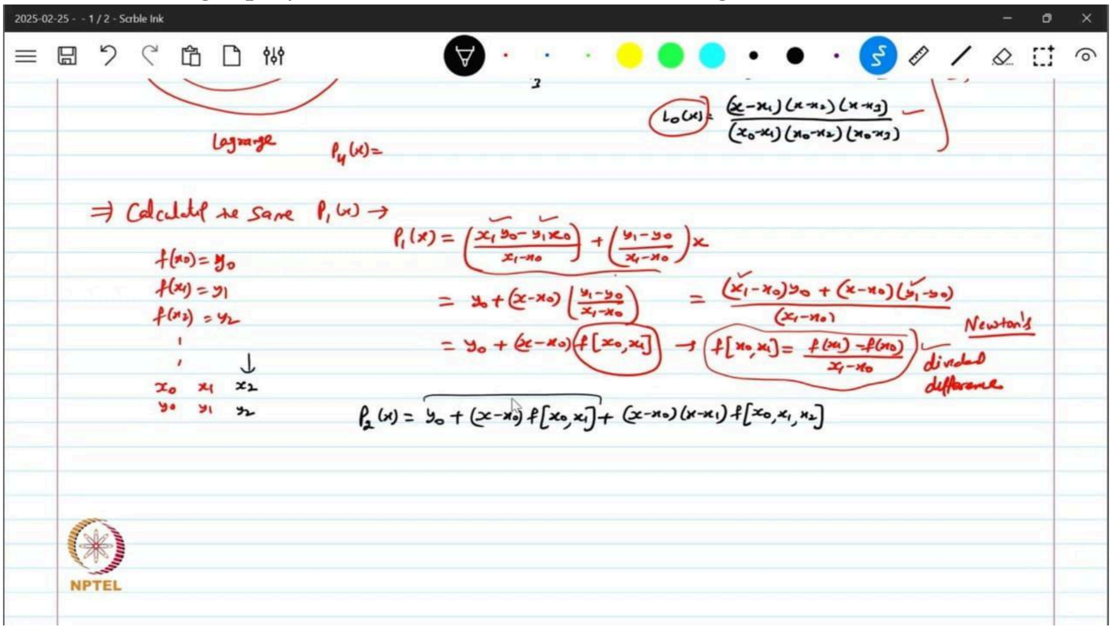
Lecture No. 23

Welcome everyone to Scientific Computing Using Python. In our last lecture, we discussed Lagrange's Interpolation Polynomial. So today, we will go beyond that and discuss Newton Divided Difference Polynomial, Interpolating Polynomial. So, let's get started. In what we did in our last lecture, we found out that we constructed the interpolating polynomial. So, we can construct it in this way. Now, what is the problem, what is the drawback of the Lagrange's? What happens in the Lagrange's is that if we have some data points, suppose the data points we have are x0, x1, x2, x3, these are four data points, and corresponding values given y0, y1, y2, y3, so what we did here the Lagrange's Interpolating polynomial, if four points then a polynomial of third degree we get p x is suppose i write l0 x y0 l1 x y1 l2 x y2 plus l3 x y3. This will come to us. Okay, now we will calculate all of this. Now, for example, if we look at l0, to calculate l0, we would need x minus x1 x minus x2 x minus x3 this will have to be calculated divide by x0 minus x1 x0 minus x2 x0 minus x3 this will have to be calculated in this way all our fundamental polynomials of Lagrange's will have to be calculated okay so suppose we have calculated it now what happened to us that someone said okay you have calculated till here now do this add one more value x4 y4 this as well we have got the data in this and you calculate the new data that has come in this it can be that suppose we had data of three or four days then we collected the data, the data of fifth day also came so we were told let's add this as well now what will we do to add this now we had to do it from Lagrange's only okay so what will happen in Lagrange's the degree of polynomial increased it became third now what happened all these things will have to be calculated again now our l zero that also got changed l1 also change happened, L2 also changed, earlier it was of 3rd degree, now L0 has become of 4th degree, so what do we have to do that we have to recalculate everything, even if we added just one data point, so we calculated again, then the next day someone said that one more data has come, x5 y5, you can add this as well, so we repeated the exercise that we had done, then this was added, then we repeated the exercise, so this work is quite difficult, that is, it is a time consuming work, so in this we are realizing that our unnecessary time is getting wasted because we have to do it again and again, so what did we do to get rid of this, now let's see that when we did the calculation, we calculated p x like this, the first linear polynomial and we calculated it like this, it came from here, basically, okay, so if I calculate this polynomial in a slightly different way, then see this is it the polynomial. So I calculate p1 x in a different form and we'll see what happens. So calculating the same p1 x. So if we look at it, our p1 x was this, because we had calculated p1 x, so the p1 x we had was x1 y not minus y1 x not divided by x1 minus x not plus y1 minus y not x1 minus x not now see, I am writing this in a different way, how do I write it like this, y0 plus x minus x0 and here I write y1 minus y0 x1 minus x0, write it like this. So now if we expand this and I want to see whether this thing is the same or not, then look, x1 minus x0 can be taken common, so it will come here, x1 minus x not y not, okay, plus x minus x not y1 minus y not divided by x1 minus x not, this is what comes now, I will calculate this if So look what is being formed, y1 x minus y not x this x will come from here and x not y not will multiply from here and will cancelled by this, okay so we have if we see divide by x1 so our this thing we have can be written in this form so what we did here we changed this in this form and we gave this a new name y0 plus x minus x0, okay and I am writing this in the form of a new representation and writing it as f x0 x1, okay f x0 x1 why because what we do generally, we assume that f x0 is y0 f x1 is y1 f x2 is y2 it is okay it is a function, so that's why we will write it in this form Okay, so we have written it like this, so now what does this quantity mean? It means f at x0 x1, which is a difference operator. So this will come, the values that we have, we have f at x1 minus f at x0 divided by x1 minus x0, so if you see, this is a difference operator which is giving information about a derivative, a slope, so we have named this factor as divided difference. So this is divided difference, whatever name we have given to it, we said that this is divided difference and this which was given to us was given by Newton, so we call it Newton's divided Difference. Now what did we do in this case, in this case we have only two points, see x0, x1 there, the y0, y1 given to us, now someone told us to add one more point, so we added an extra point, from here someone got it added to us and here, someone said that add x2 to this as well. Okay, so someone told us to add one more value to this. Now we have come to know that if we add one more value, the degree of the polynomial will increase because it will have to be increased. So what will we do now? We will use this polynomial which we have calculated. So what will be the new polynomial? I would have written it like this, which we have calculated, y0 plus x minus x0, the first divided difference we have already calculated this. Now what will I do? I will add one more to this. I will call it x minus x0 into x minus x1 because we have to make it quadratic. And the divided difference that comes here will be the divided difference of the second order. And the dependent will be the three factors x0, x1, x2. All the three data points. So now see, the first terms are exactly the same. Now the g x polynomial which was of the first degree, we added one term to it.



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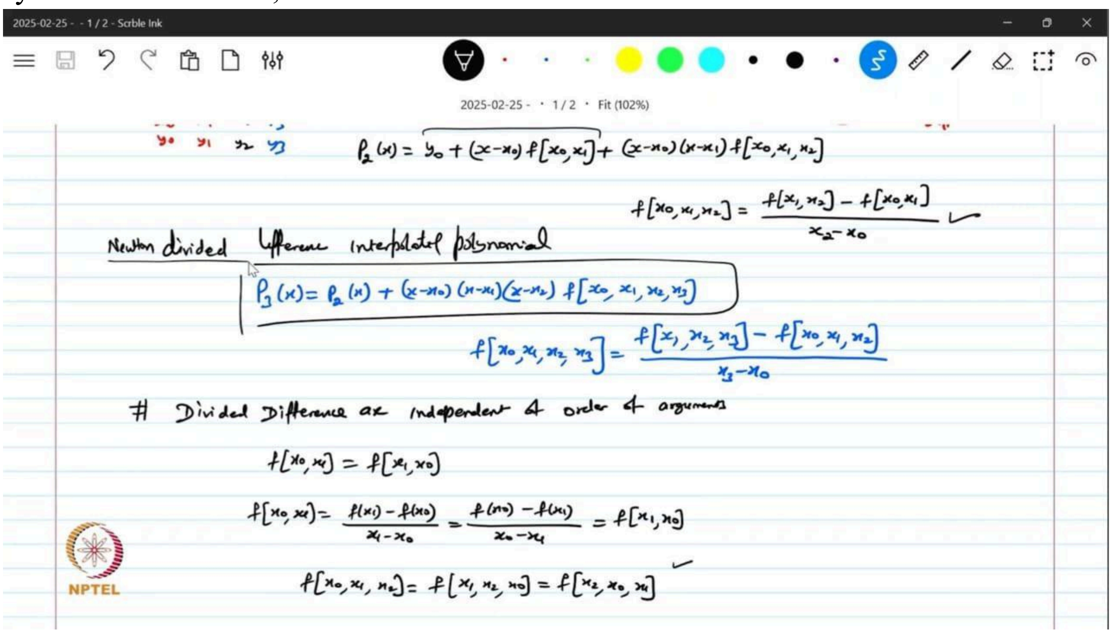
Now if someone asks us what is this, then this is the second difference operator. x0, x1, x2 What will this be? f x1, x2 the first degree which is the divided difference x0, x1 divided by x2 minus x0. So if we have this, then we will calculate it and we will add it. Now we are told to add one more point to it. So the polynomial that we have will become cubic of the third degree. But we will not leave what we have already calculated. So I will write it like this. I will write it p2 x So p2 x is the same as what we have calculated till now. Plus x minus x0 x minus x1 x minus x2 will become cubic from here and this third thing that comes will become its divided difference and this will be dependent on x0, x1, x2, x3. So where the divided difference is x0, x1, x2, x3 which we have, if we write it now, then what will come out to be f of the second order divided difference. x1, x2, x3 minus f x0, x1, x2 and divided by now see in this x3 is going to x0 so if this becomes x3 to x0 then what we have is divided difference. So with the help of this we can keep writing any number of divided difference x0, x1; x0, x2 so now we have come to know that the divided difference we will keep adding like this so now no matter how many points new points come to us if someone tells us to add more then it doesn't matter keep what we have calculated earlier plus one more term which is add new terms and there we will get a new polynomial. So according to this we will calculate it. Now there is one more property of Newton divided difference that is that the divided difference are independent of the order of elements or argument what does it mean that we have written here f x0 x1 this is the same as f x1 x0 because we know what is f of x0 x1 that is f x1 minus x0 divided by x1 minus x0. I can write it like this f of x0 minus f x1 divided by x0 minus x1. So what does this become? Our f x1 becomes x0. Okay, so in the same way we can calculate the divided difference. So here we have x0, x1, x2 I can write it in different forms. So it will be the same. I can write it like this f of x1 x2 x0 or f x2. I can write x0 x1 in this way. I can change the order but ultimately if the values we have are the same, then they are independent of order. So we got to know one thing that the polynomial that we tried to calculate by interpolating the polynomial, we wrote that same polynomial in a different way. We wrote it in a way. So it became a Lagrange interpolating polynomial. So what can we do so that we get this polynomial. Now we are getting this polynomial. We got an interpolating polynomial, so we will call it Newton divided difference interpolating polynomial. So this is its name, Newton divided difference interpolating polynomial. So now with its help we can interpolate polynomials according to any degree, and we have already seen about divided difference that divided difference is independent of order, so we can calculate it like this, so the polynomial that we have, and the interpolating polynomial that is basically by divided difference, Newton Divided Difference and this is the same.



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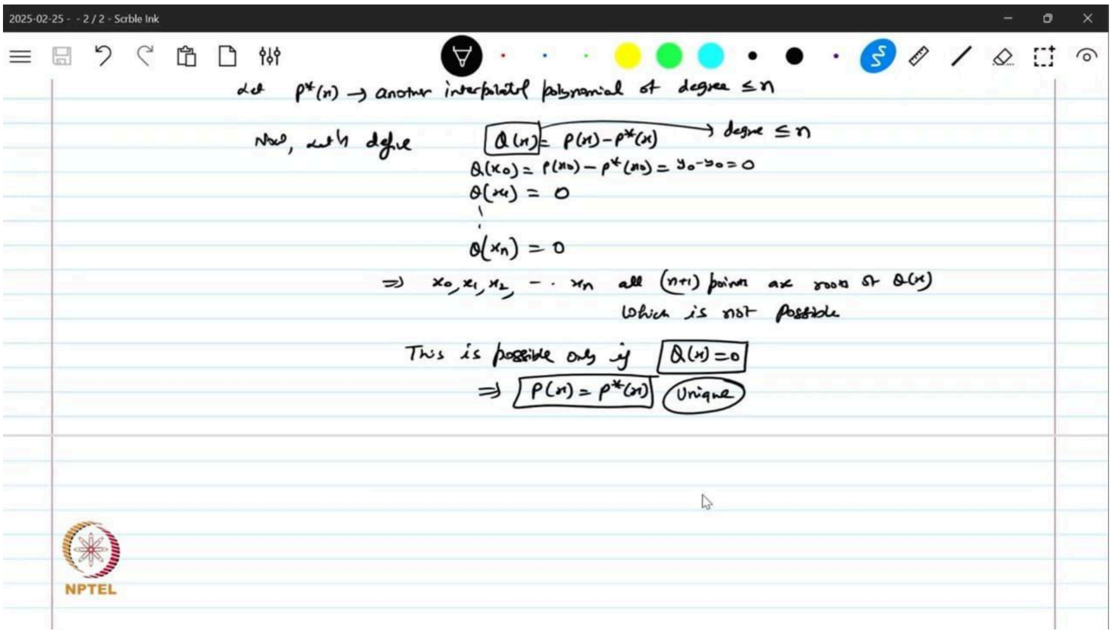
So, we can do one thing from here, the polynomial that we get, whether we do it with Lagrange's or divided difference, we can say that the polynomials that we get is unique, so it gives the guarantee of uniqueness, because initially we did the calculations give us values so we saw that in it we are getting the values a0, a1, a2 which are of n degrees polynomials are unique in that sense, so if they are being uniquely represented and calculated, then it means that we can give proof of the polynomial very easily. So let's say the p x that we had was suppose a0 a1 x, we have also calculated this as a n x, right? So we had also shown it by calculating, we have given it, okay, so we have calculated it. Now after that we can do let p star which is another interpolating polynomial, some interpolating polynomial, okay, of degree less than equal to n. Let me now what do I do, we define it, so now let's define the function q x, I have defined p x minus p star x, the difference between these two, okay, what about Q at x0, what will be P x0 minus p star x0, and we know that they will be the same value, what is p x 0 is y0, what is p star x0, what is it, it is also y0. What will happen, zero comes. Q at x1, that will also be zero, so like this If we see, that is Q x n is zero. Now p n is the polynomial of degree less than equal to n minus p star is the polynomial degree less equal to n. The difference between the two will be q x, okay, so the degree of q x will also be less equal to n. Now we have a polynomial whose degree is suppose n and the value that we saw at the n plus 1 points is zero, so this means that we can say that here x0, x1, x2 all x n all n plus 1 points are what have become the roots of q x, okay, so the n plus 1 points became the roots of q x, but what was the degree of q x that was n the maximum? So this is not possible. We know that if we have a second degree quadratic equation, it has two roots, cubic has three roots, so if it is n degree polynomial, then it should have n roots, but here the n plus 1 points which are n plus 1 roots, then which is contradiction which is not possible, so if it is not possible, then it is not possible, so what does it mean that it is possible, only then it is possible, because it is happening, this is possible only if q x is a zero polynomial in itself, then n plus 1 will always be zero at n plus two points, it will also be zero at i, so from here it is known that q is a polynomial, so from this we get that the p which is zero, p star x will be the same, so what do we get by being the same, it is uniqueness, we will see that if there is any interpolating polynomial, but the ways of writing it are different, but if we calculate and simplify all of them, then you will get the same interpolating polynomial, okay, so this is what we call unique, okay, so we have already done existence and now done uniqueness that it is unique has unique representation. Okay, so we have calculated an interpolating polynomial from Newton divided difference formula, so now I can also see it a little bit according to our code, so we can also calculate it according to the code because we have taken a lot of examples, so we see how we can calculate it with Newton divided.



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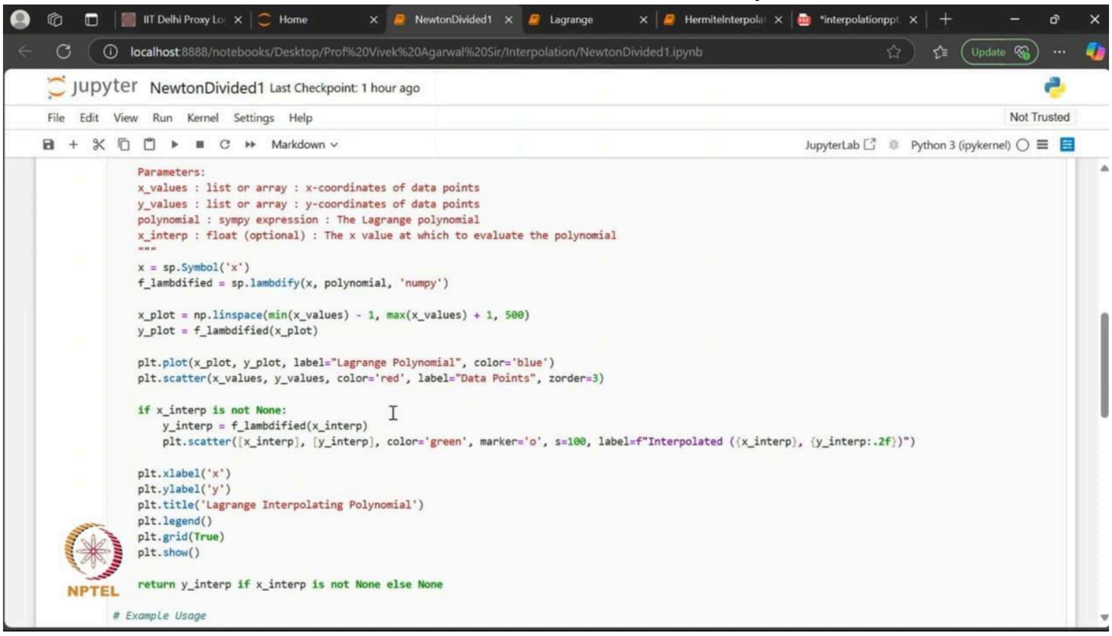
So now I see that code, it is okay, so in this we made this formula of Newton divided difference, this is our code, Python code, okay, so for this we calculated it, we need the same x value and y value, okay and if we see, the calculation is happening in the same way, just Newton divided, so if we use it and calculate, then we will see that whatever values we have, we can write it in a different way or we can do it with the help of Lagrange's , so there will not be much change in Newton divided, we will just rearrange that polynomial, okay, there will not be much calculation in it, now this work is done with us, so we can calculate it in the same way.



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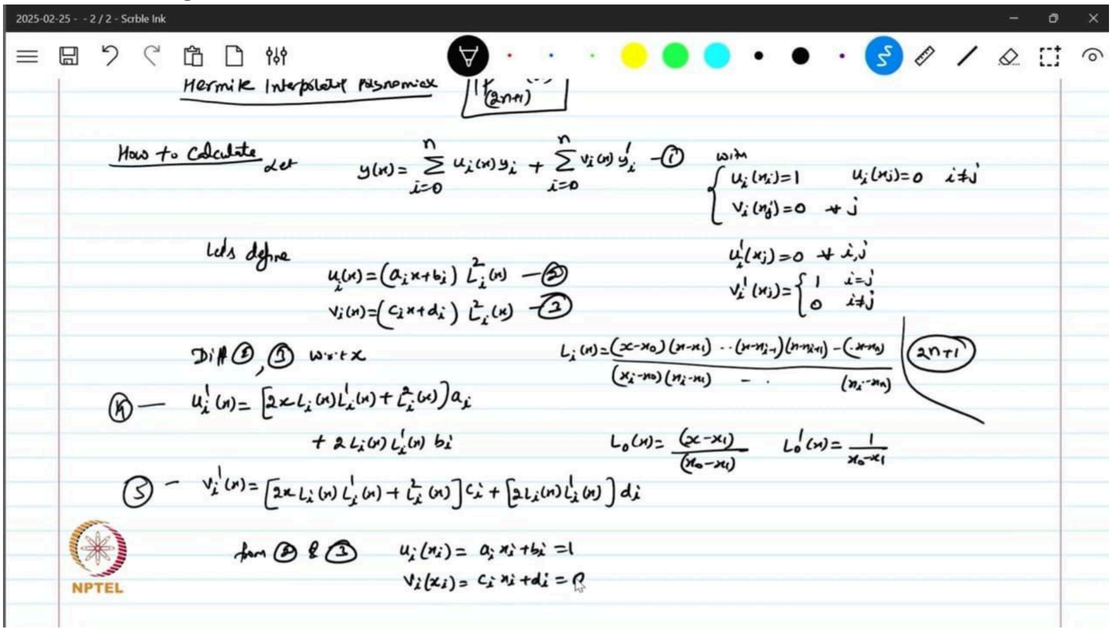
Now we have done a little bit Let us see, now suppose the data that we have is given in a different way, okay, this is done now we have it, now what happens is that suppose the data that we have is given like this x0 x1 x2 x n, okay its value is given y0, y1, y2, y n, but now what do we have, the value of their derivatives is given, okay, so the data points that we have are given like this x0, from there y0 is also given and y0 dash is also given, the second tuple that will come is x1 y1 y1 dash x n y n y n dash ad so what do we have, if we see, earlier we had n plus one points, okay, we had n plus 1 points, now we have 2 n plus 1 points so there are 2n plus 1 points, okay, so if our number of conditions has changed, now we have condition points, so that many, but condition, let us call it condition, instead of calling it point we will say what is it conditions. So, if we have a 2 n plus 2 conditions, okay, so what does it mean , it means now the poynomial , in polynomial suppose we have x0, x1 two points y0, y1, y0 dash, y1 dash so here if we see we have 4 conditions and what does 4 condition mean? which poynomial we can approximate is third degree. The polynomial of third degree has 4 coeffiecients with the help of this one, now i will define a third define polynomial a0 plus a1 x plus a2 x square plus a3 x cube. we have only two points but now we can do interpolation of third degree. I can find the interpolating polynomial of third degree with the help of this. Now we have, now if we want to do this work then we have, mean, we are going to do , its name is Hermite interpolation polynomial. Okay, Hermite interpolating polynomial and we have to calculate it, so we have come to know that its degree will increase, okay, so now in this case, the degree of the polynomial will come to 2n plus 1, so with higher degree, we can now approximate the higher degrees with same number of points, so this is the benefit of this, but what to do we should also have the values of derivatives in it, okay, so if we have the values of derivatives, then whatever data we have, we will use it to find out the polynomial, so what will we do to calculate it, so what do I do, suppose the question is how to calculate, see what is coming in the calculation, that the derivatives are also coming, so what will I do, let me have a function y x, I will write it as sum i from one to n u i x into y i, okay, plus summation i from one to n from zero to zero to n v i x and here I write y i dash like this, if we do this condition, then this condition is given now see what condition have we put with the condition, now what is the condition, u i x i should be 1 u i at x j is zero like we already had it in the Lagrange's, okay, v i x i or x j it will always be zero for all j, okay, so now we have a condition, I put a second condition, one this has come, now I am putting the second condition, now if I differentiate it, if I give the number of the polynomial to both, then if I differentiate one, then we will get u dash and v dash will come, so what second condition have I put, I put a second condition, u dash i at x j will always be zero for all i j no matter and v i dash x j is 1 if i is equal to j and 0 if i is not equal to j, okay, so I put this condition, so if we see, the condition which we had put in the Lagrange's, okay, in the same way we are doing something, so this is what we are doing we will apply this condition so let's see how we can define it that what should be the u x what should be so now what do we do let's define so which one do we have to define u x and what have we done we have defined u x and v x yes now see u x every one all the u x how do I define a i x plus b i and what do I do here we have squared our Lagrange's because in this case Lagrange's will be of n degree if we square it then it becomes 2n degree brother if we have a Lagrange its degree is n then we square it so it becomes 2n and there is a polynomial polynomial if we multiply it then its degree will become 2n plus 1 so that's why we did it like this how will we write v i x it we have written it as c i x plus d i and this is the same we have written it okay L i x we know what is it L i x will be if you see what will be L i x x minus x0 x minus x1 x minus x i minus 1 x minus x i plus 1 and in the last x minus x n will become this, right? And divided by x i minus x0 x i minus x1 and x i minus x n. So if someone asks us that, suppose I write that we have only two values. So let me take only one. So if I see L0x, what was L0x was x minus x1 by x0 minus x1. Now if I see L0 dash, see its derivative, what will happen? x0 minus x1, a value of comes. It means that the one which was before was of first order, of first degree, if we take its derivative, then one degree will be reduced. Definitely, the constant value will come. So in the same way, if we have any Lagrange's interpolating polynomial, that fundamental polynomial we will take its derivative, so we will get a polynomial of less degree. So, this thing which we know, then I will give it one, I will give it two, I will give it three. What will we do, we will differentiate them. So, differentiate 2 and 3 with respect to x. So we have u i dash x. See what will happen now if we differentiate, see the x which is coming is coming here as well and it is coming here as well this will come 2x L i x L i dash x okay plus L i square x we have taken this because if I took it with x so if we take it with x then first we left this with x I took its derivative so it became 2 L L i dash and after that if we took x then it became 1 so this is plus now b will remain the same so we will have this 2 L i dash x b i okay like this we will have v i dash same if we do it like this then what will we get 2x L i x L i dash x plus L i square x c i plus 2 L i x L i x dash d i so this is what we have four and 5 this has come okay so now let's see how to calculate this let's use the condition next because we have this thing so what will we do we will use the condition for this now look at the condition that we have, we will see from here now from 2 and 3 to u i at x i what is that It will be a i x i plus b i and this will become one for us and we have defined its condition, here u i at x i is 1 so it means that it should become one, right? And what is the second condition, u i x i is always zero, okay, so what is v i at x i c i x i plus d i always is zero, from here the condition has come, we give this six number.



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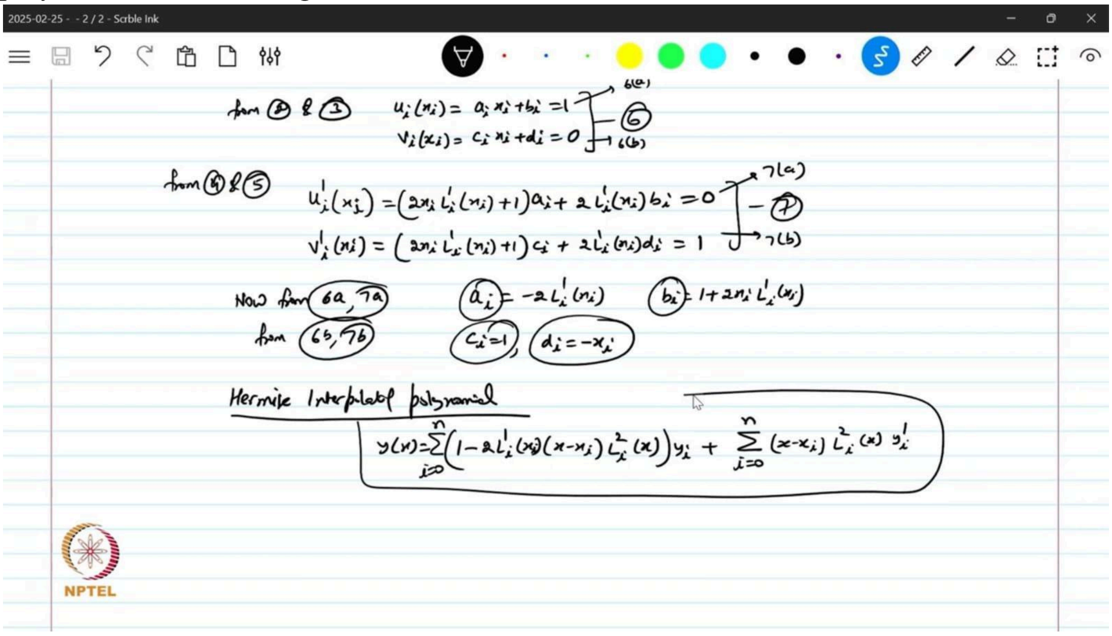
Now what will we do, we will use this condition, if we use these conditions at derivative, then we can say from four and 5 4 and 5 use it. So if we use this, then you see u i dash at x j any x i xi for i takes zero. So it means the quantity that will be left with us. And if you see 2 x i l i now l i which is one, then we will be left with l i dash x i plus 1 because this will become one because this is 1, this is 1 plus 2 L i dash x i b i. This will also be zero. This will always be zero for i j, it is always zero, so for l I can also define it like this, I can also write it as i, okay, so this always because i is not equal to j, for that it is zero. We just have to see for i, so we saw that zero will come to us like this. In the same way v dash i at x i see now v dash i x i is one so the condition that we will get is 2 x i l i dash x i plus 1 c i plus 2 l i dash x i d i and that is 1 we have two conditions. Now you will see that we have a i's b i's c i's d i's a i b i c i d i. So, if we see, we have a, b, c, d four variables which are constant values, we have to calculate them, find them out or find out parameters and we have only four conditions okay so from here if we do it so now first we take now from 6a and 7a from both means this one is 6a, this is 6b, this is 7a and this is 7b so from 6a and 7a we will calculate a i and b i, so after calculating this a little bit, we will get to know that the a that has come is minus 2 L i star one and the b i that has come is 1 plus 2 x i L i star x i from similarly from 6 b and 7 b so the one that came from that is our c i is 1 and the d i that came is minus x i ok so we will substitute all these values that we have all the parameters that were unknown a i b i c i d i we have calculated all of them so now we have got the Hermite interpolating polynomial and what does it become we have y x equal to now u i which we had calculated so what will it be we have 1 minus 2 l i dash x i which is our value x minus x i l i square x okay we will calculate this y i and we will substitute it from there we have to put here which we had defined above this is basically to write it so we substituted the a i and b i that we took out and wrote it here plus summation summation so i from zero to n i from zero to n and this is how we have x minus x i l i square x into y ok because c i's was one and d i minus x so if we see then this polynomial that we have got so we will call it hermite interpolating polynomial ok so its degree will increase a lot.



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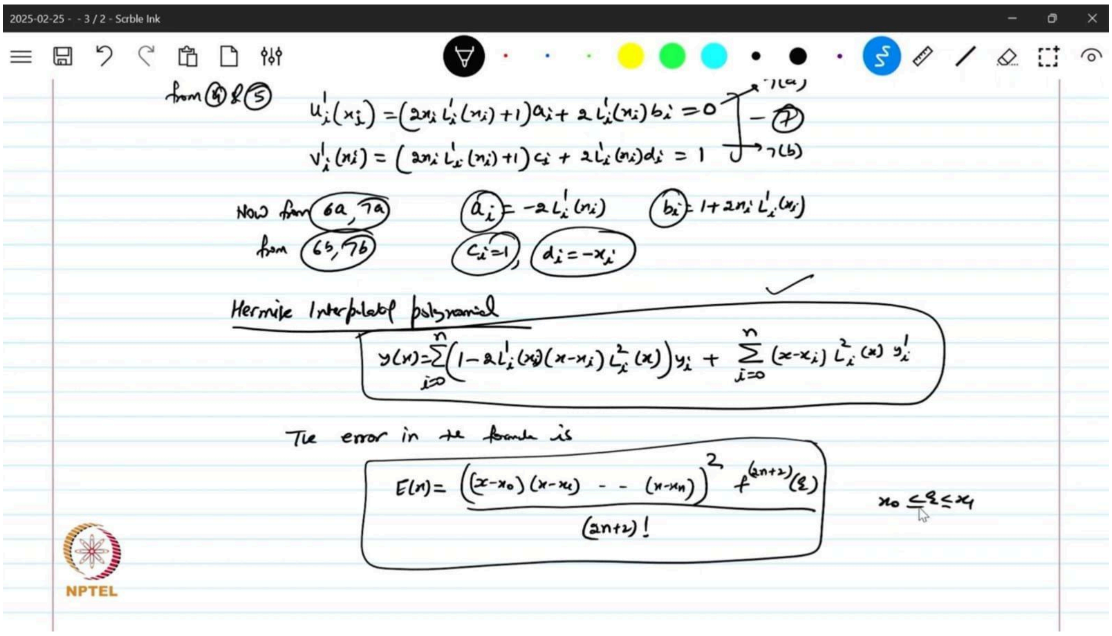
So now how much will be the error in this the error will also change so how much will be the error in the formula that we have now look the error is one let me write E x what will be it now the value that we had was x minus x0 x minus x1 x minus x n so what will be its square because the degree will increase ok so it will become of 2n plus 1 degree now this has increased now the derivative that we have here will become 2 n plus 2 then xi and here we will get 2n plus two factorial xi will belong to x0 to x1 so the error that we have here is this which is given in approximation for any x and the x that will be for interpolation is x0 to x n not 1 it is n so in between this if we take out x anywhere in the function, we can find the error bound, which we have done. So we can calculate it.



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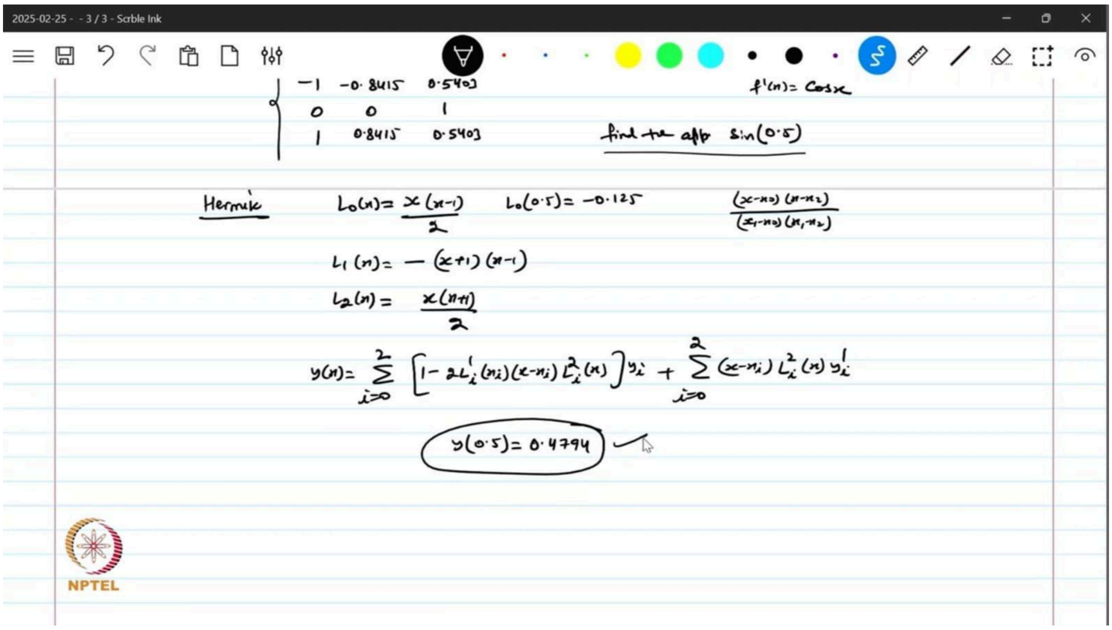
Now see what will be the benefit of this, that the degree of the Polynomial that we have has increased. So we can take an example of this. Now in the example, suppose we have the value of x given. The value of x is given. So the value of x is given minus 1, 0, 1. And it is given minus 0.8415, 0, 0.8415. Suppose we have the value given like this. We have a function, we have a function sine x it is given. So what does x mean? sine minus 1 is minus sin 1, meaning radian, okay, it will be is zero over zero. Now the y dash that we have will be 0.5403. So it means that it is y dash, it has become one because the derivative of sine is f dash x, basically cosine x is given on y and 0.5403. So we have this data given. And if someone tells us that find the approximate value of sine 0.5, tell me, okay, if we have to find out this value 0.5, then what will you do? As soon as we come to know that the derivative is given, then we will apply Hermite to it. So, to find the Hermite, we already know the formula. So, first of all, we will find l0x because we have to square it, what will be its direct result? x x minus 1 divided by this minus this minus this 2, okay, this minus this will become 2. l zero at 0.5, the value minus minus 0.125. Let's do this later. We will do it. If we leave the first one in l1, then it will come x plus 1 x minus 1, okay, right? And this will come 1 because here we will do 0 minus 1 minus 1 and 0 minus 1 minus 1 l1 comes, because in this, what was l1, our x minus x0 x minus x2 x1 minus x0 x1 minus x2 now what is x minus x0 minus 1 so plus becomes x plus 1. What is x2 and if it is 1, then this becomes x minus 1. Now 0 minus minus 1 so it becomes plus 1. And this x1 minus x2 will also have a minus sign, so this will become minus, right? If x1 is 0 minus 1 then ,minus 1 will also have a minus sign, so we can basically write this as l1 x minus x plus 1 x minus 1, right? In the same way we will get L2 x, that will become x x plus 1 by 2. So now we will take their derivatives, right? We will calculate all the derivatives and define their values squared. So ultimately our Hermite polynomial will become, in this case it will become summation i from 0 to 2, okay. It will go 1 minus 2 l i dash at x i x minus x i l i square x. This value is y i because it is a square, so if we see, it is a second degree polynomial, we will square it, so 4 degree plus x will be multiplied by 5 degrees, so we will get a polynomial of 5 degrees, summation i from 0 to 2 x minus x i l i square x y i dash. This came, okay. So we will substitute all the values in it y 0.5 which we have, if we calculate it, we will get 0.4794 this value.



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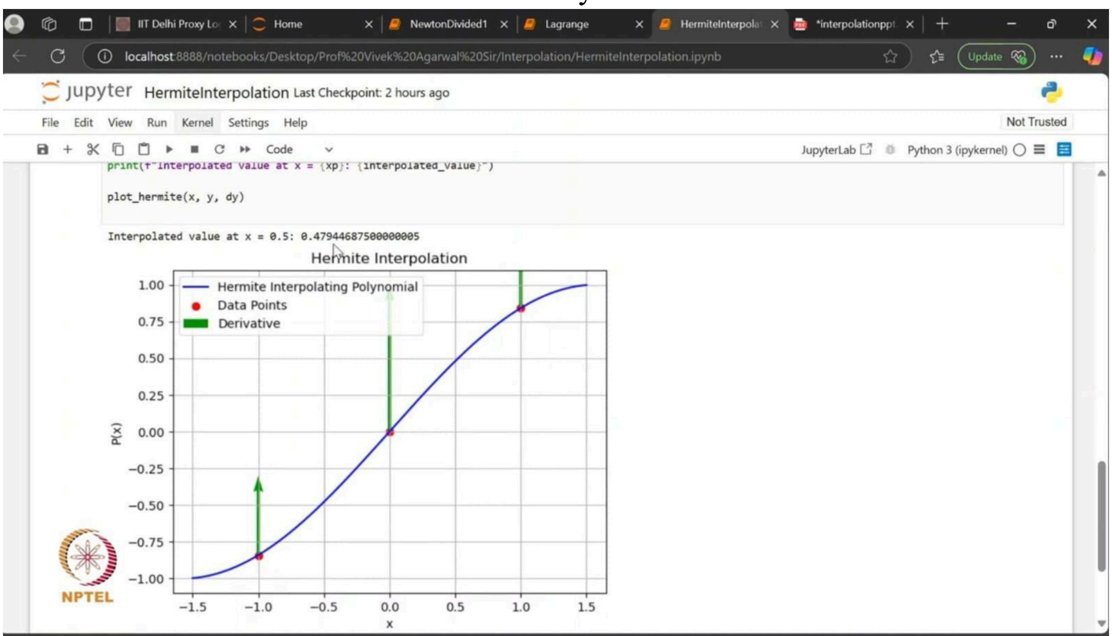
So we will do this with the help of Hermite so if we see its code, the code of this Hermite interpolation if we calculate it and we will get to know that we are calculating it, then what is its value coming out so the code of Hermite is a little bit because the derivatives in it are also involved, so the input of the function that we have to give is x0, we have to give y and also give its derivative, so we have written it with the help of divided difference, we are calculating it, okay, basically everything is happening by Lagrange's, so if we calculate it, then we saw that now this value was minus 1 ,0,1 but we were given the value, okay, so we can do it in different ways, now what do I do, let me calculate it, so it came to us, so this interpolating polynomial which was Hermite We got it. We have data points and these are its derivatives. So these data points are given to us and we saw that at x = .5, 0.4794 came and we saw the same on that as well. If we calculate it, 0.97 4794 came. The same result is there on this as well. So this is our result in this way.



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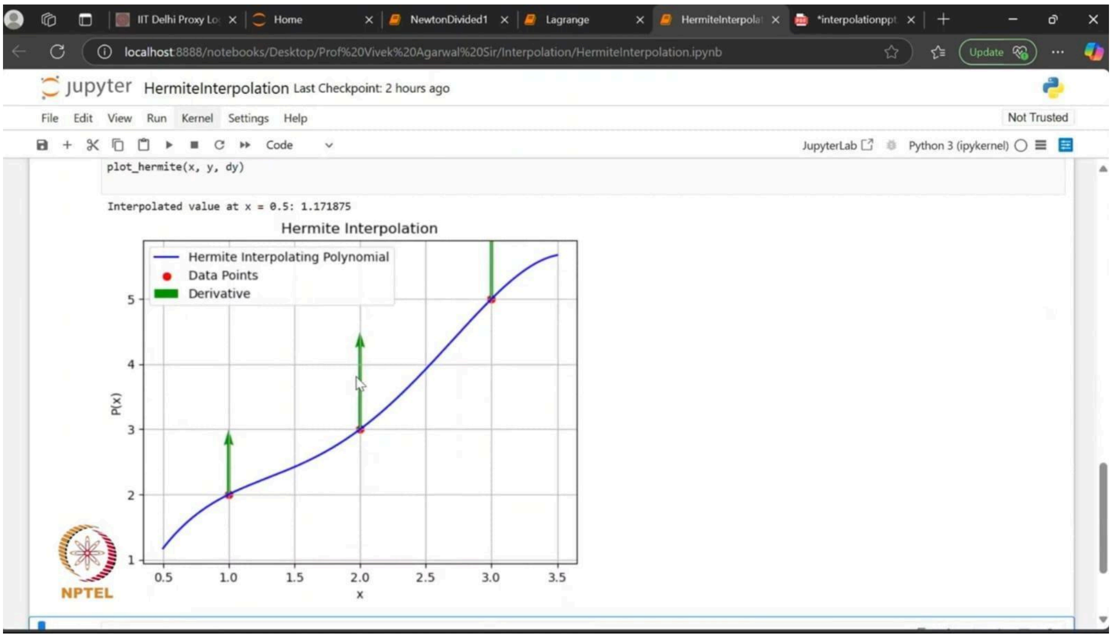
Now if I change this data, suppose we have the data that instead of giving it as cos x, what do I do? I reverse it. So what did I do? I reversed the value. And now let's see what comes out on this. So we want to find out the value of at 0.4794. So I just reversed it and took cos x instead of sin x, so its derivative will come out as minus sine x. So look, I calculated it and this is what we got. These are the values that we have, this derivative, we can see on the derivative that the derivative is coming on it, minus is coming, meaning negative direction. It is telling that this is a positive direction and this data point is okay, so we can calculate it like this and the values that are there on it will be found, the value that is coming on it is 0.7 coming, we took out the value at .5, let me see, then from 0.7 is coming ok, so the value that is being calculated in this, the function that we took was cosine x, its derivative was sine x minus sine x, so from there we got the values, so if we do that a little bit and the data came, we changed it, so this is the data of type, so we can do it for different values, okay, so we can visualize it with the help of this code, Python code, that we have got this, okay, so we have this polynomial, interpolating polynomial and this is our data point and x .5 we have 1.17 values, so in this case we have three values, so the polynomial that will come will be a polynomial of 5 degree, so with the help of this we can calculate it, okay So this is Hermite interpolation, and with its help we have calculated Hermite.



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So today in this lecture we saw that we redefined the interpolating polynomial in terms of Newton's divided formula differences and after that we saw that if we have the data points and their derivatives are also given, then how can we increase the degree of the interpolating polynomial and that is what we did as a Hermite interpolation. You must have understood this and thank you for watching this.