Engineering Mathematics - II
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Lecture 08
Surface Integral (Part 2)

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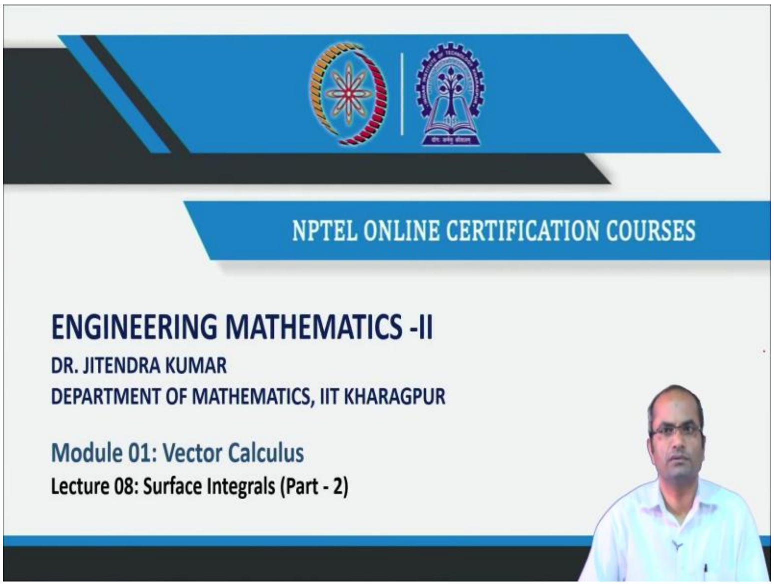

Welcome back to lectures on Engineering Mathematics 2. So this is lecture number 8 on surface integrals and we will continue, because we have already covered the surface integral for scalar functions and today we will introduce the idea of vector functions for surface integrals. So, we will cover this Orientable Surfaces first and then we will come to this Flux Integrals, which is the surface integral of vector field.

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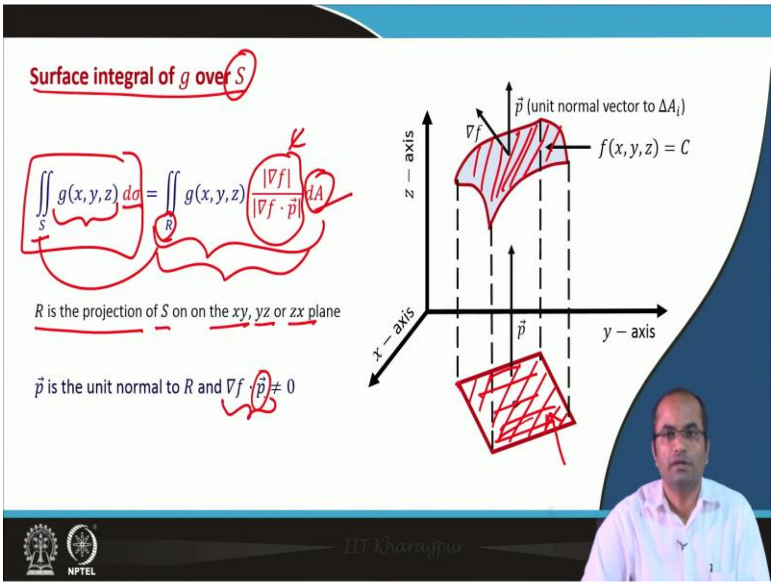

So, just to recall what we have covered in the last lecture that was the surface integral of g on over this s over the surface S. And the idea was that this function g is being integrated over the surface S. And this surface integral will be converted to this simple area integral or the double integral, where this R is the projection of this surface. So, for instance this is the surface here and this will be projected on one of the coordinate axis.

So, suppose here on this x y plane, so not the coordinate axis it will be projected on the coordinate planes. So, suppose this surface is being projected on this x y plane, then we have this R here on the x y plane. So this double integral will be evaluated on this region R in x y plane and there is an additional factor here because of the difference between this curvature of the surface and this plane area on this x y plane.

So that factor will be coming and then we have a simple double integral that is dA. So, as written here R is the projection of S on the xy, yz or zx plane as per the convenience, because we have to also look that this del F which is the perpendicular to the surface should not be perpendicular to this p, which is the unit normal vector on the plane where we have projected the surface.

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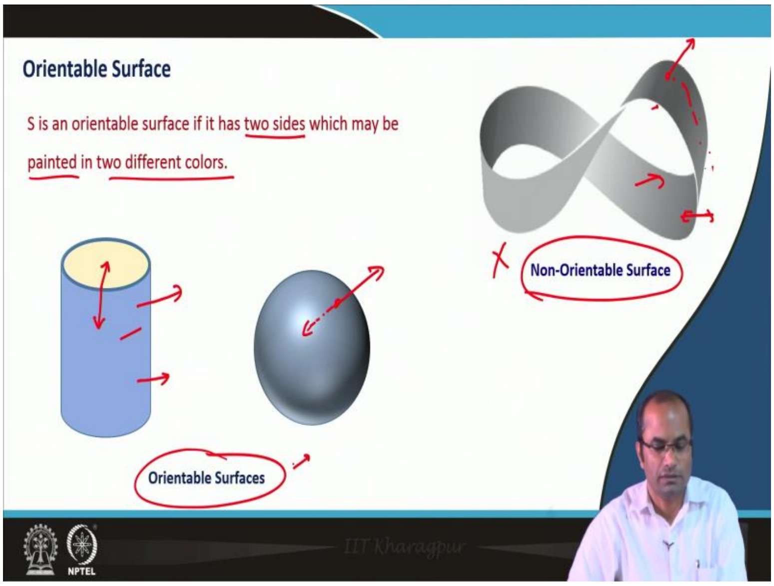

So, with this idea we can now go for more general integral more general surface integral. So, before that we have to introduce here this Orientable Surface. So it is a smooth surface which we have already discussed the only difference is that we will add here, so if the two sides may be painted in two different colours of a given surface then we will call this as a orientable surface. So, with the help of examples we will explore more. So for instance here we have these 2 orientable surfaces.

So this is the cylinder, we can paint with 2 different colours, the inside and the outside. The more formally mathematically the idea is that we can have a normal at any point and that will be in the two directions. So what one will be in the inner direction and then will be the outer normal. So, similarly here for the sphere as well we have the 2 normals at any point, one is the outward normal and the another one will be the inward normal.

So, at any point of these surfaces we have a unique normal at any point and they will have 2 directions, one will be that is the other of the direction of the previous one. But if we consider this Mobius stripe then the situation is not so what we have here for such surfaces. So, for instance here if I pick this point and then this will be kind of normal at this point, but if I go along with this and then come back then I will be on the other side of this stripe.

So, basically the idea is that this geometry here you cannot paint with 2 different colours or meaning that we do not have 2 distinguishable surfaces for such a geometry. So, we will not consider this for such surface integrals which we are going to cover today. So, always you will have 2 different, 2 distinguishable faces of surfaces of a given surface.

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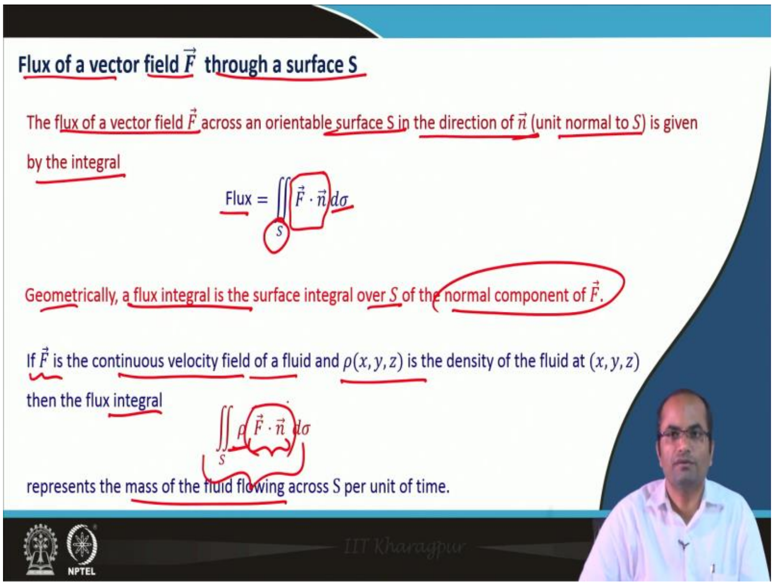

Okay, so having that we have the flux of a vector field F through a surface S which we will discuss here in this lecture. So the flux of a vector field F across the orientable surface S. So, orientation is important because we will be considering the normal in two directions. So, in the direction of this n unit normal to S and this is given by this integral here. So, we have this flux integral.

The F dot n the component of this F in the direction of the normal is being integrated over the surface S. So, this is what we call the flux integral or the flux of a vector field F through a surface S. Geometrically a flux integral is the surface integral over S of the normal component of F of the vector field F. So if this F is continuous velocity filled for example of a fluid and the rho is the density of the fluid at a point x y z. Then such a flux integral for instance here rho times this F dot n.

Because, this is exactly the component of this velocity field (across this) along this normal to the surface. So, this will tell us that the mass of the fluid flowing across S per unit of volume, so it has several other applications this integral the flux integral, so, evaluate this for some simple examples, but now the idea is that how to evaluate and it is exactly being translated from the previous lecture, where we have discussed already that how to evaluate surface area, how to evaluate a scale function over the surface.

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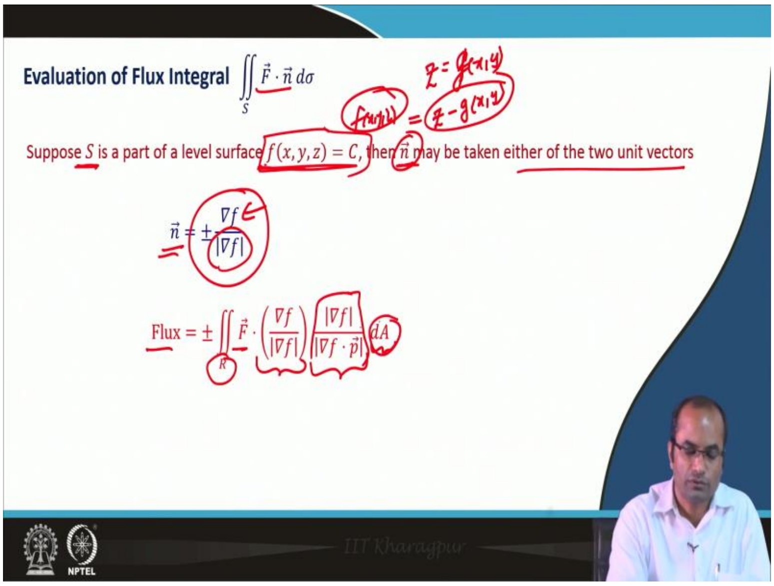

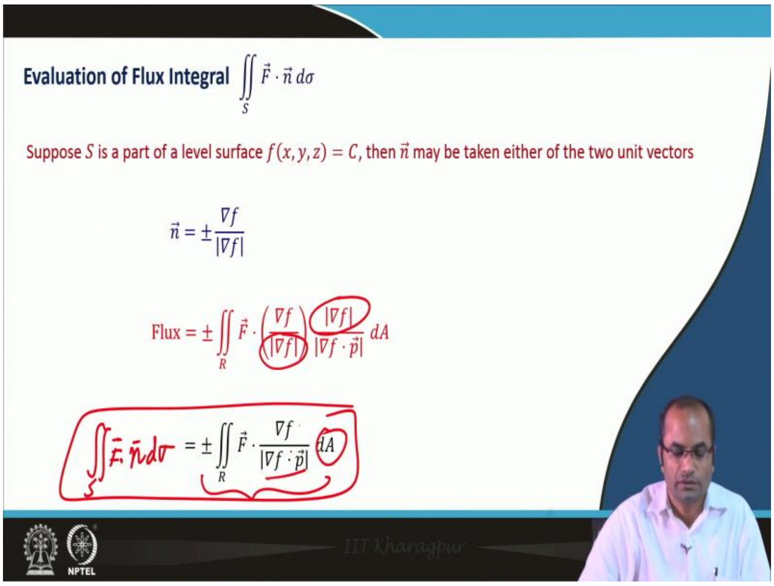

And here we will continue this again for such more general integral. So the evaluation we assume that the S, the surface S is a part of this level surface fx yz equal to C. So, whatever if our surface is given z is equal to f xy we can write or let us say g xy, then we can define z minus g xy as the new function this f xyz and for this f xyz function we will consider equal to 0 as a special case of this level surface.

So, that will exactly represent the given surface, the equation of the given surface because here once we have taken this equation of the surface f xyz equal to constant then this n the unit or the normal or the unit normal, maybe taken either of the 2 unit vectors as I discussed before that there will be 2 directions for the unit vectors.

So this we can compute because we know that with the help of the gradient, so gradient of f will pointing out in the normal direction of the given surface and then we can normalize to have the unit vector. So, this n is nothing but the plus minus the grade f, gradient f and its magnitude. If you divide by the magnitude this is the unit vector and the flux now, so F dot n so this is the n for the given surface f, gradient f and its value there to have this unit vector.

And this is the factor which we have already discussed, because now we are integrating over the projected area which is being projected on one of the coordinate planes. So, this factor will be the additional factor to compensate that curvature of the surface and then this is being integrated over this projected R.

So, this integral we can evaluate, we can simplify it more because we have the steps, the magnitude grade f and the magnitude grade f. So, this can be further simplified. So, the flux integral which was written there that dot this n d sigma can be evaluated with the help of this F dot grad f over grad F dot p and integrated over this projected region R.

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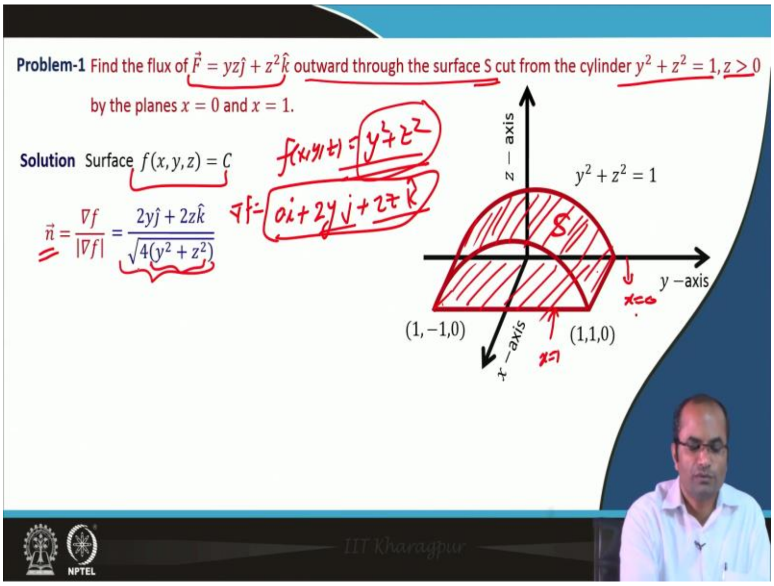

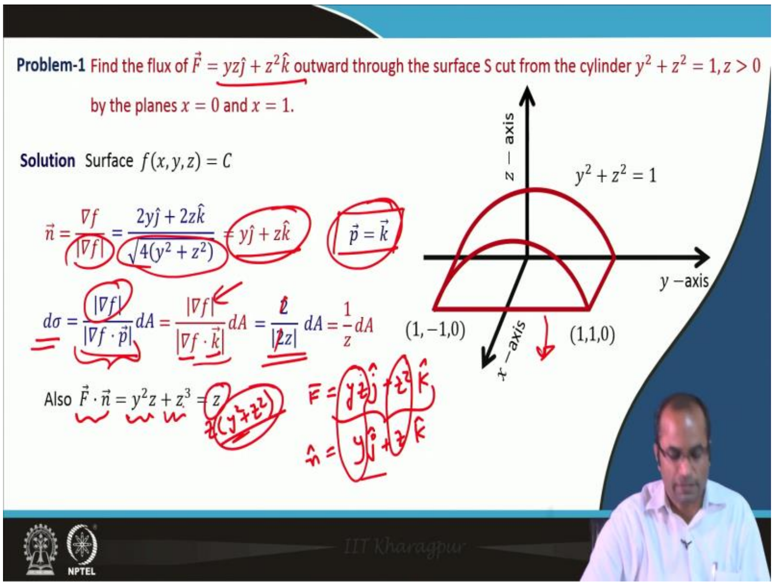
Problem-1 Find the flux of $\vec{F} = yz\hat{j} + z^2\hat{k}$ outward through the surface S cut from the cylinder $y^2 + z^2 = 1, z > 0$ by the planes $x = 0$ and $x = 1$.
Solution Surface $f(x,y,z) = C$
$\vec{n} = \frac{\nabla f}{|\nabla f|} = \frac{2y\hat{j} + 2z\hat{k}}{\sqrt{4(y^2+z^2)}} = y\hat{j} + z\hat{k}$
$\vec{p} = \vec{k}$
$d\sigma = \frac{|\nabla f|}{|\nabla f \cdot \vec{p}|} dA = \frac{|\nabla f|}{|\nabla f \cdot \vec{k}|} dA = \frac{2}{|2z|} dA = \frac{1}{z} dA$
Also $\vec{F} \cdot \vec{n} = y^2 z + z^3 = z$
z − axis
y − axis
x − axis
$y^2 + z^2 = 1$
(1, −1,0)
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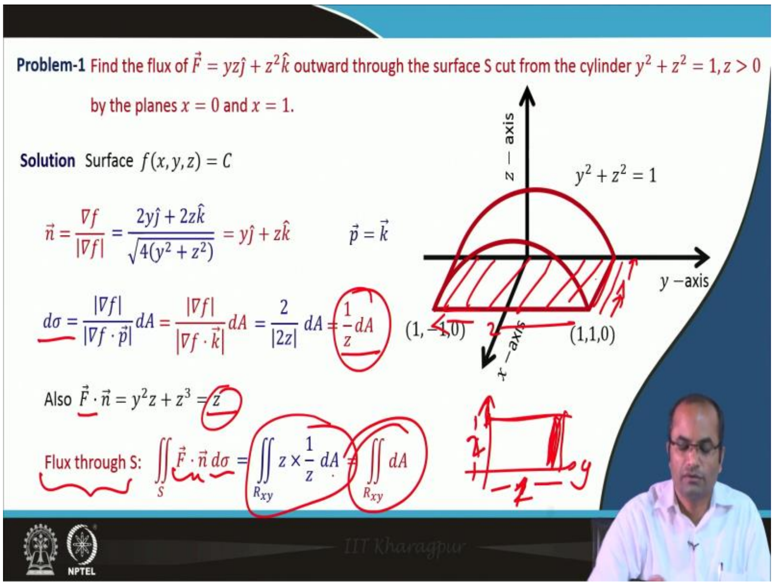
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$\vec{p} = \vec{k}$
$d\sigma = \frac{|\nabla f|}{|\nabla f \cdot \vec{p}|} dA = \frac{|\nabla f|}{|\nabla f \cdot \vec{k}|} dA = \frac{2}{|2z|} dA = \frac{1}{z} dA$
Also $\vec{F} \cdot \vec{n} = y^2 z + z^3 = z$
Flux through S: $\iint_S \vec{F} \cdot \vec{n}\, d\sigma = \iint_{R_{xy}} z \times \frac{1}{z} dA = \iint_{R_{xy}} dA$
z − axis
y − axis
x − axis
$y^2 + z^2 = 1$
(1, −1,0)
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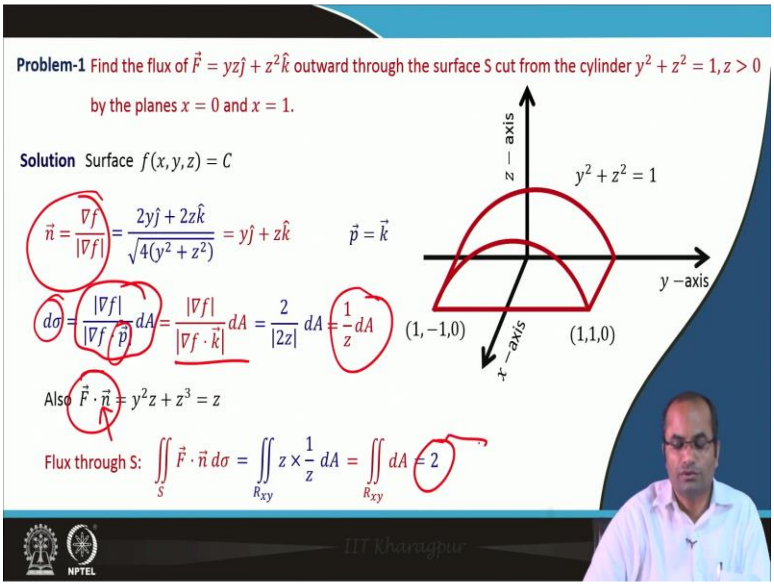
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Also $\vec{F} \cdot \vec{n} = y^2 z + z^3 = z$
Flux through S: $\iint_S \vec{F} \cdot \vec{n}\, d\sigma = \iint_{R_{xy}} z \times \frac{1}{z} dA = \iint_{R_{xy}} dA = 2$
z − axis
y − axis
x − axis
$y^2 + z^2 = 1$
(1, −1,0)
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Okay, well so we can now go through the problems. So, the first problem we want to find the flux so the F is given as y z plus z square, an outward through the surface S, so they will be inward and the outward direction. So here we are going to cover the outward, in the outward direction. So we have the F the vector field is given, and the surface S is cut from the cylinder.

So we have i square plus z square cylinder that means it is parallel to the x axis, the axis of the cylinder is the x axis and only in the positive, for positive values of z, so we are not going the negative values of z. So it is like a half cut of the cylinder and then to get the solution so this is a situation here we have the x axis. So this is the cylinder which is cut from this x is equal to 0, 2x is equal to 1.

So here we have this x is equal to 1 and this is of course the x is equal to 0 line. So then this is the surface S and as discussed before, to get this surface integral, we have to project it on one of the coordinate plane. So here naturally we will project here on the x y plane, because if we project on the other planes that condition which we have in the denominator the dot product that may become 0, so we will or will become 0 actually.

So here in this case we will project on the x y plane. And so we have the surface f xyz equal to C So, f xy is basically this (x) y square plus z square minus 1 or F we can take here x y z is equal to y square plus z square, that surface equation. So, having this we can compute the outward normal. So, with the plus sign it will be the outward normal and we take the minus sign that it will be the inward normal that we can realize from this expression.

So the grad f will be the partial derivative of this with respect to x which is 0. So we have the 0 ith component then we have the partial derivative with respect to y that means 2 yj and then we have 2 z and k. So this is the grad F. And then we can divide by its magnitude. So we have 2ij and 2 zk so the magnitude is 4 y square plus z square and this y square plus z square on the surface on the cylinder that is given 1, so we have here, just 2 and then that means the 2-2 will also get cancel. So we have this outward normal this yj and plus zk.

So the P, concerning that vector p which is the normal to the projected plane, so, here we are talking about that this surface is being projected on the x y plane. So that means this P is going to be k. So, this is in the direction of the z axis. So, P is simply K and then we can compute this d sigma that is the differential element on the surface with the help of this expression. So, we have grad F and the again magnitude of grad F dot p.

So the grad F magnitude and this grad F the dot product with the k in this case, so the grade F is already given y j plus z k. So here in the denominator, we are going to have 2 z, so grad F is 2yj plus 2 zk and then when we do this multiplication with k, so this second component will survive that means the component with the k that is 2 z, so gradient 2z and then this is already we have seen that the magnitude of this grad F is 2.

So we have 2 over 2z and z we are talking about the positive and going for the negative direction there. So, it is just positive, so we have (2 over) oh sorry 1 over z 2 will also get cancelled 1 over z and this dA, dA is the differential element on the x y plane. Well, so we need to get also the F dot n. So the F is given y z plus z square. So we have F as, y z j plus z square k and it is dot product with this n the unit normal that is given here, y j plus this z k.

So, if we put the dot product we have this y a square z and then here we have z cube. So that means, here if we take the common z then we have y square and plus z square and on the surface this is y square plus z square is 1. So, we have simply z there. So having this F dot n z and d sigma is also given now we can compute the surface integral. So, the flux through this S that is F dot n and d sigma.

So, d sigma element is 1 over z and this F dot n is z. So, if we multiply it with 2 z and then 1 over z we have just 1. So, this will be integrated over the projected plane. So, what is the projected plane? Projected plane is in the x y plane and this is just, the square with 1 and the 1 here. So, we have this is from here to here 1 it is actually rectangular, because here you have the minus 1 as the y component here 1, so, this is 2.

So, this is one in the direction of x and then in the direction of y this is 2, so this is 1 here and this is 2 there. So this projected area will be simply the area of the projection. So that is the area of this rectangle. So without a further calculation we can get this area of the rectangular which is 2 here. So, it is clear we need to compute this F dot n for that we need n and can be computed from the given equation of the surface.

We need this differential element which is standard we have done in the last lecture as well. So, this again with the help of this F and this p which depends on which coordinate plane we have taken considered for the projection and then this is one that was a dA, so the value is 2 of this flux.

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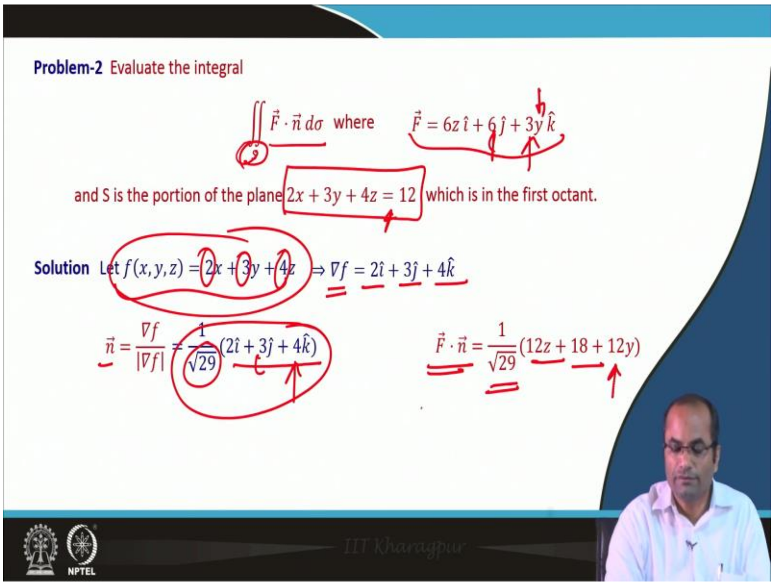

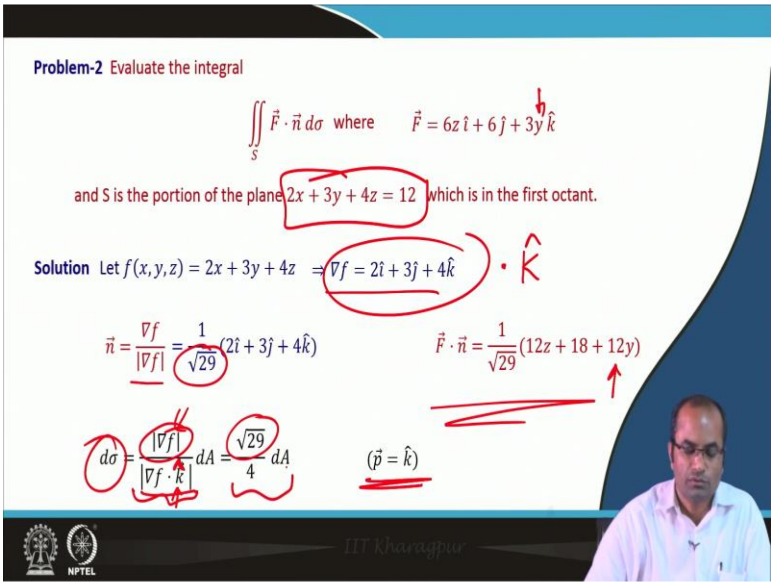

So, the next problem if we want to evaluate this integral here, so, if we want to evaluate this integral F dot n d sigma over this S, where this S is the portion of this plane which is in the first octants. So we have this plane and we are considering only the first octant of that plane for this surface. So, again the same process the F is also given, we need to compute F dot n, we need to compute this differential element etc.

So the solution we take f the surface here 2x plus 3y plus 4z and the given surface is just the level surface of this function. So the grad F we can compute now the partial derivative of this with respect to x that is 2 here their partial derivative 3 then partial derivative with respect to j z is 4, so 2i, 3j and 4k is the gradient of f and the unit normal vector, so gradient of f and we have to divide by its magnitude, that is the square root 29 and we have this vector 2i, 3j plus zk.

Then we also need this F dot n, so F is given there and n is here, so we can compute this F dot n now, so 1 over square root 29 will be coming and then here 2 there we have 6 or 12 z we have 3 here and then 6 there. So we would have 18 there and we have 4 and we have 3. So, we have 12 and this y from there. So this is F dot n and the next we have this element here d sigma which is gradient F divided by this product. So, we need to compute this as well.

So, the gradient F is computed here we have already computer its magnitude that is square root 29 and then the gradient F with this dot product with the k because here we are projecting on this x y plane again so, that is the normal to the x y plane we have here the k. Here we have the possibility to project this plane on either y z plane or z x plane or x y plane. So, we have taken this x y plane in that case we have taken this k here, if we take the other coordinate plane accordingly this vector p which is taken here k will change.

Okay, so this is usually we denote the unit by its hat. So, this p is taken this k hat in this case and we got this element square root 29 by 4 and dA.

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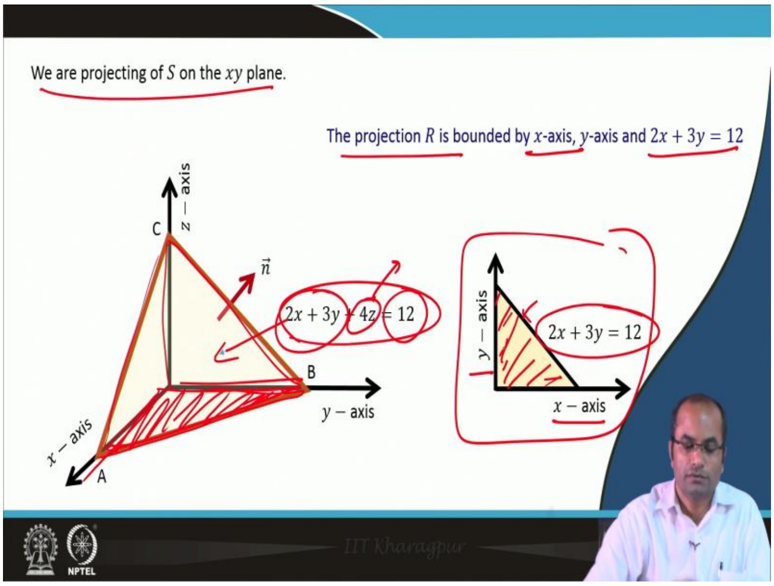

Now, moving further, so we are projecting on the x y plane as discussed. So, what is the projection on the x y plane? So this was the situation this is the plane here 2x plus 3y plus 4z equal to 12 and then, we can project this (or) either on this x z plane or we can project on the y z plane and now in this present situation we are projecting on the x y plane.

So as you can see, this projection is going to be a triangular shape that means, the projection R will be bounded by this x axis, will be bounded by the y axis. And of course, from here because we have projected already, so z will become 0 there and we have 2x plus 3y equal to 12 that will be the line here connecting the 2 axis. So, we have this situation now for the projection, this is the x axis and then we have y axis there and this is the line 2x plus 3y equal to 12. So, we will be doing the integral over this triangular shape of for the surface.

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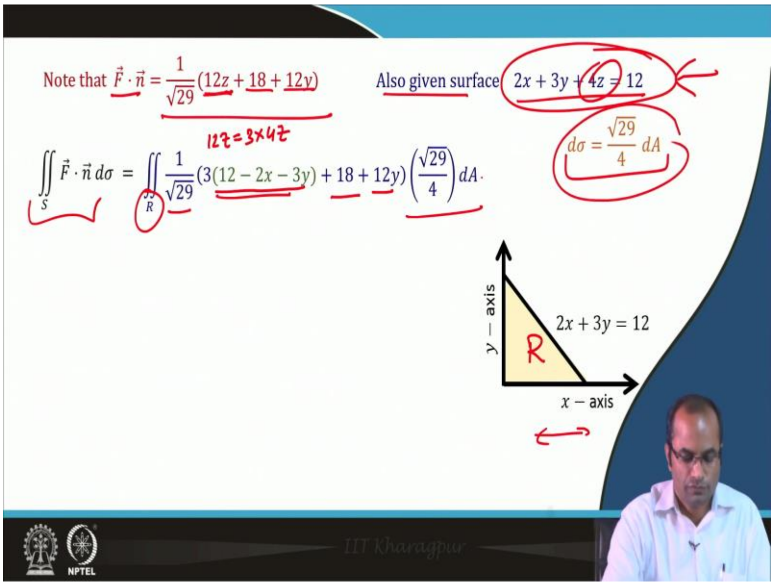

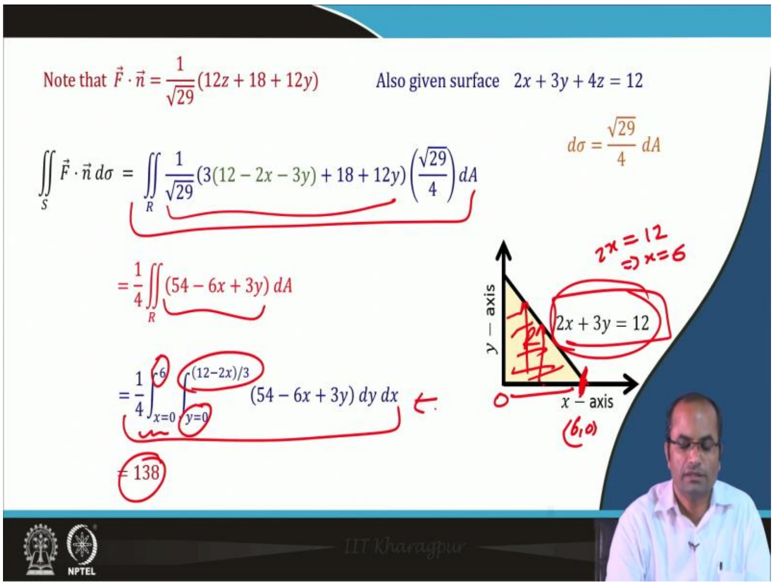

So, we have F dot n is given by this we have valuated already 1 over a square root 29 and 12 z plus 18 plus 12 y we have also the surface that is the given surface, this element also we have evaluated d sigma is equal to square root 29 by 4. So, these evaluation we have done, so we are ready now to compute this flux integral.

And the domain also we have discussed that this is the projected domain where we will be integrating now double integral. So, this surface integral over this R, R is this region given here the projected one. So, we have 1 over 29 that is F dot dn, F dot n here and 1 over 29. So this 12 z because we are projecting in x y plane, the z has to be replaced from the given surface. So we have this 4 z as 12 minus 2x minus 3y, 12 minus 2x minus 3y.

So this 12 (x) z was written as 3 into 4 z and this 4 z from this equation of the surface is replaced by this 12 minus 2x minus 3y. And then we have 18 and we have this 12 y, then this differential element is replaced by the square root 29 by 4 dA. So we have the simple double integral which has to be evaluated over this given region R.

So concerning the limit, we have to propose the limit first, this simplification will lead to this expression 54 minus 6x plus 3y and then the limits, so first let us talk about the limit of the y. So y is moving from the 0, so y from 0 to from here we can get that means 12 minus 2x and divided by 3, so 12 minus 2x divided by 3 and for x, we have from 0 to this is what we will get from here, so when we put this y zero we have 2 x equal to 12 that means your x is equal to 6.

So this is 6 0 point, so the x is moving from 0 to 6, x is from 0 to 6 and y is from 0 to this line, where y we have computed as 12 minus 2x divided by 3. So this double integral one can evaluate and the value is coming as 138. So this is the simple double integral which we have evaluated already in integral calculus.

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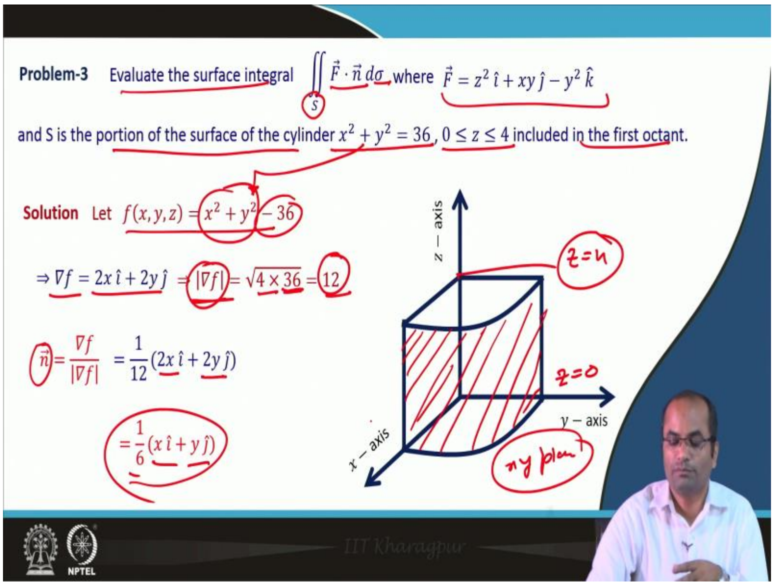

Okay, so another example where we will evaluate this surface integral over this S, S is a portion of the surface of the cylinder. So we have again a cylinder x square plus y square is equal to 36 and the z direction the x axis is parallel to the z axis. So, we have this (4) sorry 0 to 4 the z is varying from 0 to 4 and it is included in the first octant.

So here we want to evaluate the surface F dot n surface integral d sigma where F is given by this expression. So, again we will consider, we will take this F the given surface x square plus y square that is also fine if you take x square plus y square 36 is not playing any role for the evaluation of the gradient for instance.

So, here we have the situation we have x axis, we have y axis, we have z axis and this is the cylinder which is along the z axis and it is varying from z equal to 0 to z equal to 4. Well, so then we have grad F is 2x the ith component 2i, the jth component and the absolute value of this grad F we can evaluate. So 4 x square plus 4 y square and x square y square is 36. So 4 into 36 which is coming to 12, so the grad F is 12, the magnitude of grad F is 12.

Then the unit normal vector again the grad F and divided by this magnitude 12. So grad F we have this 2 xi and 2 yj divided by 12, so this is 1 over 6 xi plus yj so this is the n. Now about the projection because the surface this cylinder is in the direction of z, so it should be noted that that we do not have possibility now projecting this to any of the coordinate planes, because if we project for instance on x y plane, what will happen?

X Y plane, the P the perpendicular to this x y plane is the k and when we do this product of n and k which is needed there, so this will become 0 and that is coming in the denominator in our integral.

So, this is not going to work and again you are looking at the projection also of this cylinder on the x y plane, this is going to be just a part of the circle there will be no indeed a surface there. So whatever ways the formula will not work and here also this is not, we cannot project it on this x y plane because we will not get any surface on this, any plane on the coordinate planes. So, here we have to project either on this x z plane or we can project on this y z plane then we can evaluate.

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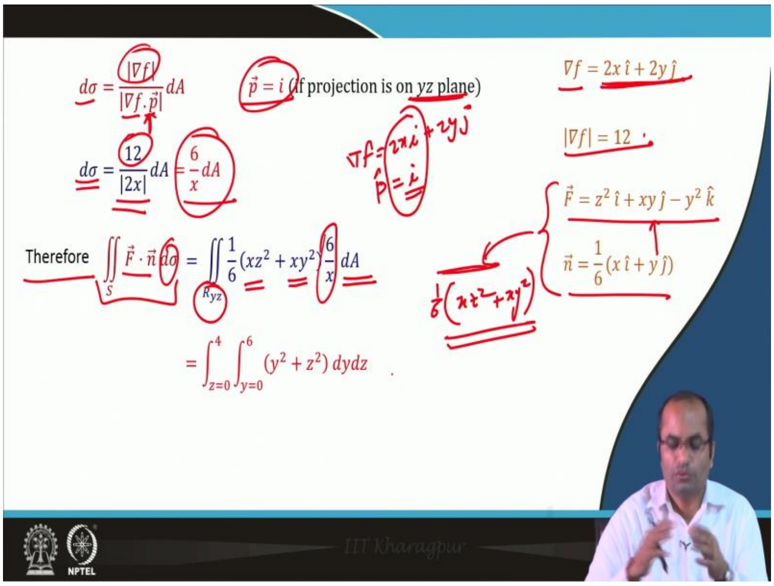

So, what we have now? To summarize first the grad F we have evaluated that is 2 xi plus yj it is absolute, the magnitude is 12 the given F is there and the unit normal vector also we have now 1 over 6 xi plus yj. So d sigma that element on the surface we can evaluate here, the gradient f and p. So this gradient f is 2 xi plus 2 yj and the P. So, here we are talking if we take on y z plane, if we take the projection on the y z plane, that means the P will become i.

We can do on x z plane as well, but it does not matter. So, we have taken here y z plane, the p is i so, this p is given now. So, we have d sigma equal to that 12 is given but we have replaced here the gradient f is 2 xi plus 2 ij and so the gradient f was 2 xi plus 2 yj and then P here this vector unit vector to the projected plane is i we have taken.

So the dot product will give just the 2x and its magnitude 2x. So, that means, we have 6 over x and this dA okay therefore, what we have now, this surface integral F dot n d sigma can be evaluated with this. So, F dot n, F dot n, so from here we need to compute F dot n that means, 1 by 6 will be there and then we have x z square and from here we have x y square.

So 1 over 6 x z square and x y square and this d sigma where d sigma is we have evaluated that is 6 over x and then we have dA and we are projecting on y z plane.

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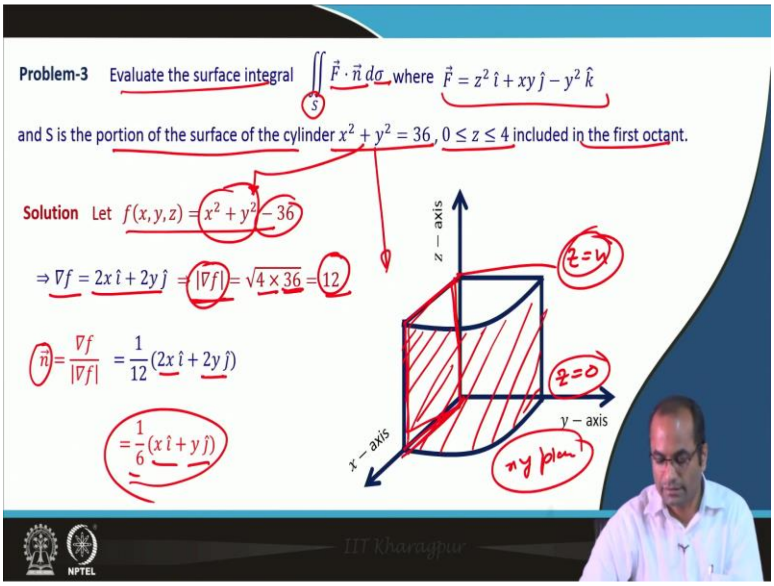

So, the projection if we have to see that again it will be a rectangular shape because we have already seen one of the previous example, we have the circular, we have the cylinder there and if it is projected on the y z plane, it is going to be a rectangle here. So, for example, x z plane we have chosen so this is going to be rectangles z is varying from 0 to 4 and then from the given equation we can find the coordinator of x as well.

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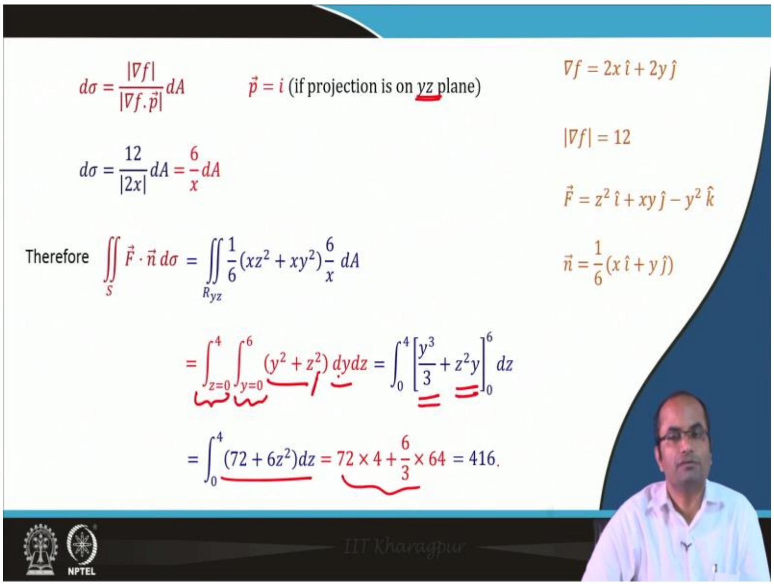

So, having this we can now go with this limits. So limits for z, 0 to 4 and the y we will compute from this y axis. So that is for y, we are projecting on y z plane but same thing can be done for x z. So y is moving from 0 to 6 and then we can evaluate this. So with respect to y we have y cube by 3 and then we have z square and then y there with respect to y after this evaluation we can then evaluate with respect to z and finally we will arrive here, the answer which is 416 in this case.

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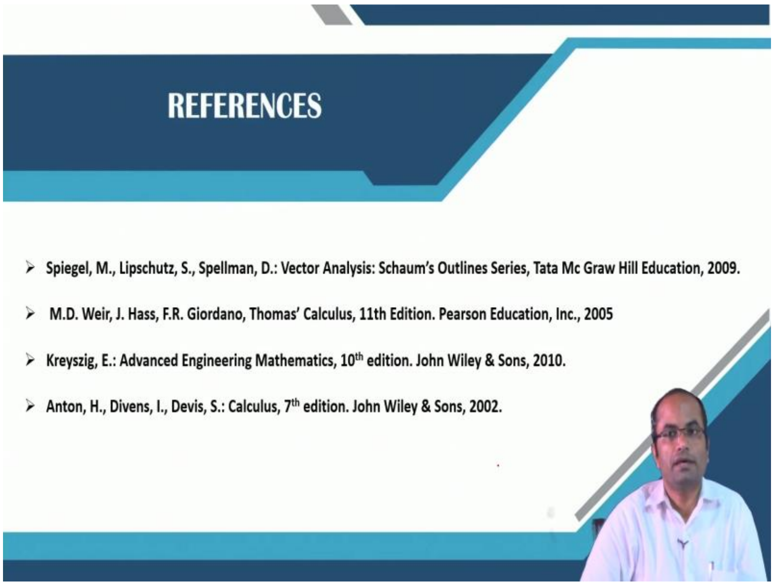

Well, so, these are the references we have used for preparing these lectures.

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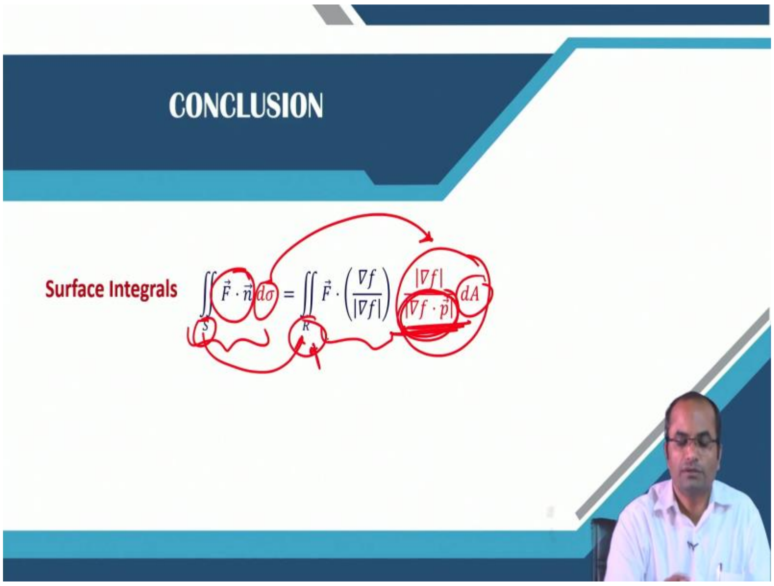

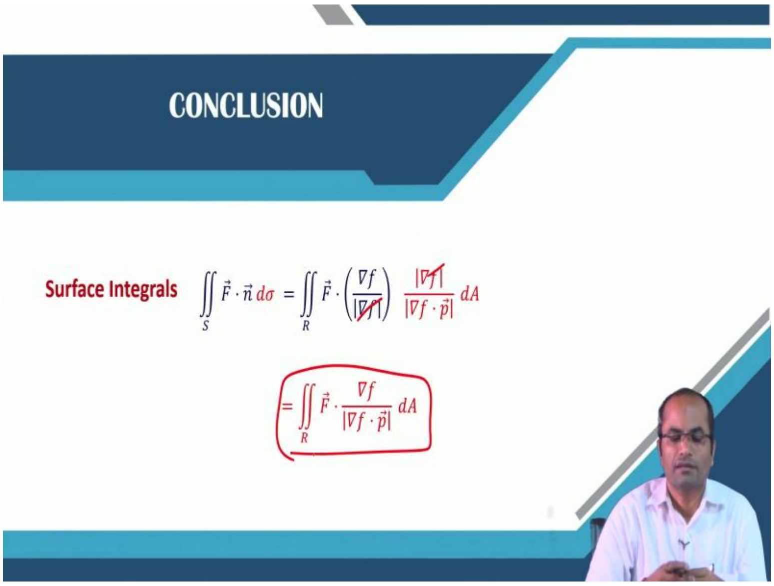

Coming to the conclusion, so we have this surface integrals mainly this F do n this flux in the direction or the component of F in the direction of n is integrated over the surface S and it has several applications some of them we have named already in the lecture and how to evaluate? The idea is exactly what we discussed for evaluation of the surface area indeed, and this d sigma will be replaced by this grad F over grad F dot p, the p will be the normal to the projected plane.

So we have to project the given surface in one of the coordinate planes and that depends on so, that this grad F dot p, the grad F dot p should not be 0. So the gradient of F is normal to the surface. So normal to the surface should not be perpendicular to the normal to the projected plane otherwise, this will become 0 and we cannot use this formula.

So that we have to see that which one is convenient and also this should not be 0 (for the) during this evaluation of this area integral. So this surface integral is converted to the simple double integral, which we studied already. And then this simplification leads to another simple formula, which can be used for the surface integrals. So, so far we have learned the line integrals and the surface integral.

The last, these 3 lectures were devoted one for the line integral and now for the surface integral. So, the next lecture is going to be again the conversion from the line to the surface integral if you remember we have already discussed the Greens theorem, where such a conversion took place that we have the curve integral, the line integral and which was transformed to the double integral.

And now we have more general instead of double integral the idea of the surface integral. So, we will generalize this Greens theorem which is called Stokes theorem, one of its generalization that will be discussed in the next lecture. So that is all for this lecture and thank you very much for your attention.