Introduction to Probability & Statistics
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Week - 4
Lecture - 14
Expectation of a Discrete Random Variable

Ham apne CDF sanchayi manphalan ki baat ko aage badhate hue note karte hain ki X ek discrete random variable hai jiska PMF small p(x) aur CDF capital F(x) hai. Agar A aur B koi bhi do real numbers hain jisme A < B, to hum do ghatnaon ko dekhte hain: pehli ghatna X ≤ A, aur doosri ghatna A < X ≤ B. Yeh dono ghatnaayein mutually disjoint hain, kyunki X ek hi time par A se chhota ya barabar bhi nahi ho sakta aur A se strictly bada bhi nahi ho sakta. Inka union hota hai ghatna X ≤ B. Finite additivity ke hisaab se hume milta hai: P(X ≤ A) + P(A < X ≤ B) = P(X ≤ B). CDF ki paribhasha se P(X ≤ A) = F(A) aur P(X ≤ B) = F(B), to isse nikalta hai: P(A < X ≤ B) = F(B) – F(A). Ab doosri ghatna ko dekhein jisme hum A < X ≤ B mein ek aur disjoint event ko add kar dete hain: event X = A. Ab in dono ka union ban jaata hai A ≤ X ≤ B. Phir se finite additivity lagate hue: P(A < X ≤ B) + P(X = A) = P(A ≤ X ≤ B).

$$Let\ a < b$$

$$\{x \leq a\} \cup \{a \leq x \leq b\} = \{x \leq b\}$$

$$P(x \leq a) + P(a \leq x \leq b) = P(x \leq b)$$

$$P(a \leq x \leq b) = F(a) - F(b)$$

Yahan pehle term ko hum F(B) – F(A) likh chuke hain, aur P(X=A) hamara PMF ka maan p(A) hota hai. To hume milta hai: P(A ≤ X ≤ B) = F(B) – F(A) + p(A). Yeh formula tab hi simplify hoke sirf F(B) – F(A) ban jaata hai jab p(A) = 0 ho, yani jab A random variable X ki koi possible value na ho. Agar A X ki possible value hai tab hi p(A) > 0 hoga; agar A X ki value nahi hai to p(A) = 0. Isliye P(A ≤ X ≤ B) = F(B) – F(A) tabhi sahi hai jab A X ki value nahi ho. Kaafi kitaabon me isko ek general form me likhte hain: P(A ≤ X ≤ B) = F(B) – F(A–) jahan F(A–) ka matlab hota hai A ke just left waali limit (ya discrete case me F(A–1)). Is tarah CDF ko use karke hum kisi bhi ghatna ki probability aasani se nikal sakte hain. Humne jo teen CDF wale formula dekhe the unka istemaal aur discrete random variable ke CDF ke step-function waale graph ka idea ek example ke zariye dekhte hain. Is example mein hume PMF nahi diya gaya, balki seedha CDF diya hua hai: F(x) = 0 agar x < 1; 1/4 agar 1 ≤ x < 2; 2/4 agar 2 ≤ x < 3; 3/4 agar 3 ≤ x < 4; aur 1 agar x ≥ 4. Jaise hi x ek interval se doosre interval me jaata hai, CDF ki value jump karti hai, jiss se pata chalta hai ki random variable X kin points par actual values leta hai. Agar hume probability 1 < X ≤ 3 chahiye to hum seedhe formula P(1 < X ≤ 3) = F(3) – F(1) laga sakte hain. F(3) 3/4 hai aur F(1) 1/4, to answer hai 1/2. Isi tarah probability 2.1 < X ≤ 3.9 ke liye hum F(3.9) – F(2.1) likhenge; 3.9 jis interval me hai usme F=3/4 aur 2.1 ke liye F=2/4, to answer 1/4 aata hai. Agar hum iska graph banayein to har jump X = 1, 2, 3, 4 par hota hai, kyunki F(x) inhi points par badalta hai. In jumps ka height P(X = a) ke barabar hota hai jaise X=1 par jump 0 se 1/4 tak hai to P(1) = 1/4. Isi tarah P(2) = (2/4 – 1/4) = 1/4, P(3) = 1/4, aur P(4) = 1/4. Ye chaaron probabilities ka sum 1 hai, to yehi chaar X ki possible values hain aur sabki probability 1/4 hai. Is example ka main point ye hai ki agar CDF diya ho to PMF aasani se nikal sakte hain—bas CDF ki value jahan-jahan change hoti hai un points par jump ka size nikaal kar. Ab aage badhte hue ek aur example lete hain. Maal lijiye humne random taur par 100 log chun kar unse pucha ki garmi ki chhuttiyon me unhone theatre me kitni movies dekhi. To random variable X un movies ki sankhya ko darshata hai. Kuch logon ne zero movies dekhi, kuch ne ek, kuch ne do ya teen. To 100 logon ke responses se X ka PMF is tarah diya jaata hai Is example mein humne dekha ki 0.2, 0.2, 0.35 aur 0.25 ka matlab hai ki 100 logon me se 20 log aise the jinhone ek bhi movie nahi dekhi, 20 logon ne ek movie dekhi, 35 logon ne do movies dekhi aur 25 logon ne teen movies dekhi. Yaani poore 100 log cover ho gaye. Ab agar hum se poochha jaye ki theatre me jaakar logon ne average kitni movies dekhi, to sirf 0, 1, 2, 3 ka ordinary average lena galat hoga, kyunki har value ki probability alag hai. Sahi tareeka ye hoga ki hum tickets (ya movies watched) ka total nikaale: 20 log × 0 movies = 0, 20 log × 1 movie = 20, 35 log × 2 movies = 70, aur 25 log × 3 movies = 75. In sabka total 165 hota hai, aur agar ise 100 se divide karein, to average 1.65 milega. Isi ko hum mathematically likh sakte hain: 0×0.2 + 1×0.2 + 2×0.35 + 3×0.25 = 1.9. Ye hi expected value ya mean hai jise hum E[X] ya μ kehte hain. General paribhasha ye hai ki agar X ek discrete random variable hai jiska PMF P(x) hai, to uska mean ya expected value hota hai: E[X] = Σ x·P(x),

$$E(X) = \mu_X = \sum_x xP(x)$$

jahaan summation un sabhi x ke liye hota hai jinke liye P(x) > 0. Phir humne dekha ki Bernoulli random variable jo sirf 0 aur 1 values leta hai, uska expectation hamesha p hota hai, kyunki E[X] = 0·(1–p) + 1·p = p. Isi tarah agar coin-toss game me heads par +1 aur tails par –1 milta hai, to expected gain hota hai 2p – 1, jo fair coin (p = 0.5) ke liye 0 ban jaata hai aur biased coin ke liye positive ya negative ho sakta hai. Geometric random variable jisme X = number of tosses till first heads, aur PMF hai P(X=x) = p(1–p)^(x–1), uska expectation 1/p aata hai — iske derivation me hum geometric series aur differentiation ka use karte hain. Aakhir me ek cautionary example tha jahan P(X = x) ∝ $1/x^2$ hota hai. Yeh PMF to exist karta hai, kyunki $\Sigma(1/x^2)$ convergent hai, lekin expected value $E[X] = \Sigma\, x\cdot(k/x^2) = \Sigma\,(k/x)$ diverge karti hai kyunki harmonic series 1 + 1/2 + 1/3 + … infinite hoti hai. Isliye is random variable ka mean exist nahi karta. Yeh dhyaan rakhne layak baat hai ki kuch discrete random variables ke liye expected value infinite ho sakti hai ya bilkul exist hi nahi karti.

$$E(X) = \mu_X = \sum_x xP(x) = \sum_{x=1} x * p(1-p)^{x-1}$$

$$= p\sum_{x=1}^{\propto} x(1-p)^{x-1} = P\sum_{x=1}^{p}(\frac{-d(1-p)^x}{dp})$$

$$= (-p)\frac{d(\sum_{x=1}^{\propto}(1-p)^x)}{dp}$$

$$= (-p)\frac{d}{dp}\left(\frac{1-p}{1-(1-p)}\right)$$

$$= (-p)\left(-\frac{1}{p^2}\right)$$

$$E(X) = \frac{1}{P}$$