Linear Algebra
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Lecture – 44
Results on Orthogonality

Alright. So, in the previous class, we realized that we need to find out given linearly independent set how do I make it into an orthonormal set or orthogonal set, or how do I make them perpendicular to each other, alright. So, let me give you a definition of orthonormal set, orthogonal set I just studied them.

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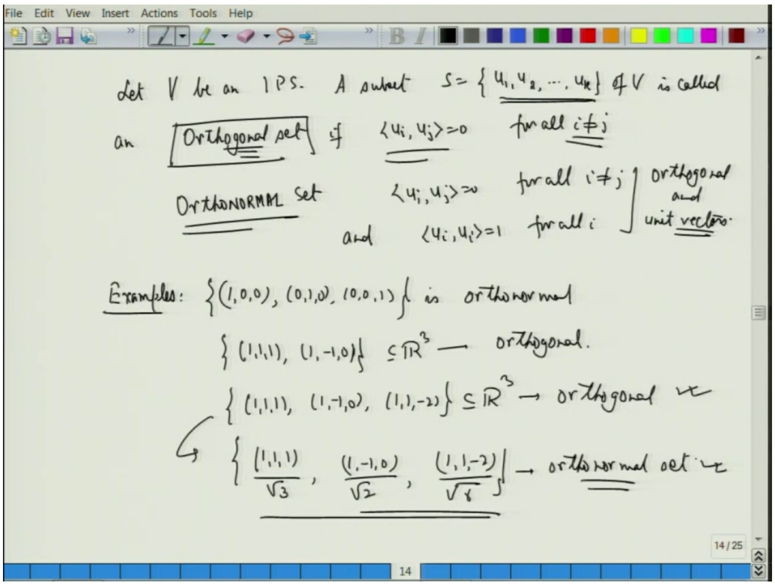

So, let V be an inner product space, fine or collection of sets a collection or a subset S which is equal to u 1, u 2, u k of V is called an orthogonal set, if inner product of u i and u j is 0 for all i not equal to 0, alright.

So, I am talking of orthogonal, set orthogonal means they should be perpendicular to each other, they should be orthogonal to each other. Then there is a word what is called orthonormal set. So, when I talk of orthonormal set then it is orthogonal. So, I need u i, u j to be 0 for all i not equal to j and the length of each vector should also be 1 for all i. So, here it is orthogonal and unit vectors.

So, I have got all the unit vectors. And they are perpendicular to each other, they are orthogonal to each other, is that ok. The two things that are important, one is just looking at orthogonality that they are perpendicular to each other and there is that you are looking at only unit vectors, fine.

So, you have to be careful about them. So, example, so when look at these vectors 1 0 0, 0 1 0, 0 0 1, this is orthonormal. Even if I look at this vector 1 1 1, 1 minus 1 0, so this is a subset of R 3. This is orthogonal, fine. Even these vectors 1 1 1, 1 minus 1 0, 1 1 minus 2, subsets of R 3 this is also orthogonal, fine.

Now, if I want to look these from the point of view of orthonormality, I have divided by the unit vector, so make it unit vector, so 1 1 1 divided by root 3; 1 minus 1 0 divided by root 2; 1 1 minus 2 divided by root 6. This will give me an orthonormal set, alright. So, I have given you enough examples of orthogonality and orthonormality here, fine.

Even though this looks cube or sum to you because of root 3, root 2, root 6, this is much more desired as compared to orthogonality, alright as far as we are concerned; because everything is in terms of unit vectors, so calculations and ideas are more fruitful as per as orthonormality is concerned, fine.

So, let me just prove the first statement. So, we have proved that we have given the statement that a collection of vectors u 1 to u k, they are called orthogonal if something happens, is that ok. That is more important. That, orthogonality means there is perpendicularity between them. Now, the question is that, given an orthogonal set is that set linearly independent?

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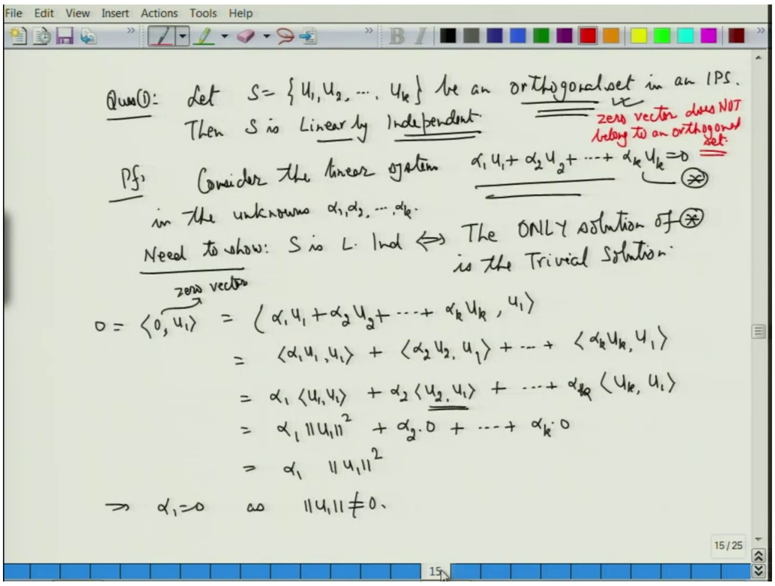

So, question 1, let S is equal to u 1, u 2, u k be an orthogonal set in an inner product space, fine, some inner product spaces. Then, S is linearly independent. So, let us try to prove this part proof or the solution of this, fine. Now, if I want to prove something is linearly independent or dependent, I need to look at the system of equation. So, consider the system, consider the linear system, system alpha 1 u 1 plus alpha 2 u 2 plus alpha k u k is equal to 0 in the unknowns alpha 1, alpha 2, alpha k, fine.

Need to show S is linearly independent which is same thing as saying that that the only solution of star is the trivial solution, alright. I do not have any other solution except the trivial solution, alright. So, let us try to compute it. So, let us look at inner product of the 0 vector. So, this is 0 vector with any say with u 1, suppose I want to look at inner product of these two, alright, fine.

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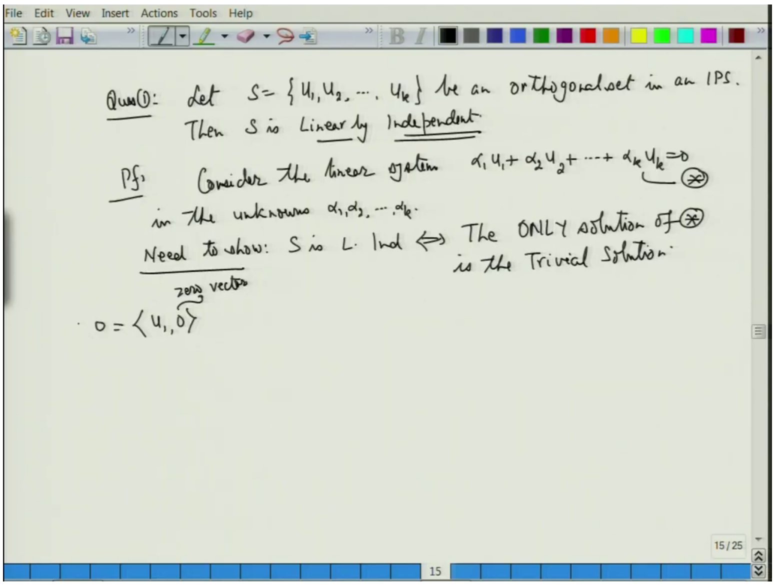

Or let me write the other way around I think, otherwise you have; so, I want to look at u 1, and the 0 vector; I want to look at this part. So, this is my 0 vector, fine. So, I know that inner product of any vector was 0 is 0. So, I know that this is 0, but this is also equal to I think I written it correctly; I written it earlier itself that was correct, so 0 comma u 1. So, this is my 0 vector, fine.

So, this is same as I am looking at the system which is same as alpha 1 u 1 plus alpha 2 u 2 plus alpha k u k. So, 0 is this vector. And I am looking this with inner product with u 1. So, by definition of inner product this is same as alpha 1 u 1, u 1 plus alpha 2 u 2, u 2 plus alpha k u k, u k; u 1 here, u 1 here which is same as alpha times u 1, u 1 plus alpha 2 times u 2, u 1 plus alpha k times u k u 1. This is same as alpha times u 1 square plus alpha 2 times my set was orthogonal.

Now, since the our set was orthogonal, it means that inner product of u 2 with u 1 is 0 and so on till alpha k times 0. Oh, this is alpha 1 here, alpha 1 here, as usual; same as alpha 1 times length of u 1, square, fine. So, this implies alpha 1 is 0 as norm of u 1 is not the 0 vector, alright, fine.

So, you have to be important, you have to understand this here. When I am saying that something is orthogonal set I did not say in the beginning I think I forgot to state here that 0 vector is not allowed here. So, 0 vector is not allowed in the orthogonal set; vector does not belong or whenever you want to say that a vector is orthogonal to somebody else, 0 you do not want to include because 0 is orthogonal to every vector, alright.

So, therefore, you do not consider 0 vector as such, is that ok. So, now we have shown that there was the question, but where is the definition. Definition of orthogonal I gave before, is it no. I gave it here itself, fine. So, what we have shown here is that if I have an orthogonal set then that set is linearly independent, fine. That is one thing.

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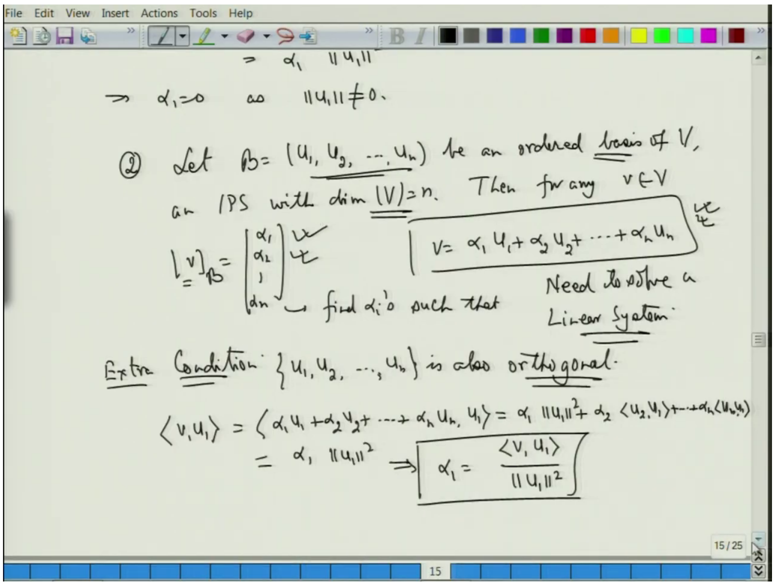

Second thing which is again very important is that, let B is equal to u 1, u 2, u n be an ordered basis of any vector space, basis V; an inner product space with dimension of V is equal to n, alright. See the dimension of V is n this is indeed a basis, therefore, I wrote it as ordered basis.

So, there is this then what we know is then, if I want to compute then if I want to write any V, then for any V belonging to V, if I want to compute coordinates of V alpha 1, alpha 2, alpha n, if I want to compute find alpha i's such that V is equal to alpha 1 u 1 plus alpha 2 u 2 plus alpha n u n.

If I want to find this alpha i's, so that I can write coordinates of v with respect to basis B as, alpha 1 to alpha n, then I need to solve a linear system, alright. And so, I will have to put the value of what the vector u 1 is, what the vector u 2, what the vector u n is and form a system of equation solve it, it takes time, but I can do it, alright.

Because u 1 to u n is linearly independent, so there will be a unique solution and therefore, I can write it. But to get this alpha i's it takes time, I have to do lot of extra work, fine. But suppose I assume extra condition, so extra condition as, extra condition u 1, u 2, u n, this set is also orthonormal or orthogonal it is; suppose I want to say it is orthogonal, alright.

So, say it is a orthogonal then things are much simpler. Why? Because let us look at if I want to say it is orthogonal then I want to compute what is alpha 1 for example, alright. So, what I will do? I have been given this V, I know that V looks like this and look like this means I have to find alpha i's, fine.

So, let us look at inner product of V with u 1, inner product of v with u 1 will be alpha 1 u 1 plus alpha 2 u 2 plus alpha n u n comma u 1. So, if you go look at the previous one it is same as alpha 1 times norm of u 1 square plus alpha 2 times inner product of u 2 with u 1 plus so on plus alpha n times u n and u 1, which is same as alpha 1 times norm of u 1 a square, alright.

So, what it tells me is that this will imply that alpha 1 is nothing but inner product of V with u 1, divided by length of u 1 square that is all. So, in place of solving a linear system and getting the coordinates I can get directly as such. This is more important for me that I do not have do any calculation, I just have to find out the inner product to get my coefficients here, is there ok. That is one thing.

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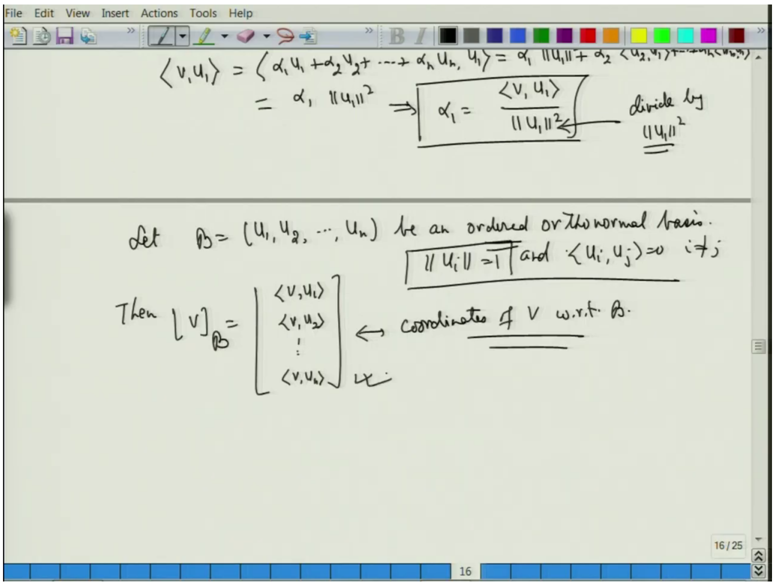

Something more, so let this u 1, u 2, u n be an ordered orthonormal basis. So, ordered orthonormal basis means ordering is there and each of these vectors are u i, length of u i is 1 and inner product of u i with u j is 0, i not equal to j, alright. So, all these things are given to me. Ordering is given to me, orthonormality means length 1 and perpendicular, alright; this is given to me.

Then, if I want to compute, then if I want to write v with respect to B then this is nothing, but just look at v with u 1, inner product of v with u 1, inner product of v with u 2, v with u n. So, just compute the inner product and you get the coordinates directly. So, coordinates; coordinates of v with respect to B, alright.

So, there is nothing, you do not have to solve any system, do not have to do any work. The length is already 1, so you do not have to divide anything, you just have to look at the inner product to get your answers. In the previous one, what we had was where to divide here divide by norm of u 1 square, we have to divide it because we did not say the length was 1.

Now, here we have assuming that the length is 1, so it is just those inner products, is that ok. Now, what we are trying to say here is understand it very important; what we are saying is that I am not wasting my time trying to find out things. So, in some sense we are trying to look at this here. So, let me just write it.

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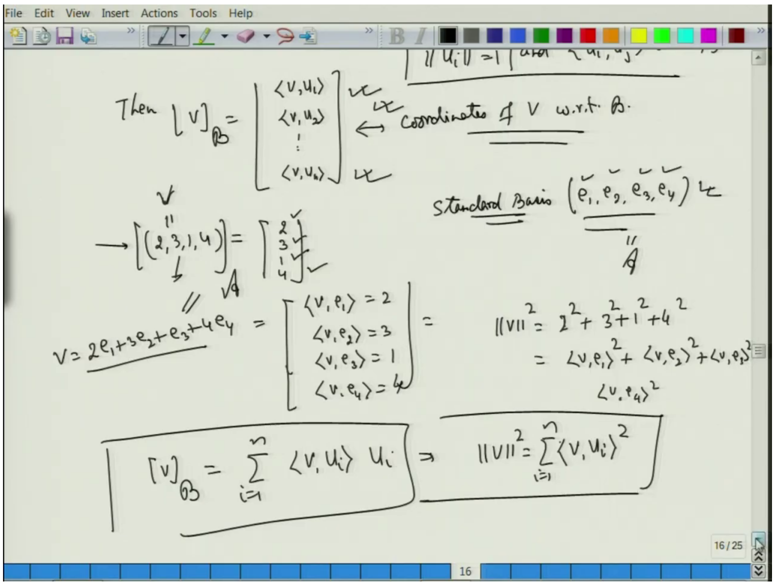

So, if I am writing the vector say 2, 3, 1, 4 as; so, 2, 3, 1, 4 is nothing but it is 2, 3, 1, 4; so, 2. So, this is with respect to the standard basis. Let me write it a standard basis. What is the standard basis? e 1, e 2, e 3, and e 4. They are standard basis.

So, if I look at there is a standard basis as my say be A here, then this with respect to A is basically B. Why it is this? Because this is nothing, but see if I write this vector as v, then v comma e 1 will give me 2, v comma e 2 will give me 3, v comma e 3 gives me v 1, v comma e 4 gives me 4, alright. This is what I am writing here, fine. So, in some sense what we are saying is that whatever we could do for a standard basis.

In a standard basis everything was nice, I just have to look at the first component, second component, third component, fourth component and they came as inner product with respect to the standard vectors e 1, e 2, e 3, and e 4, fine. What we are saying here is that forget about e 1, e 2, e 3, e 4, if you have any ordered orthonormal basis, any ordered orthonormal basis, then a similar thing can be done, fine, is that ok, that is one thing. You can just do it.

Second thing was, if you want to find the length of the vector, if I want to find the length of v, so this one is v. If I want to find the length of v square length of v square was nothing, but 2 square plus 3 square plus 1 square plus 4 square which is same as v comma e 1 square plus v comma e 2 square plus v comma e 3 square and v comma e 4 square. This is also possible here.

So, even with respect to this, so with respect to this B, I still have that it is summation, so this is same as writing as v comma u i u i, i going from 1 to n, fine. Here it was v was; so, I wrote it I somewhere no I did not write it 2 times e 1 plus 3 times e 2 plus e 3 plus 4 times e 4, alright.

Similarly, we are writing it here. I write v as this, and this implies that length of v a square is same as look at v comma u i a square it, i going from 1 to n, I get this part, is that ok. Let us look at the proof of this. Proof is very simple. So, I will just do it which is very easy, so I will do it.

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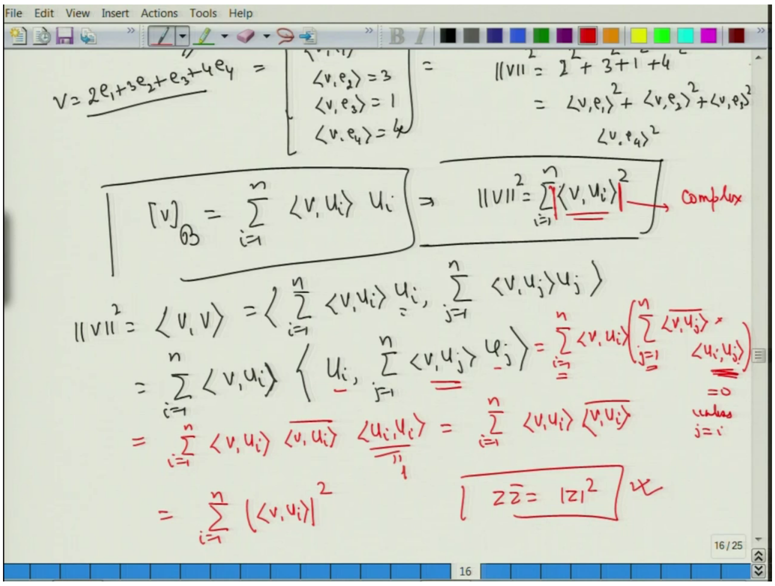

So, if I want to look at norm of v square, it is nothing but inner product of v with v itself which is same as I going from 1 to n, v comma u i, u i comma j is equal to 1 to n v comma u j, u j which is same as i going from 1 to n v comma u i, fine; small one, this big one will come for this u i comma j is equal to 1 to n v comma u j, u j, alright.

Now, there is a small thing that we want to be careful. So, let me write with the red here, that if I am looking at complex numbers not real numbers then this will not make sense. I will have to put a bar here, if I am looking at complex, fine. If I am looking at real there was no problem.

Now, see here this is what is going to happen. So, this will be equal to, let me write it here itself first line, i is equal to 1 to n, you have v comma u i. Now, if I want to take this out, this is the second coordinate. So, I will have to write it in terms of j is equal to 1 to n v comma u j bar, I will have to do that because it is the second component and then it is u i u j, is that ok, u i with u j, fine.

This into u i u j; this is what I have. Now, what we know is that this u i's are orthonormal, so they are perpendicular, so they are 0 here. So, if I am looking at i is fixed here, alright, j is a variable j is going from 1 to n, so I will get some contribution from here only when j is same as i otherwise it will be 0.

So, therefore, I can write it as summation i going from 1 to n, fine, v comma u i. As I said this is 0 equal to 0 unless j is equal to i; so, I will left out only with here v comma u i bar, fine, and I will be left out with u i u i. But this is also 1 because the length is 1, I have taken it has an orthonormal vector. So, I am left out with i is equal to 1 to n v comma u i and v comma u i bar, complex conjugate.

And therefore, this is same as i equal to 1 to n v comma u i this square, alright; because z z bar is nothing, but mod of z square, alright, for any complex number z. So, therefore, when I am looking at real numbers, I do not need to put this absolute value square, but when I want to look at complex I have to put this extra line and the square of this, alright because of this part.

So, you can see that whatever you could do for Pythagoras theorem, Pythagoras theorem was this I have a similar result here for the general setup also, fine. That is very important for us. So, you have to keep track of that, fine. Now, the next idea using this, which is again important is this part. So, let me write that part.

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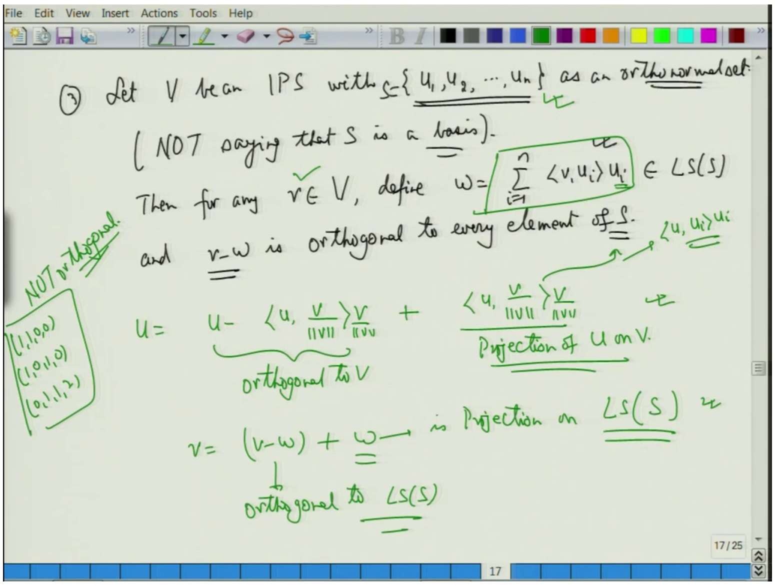

So, let V be an inner product space with u 1, u 2, u n as an orthonormal set, fine. Not saying that S is a basis. We are just saying it is an orthonormal set, we are not saying it is a basis, fine.

Then, what we are going to do is let us look at this part. So, then for any v belonging to V, this vector v is equal to, so for any v define w is equal to summation i is equal to 1 to n v comma u i u i, then this belongs to linear span of S, and v minus w is orthogonal to every element of S, right. So, this is very important this concept. So, what is the concept, understand it nicely.

I have something which is an orthonormal set, from there I am getting a w, fine and then I am looking at v minus w. We are saying that v minus w is perpendicular to every element of S and therefore, it will be perpendicular to all the linear combination and hence it is also perpendicular to w, fine. So, it is similar to recall what we had that I had a projection. So, there was this vector.

So, I had u, we wrote u as u minus u comma v divided by length of v into v. This was orthogonal to v, plus we had u comma v upon length of v into v this was the projection of u on v, alright. So, understand it nicely; we decomposed earlier u in terms of something which is perpendicular to v and which was parallel to v.

We are doing the same thing here. We are decomposing v into v minus w plus w. What is w? w is projection. So, w is projection on linear span of S, alright. And this is orthogonal to linear span of S right. So, be careful understand it very nicely. The proof is very simple, but I want you to understand this, fine. So, let us think about it what we are doing. Just the generalization of what we had.

So, recall; I wanted projections to be on each of them. So, recall that example which was I think 1 1 0 0, 1 0 1 0, and 0 1 1 2. So, there these vectors were not orthogonal. So, we could not talk of projection onto each one of them and then add it, alright. But here were starting with u 1 to u n which are perpendicular say it is perpendicular, so I am able to add those projections.

So, there I am able to projection. So, look at this. Since, u i's are orthogonal here, so if I want to write it in terms of this part here it will be u comma u i and u i itself, alright. This what it is, so each of these components here each of them is projection onto u i, fine.

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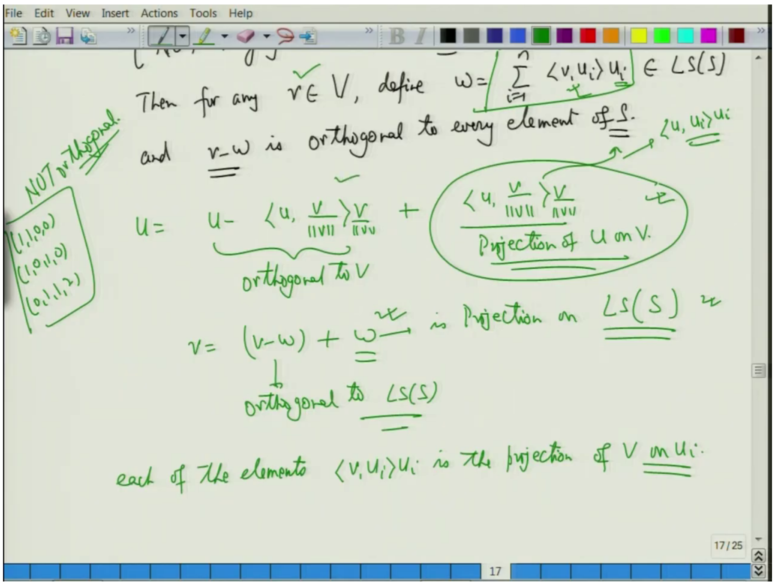

So, each of the elements v comma u i u i is the projection of v on u i, is that ok. So, I am adding all the projections and then getting in the vector w, is that ok. So, I am just writing a component of this part as this which I wrote it as w here and which is orthogonal to it, fine, alright. So, I will just end the lecture here itself.

Thank you.