Advanced Comput ational Techniques
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Lecture 13
Initial Value Problems

Now so far, we discuss about the Euler method and these general class of method. So, that is which use (p + 1) previous steps and this is an implicit methods.

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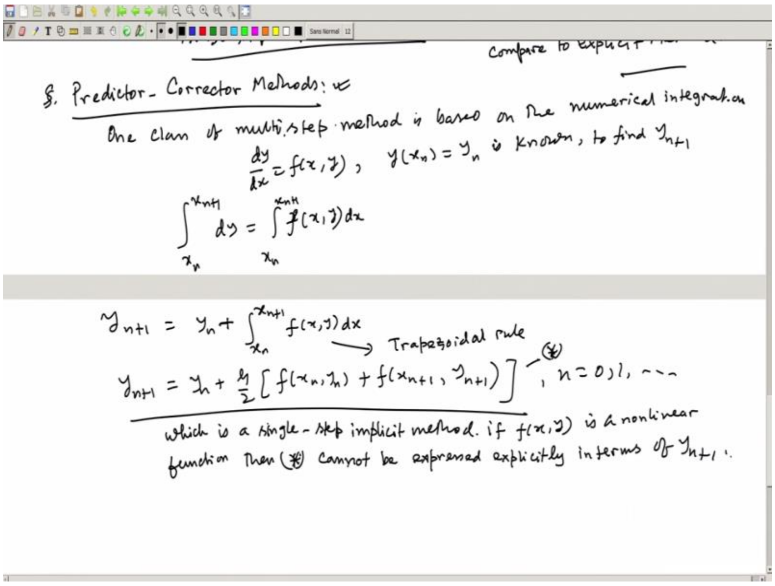

Now, these implicit methods what we do is a class of methods which is called the predictor corrector method. So, this is the thing we will talk about. Now, in the predictor corrector method what we do is we have a corrector scheme which are based on the implicit method is used to get a corrected solution from the best which is based on a explicit method. So, how it is done? So, mostly this is based on the numerical integration.

So, one class of method multistep method is based on the numerical integration. So, you have

$$\frac{dy}{dx} = f(x, y)$$

and suppose I know $y(x_n) = y_n$ is known. So, we need to find y_{n+1}. So, what we do? We integrate between x_n to x_{n+1}

$$\int_{y_n}^{y_{n+1}} dy = \int_{x_n}^{x_{n+1}} f(x, y)\, dx$$

So, if we integrate between x_n to x_{n+1}. So, what I get is

$$y_{n+1} = y_n + \int_{xn}^{x n+1} f(x, y)\, dx$$

So, we integrate this by trapezoidal formula. So, that means what we do is we are using a linear interpolation polynomial and integrating this. So, we get

$$y_{n+1} = y_n + \frac{h}{2} [f(x_n, y_n) + f(x_{n+1}\, y_{n+1})] \qquad \text{-(*)}$$

So, this is y_{n+1}. So, obviously this is a method, but this method is an implicit method which is a single step. Now, single step method is always has advantage that you can start from n = 0, 1, 2,.. like this way that means what we need only the initial correction, it is kind of self starting.

Whatever the initial condition y_0 is given from there onwards one can find out the step by step all these solutions. So, which is a single step implicit method because the y_{n+1} is involved in both cases if f(x, y) is a non-linear function. So, y_{n+1} cannot be then this say (*) cannot be expressed explicitly in terms of y_{n+1}. So, what we do is that in this case we do adopt now what is the advantage?

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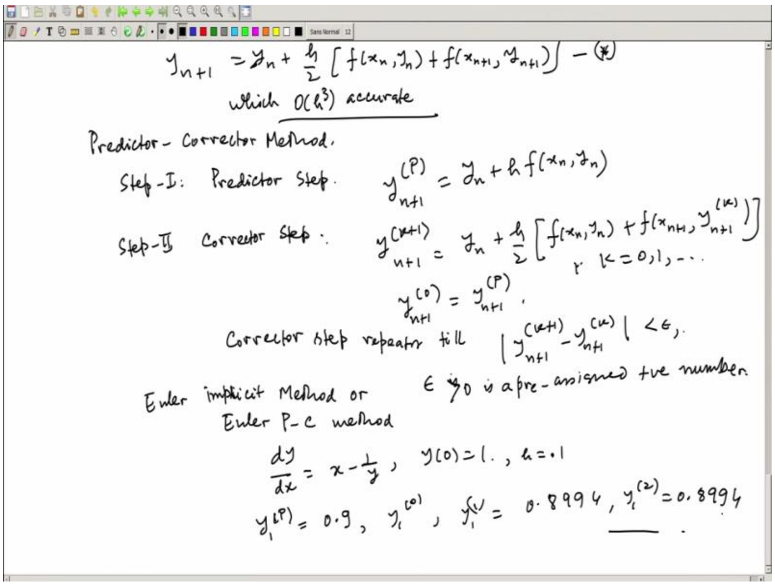

One of the advantage is this formula. Now this formula has advantage is that this is of order h^3 . So, compared to the previous one so which is $O(h^3)$ accurate. So, compared to the previous method whatever we discussed that was first order accurate, but compared to that what we find that this is $O(h^3)$ accurate. So, that is very encouraging.

Now, what we do we solve this steps by a predictor corrector manner.

Step 1: what we call the predictor step. So, there we find out the y_{n+1} a predicted value by the simple Euler method x_n y_n then

$$y_{n+1}^{(P)} = y_n + h\, f(x_n , y_n)$$

Step 2 : this is called the corrector step there we solve this (*) in a repeatedly on this manner

$$y_{n+1}^{(P)} = y_n + \frac{h}{2}[f(x_n , y_n) + f(x_{n+1} , y_{n+1})]$$

$k \geq 0$ and $y_{n+1}(0) = y_{n+1}(p)$ and till what I get is

$|y_{n+1}(k + 1) - y_{n+1}(k)| \leq \xi$ pre - assigned.

So, till the desired accuracy is achieved we repeat this corrector steps for number of times. So, in this case k = 0, 1, 2 like that way till we get an alignment or satisfaction that is two values are very close. So, this is the basic principle of solving the predictor corrector method and this predictor corrector method is based on the Euler first order approximation.

This is also called the Euler implicit method or Euler PC method predictor corrector method. So, for example, $\frac{dy}{dx} = x - \frac{1}{y}$, $y(0) = 1$

if we choose what we can get from here for any choice of h is 0.1. So, $y_1^{(p)} = 0.9$ and $y_1^{(0)}$ Similarly, one can find out $y_1^{(1)}$ that it comes to be 0.8994 , $y_1^{(2)} = 0.8994$ and so on. So, that is how the predictor corrector method Euler predictor corrector method goes.

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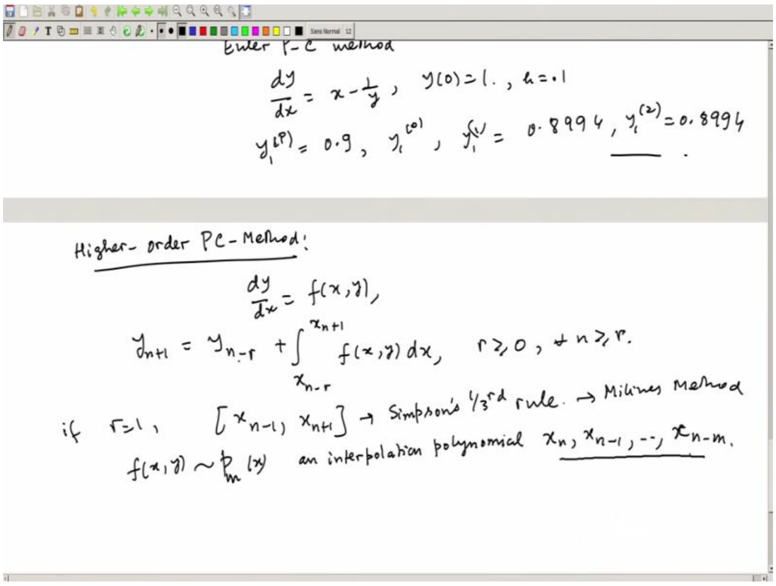

Now, there are several other predictor corrector methods like Adams-Moulton method, higher order predictor corrector method. So, what we do in the higher order predictor corrector method is that this integration $\frac{dy}{dx}$ = f (x, y). So, up to y_{n+1} up to y_n is supposed to be known. So, in that

$$y_{n+1} = y_{n-r} + \int_{x_{n-r}}^{x_{n+1}} f(x, y)\, dx \quad r \geq 0 \text{ and for all } n \geq r.$$

So, the methods are derived so either like the previous one where we have taken r = 0 and we have used trapezoidal formula because trapezoidal formula you need two points. Similarly, we can replace say [x_{n-1} , x_{n+1}] if we call r =1.So, in that case we can use Simpson's 1/ 3 rd rule and a formula can be derived and also so this is called that is the method is referred as a Milne's method.

Then there are another methods where the f (x, y) is approximated by f (x, y) you approximate by a interpolating polynomial $p_n(x)$, f (x, y) ≈ $p_n(x)$ and interpolation polynomial using this points. So, m + 1, $p_n(x)$ or rather if I say $p_m(x)$ because this n may not be a same as the degree whatever we are talking about. So, p_m (x) so this m and n may not be same. Now, this p_m (x) interpolating polynomial so x_n , x_{n-1} up to x_{n-m}. So, if it is a degree of m so we need to have (m + 1) points. So, x_1, x_{n-1},..., x_{n-m}.

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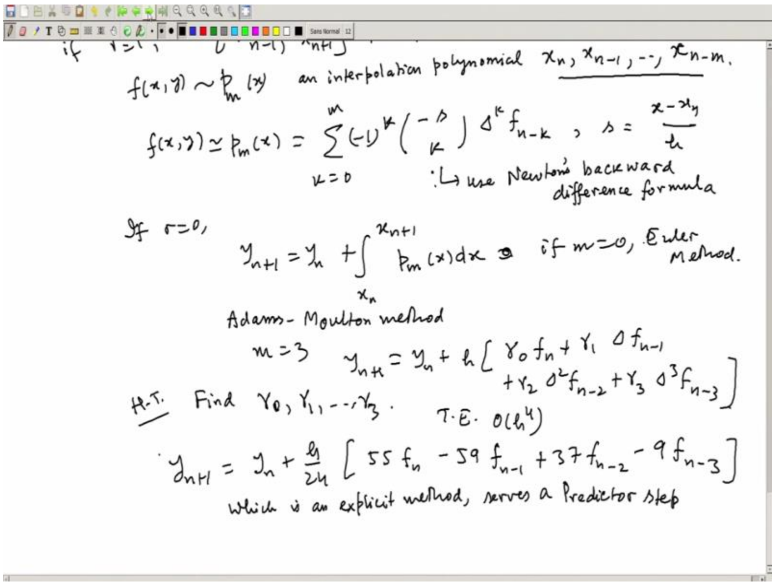

And in that case if we replace this say f(x) by a formula as p $_m$ (x). One can write say - 1 to the power k a backward difference if we use backward difference formula as

$$f(x, y) \approx p_m(x) = \sum_{k=0}^{m} (-1)^k \binom{-s}{k} \Delta^k f_{n-k}, \quad s = \frac{x - x_n}{h}$$

So, you know this is we use Newton's backward difference formula and then this integrate between r = 0 say so if we integrate y if we choose r = 0 and

$$y_{n+1} = y_n + \int_{x_n}^{x_{n+1}} p_m(x)dx$$

And f(x,y) is replaced by p $_m$ (x) and we get a formula which will be a explicit in this case because all these things are integrated. So, we get a formula and that will be much higher accurate compared to the previous one. So, that is what is referred as the Adams-Moulton method. So, if we choose say m = 0 of course m = 0 this is again the Euler method. So, a constant.

And if we choose m = 3 so in this case we get

$$y_{n+1} = y_n + h [\gamma_0 f_n + \gamma_1 \Delta f_{n-1} + \gamma_2 \Delta^2 f_{n-2} + \gamma_3 \Delta^3 f_{n-3}]$$

So, one can find out,so this is a home task to find γ_1, γ_0, $\gamma_1,\ldots,\gamma_3$ what are the constants will be? So, in this process finally I can give the final answer.

So, the truncation error will be order h to the power 4 because we are choosing degree 4 points and then we are integrating. So, the order truncation order will be of O (h^4) much higher accurate and the method finally will look like this

$$y_{n+1} = y_n + \frac{h}{24} [55 f_n - 59 f_{n-1} + 37 f_{n-2} - 9 f_{n-3}]$$

So, this is obviously an explicit method serves as predictors step. So, higher order accurate. Now, as we said that explicit method has a problem of instability of the solution. So, that is why the corrector formula.

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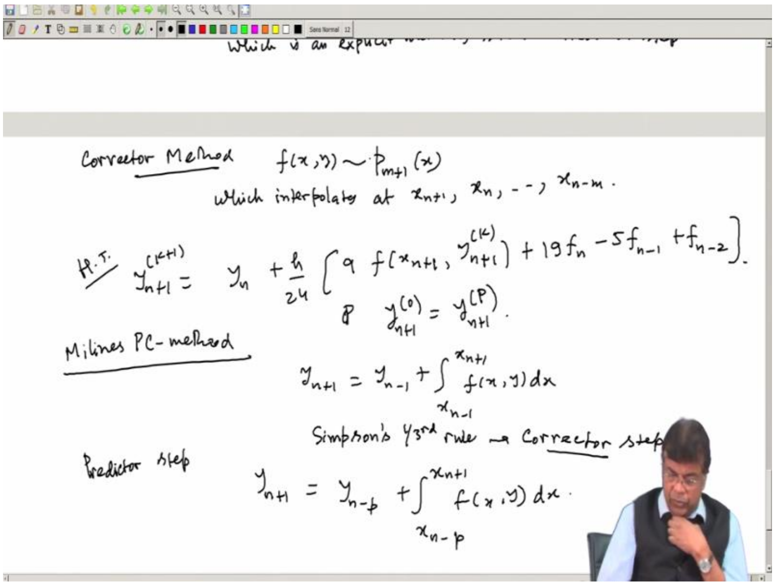

So, what I do is in the corrector formula; to get the corrector formula we go for corrector method, we go for a p_{m+1}. So, f(x, y) we are replacing by a polynomial $p_{m+1}(x)$

$$f(x, y) \approx p_{m+1}(x)$$

which interpolates at x_{n+1} we are introducing now x_{n+1} , $x_n, \ldots, x_{n-m}$. So, this (m + 2) points now we have creating a degree as $p_{m+1}(x)$ and that becomes an implicit method. So, one can derive also this implicit methods as this form

$$y_{n+1} = y_n + \frac{h}{24} [9 f (x_{n+1}, y_{n+1}) + 19 f_n - 5 f_{n-1} + f_{n-2}]$$

Again this is a home task for you . So, this is an implicit method so that means this need to be solved and $y_{n+1}(0) = y_{n+1}(P)$.

Now, another class of methods predictor corrector methods, Milne's corrector method. So, for example, so what is done is if we use say

$$y_{n+1} = y_{n-1} + \int_{x_{n-1}}^{x_{n+1}} f(x, y)\,dx$$

we will use Simpson's 1 / 3 rd rule to get integrating formula.

So, that gives you a corrector step and for the predictor one so this is a corrector step for predictor step one can use formula let us say

$$y_{n+1} = y_{n-p} + \int_{x_{n-p}}^{x_{n+p}} f(x, y)\,dx$$

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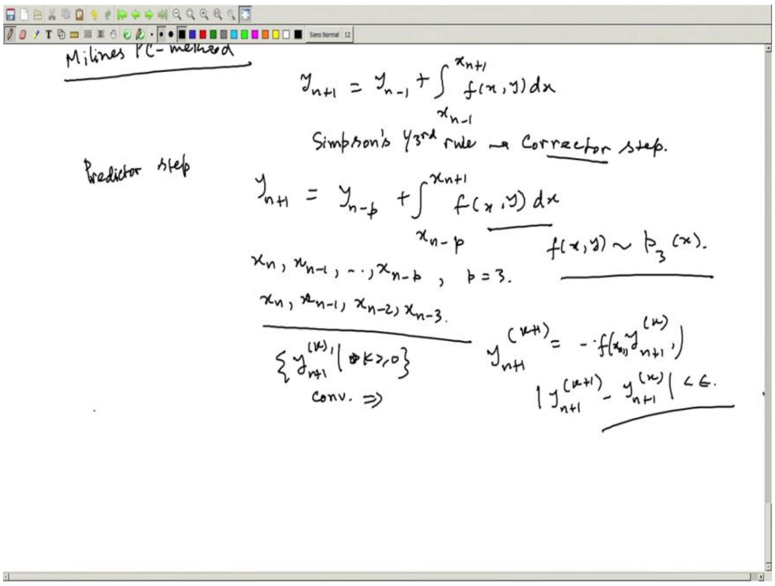

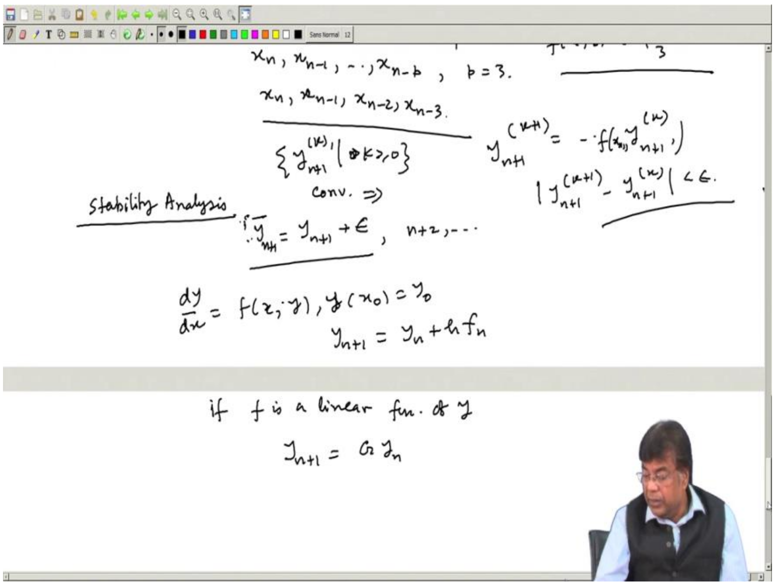

So, what we do is we can now choose the interpolating points as x_n ,x_{n-1} , …, x_{n-p}. So, that means (p + 1) if we choose then instead if we take p=3 so that means we have x_n, x_{n-1}, x_{n-2}, x_{n-3}. So, this gives a polynomial formula which involves the third degree polynomial third order polynomial not a third degree may not be a good way of expressing.

So, this kind of formula so this will be an explicit formula because f(x,y) we are now replacing by a polynomial of known points x_{n-1} or rather using y_n, y_{n-1}, y_{n-2}, y_{n-3}. So, this will be a polynomial of $p_3(x)$ and that give you a correct predictor formula and which can be used to obtain the initial approximation for the corrector step. So, this is how the predictor corrector step goes.

That means you first solve the predictor step to get an approximate value of the unknown y_{n+1} and then we will solve this corrector step in a repeated fashion to get the every step we are refining. So, that means we are refining like $y_{n+1}^{(k+1)}$ and then this is involving in the function

$$y_{n+1}^{(k+1)} = f(x_{n+1}, y_{n+1}^{k})$$

So, that means we are repeating till we get the successive approximations are very close.

So, this is why Cauchy's principle of convergence. If this sequence y_{n+1}^{k} $k \geq 0$ is the sequence of iterates, if it converge that implies that these Cauchy's criteria is holding good. So, Cauchy's criteria holds provided you have the convergence, rather if the sequence converge then we can say that the difference between successive approximates are quite close that is what is the Cauchy's criteria says.

Now, one thing we have not talked about so this is the way we can go for a higher order method and that is an implicit scheme. Now, this stability analysis we have talked about the stability of a system so stability analysis. So, that means as we mentioned before that if any small amount of error say $y_{n+1} + \xi$ is small amount of error is incorporated in the system.

So, let us call this is the erroneous value is y_{n+1} . So, since at x_{n+1} if it is incorporate in the n + 1 steps. So, obviously the solution for higher that is n + 2 onwards are will be all corrupted. So, now we have to see by the stability analysis whether that the perturbation whatever is incorporated is growing or amplifying. Now, these analysis is very difficult and only for the linear situations one can do little bit of analysis.

I just show a simple example. So, suppose you have a linear equation

$$\frac{dy}{dx} = f(x, y) \quad , y(x_0) = y_0$$

a single step so $y_{n+1} = y_n + h f_n$. Now, if this is a linear one if f is a linear function of y then one can write $y_{n+1} = G y_n$.

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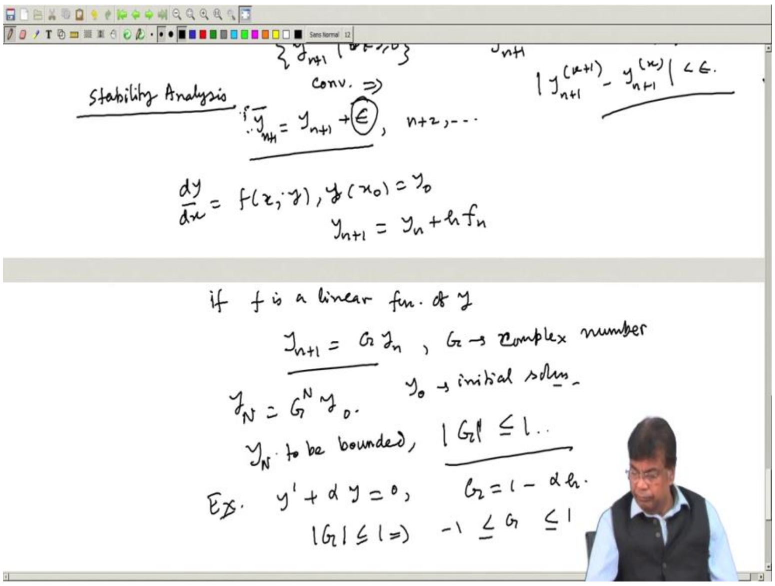

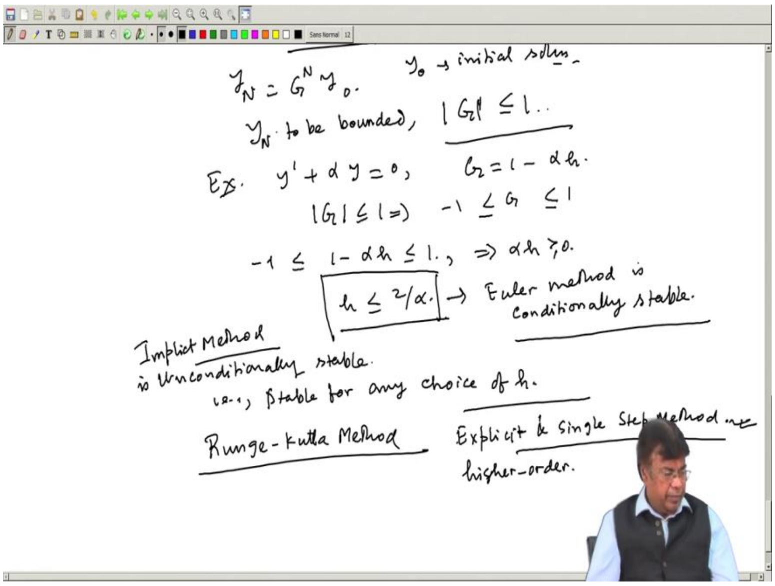

So, this G is referred as the which can be a complex number. So, this is called the amplification factor. So, that means since this is a linear equation so error also will satisfy the same way. So, that means now if I now substitute since this is the solution of the given difference equation. So, this error ξ also will satisfy the same kind of equations. Now, if I repeat so I can write after N step is $y = G^N y_0$.

So, y_0 is the initial from where initial solution from where we have started. Now, obviously this y_N to be bounded then what we need is $| G | \leq 1$.

Ex : $\acute{y} + \alpha y = 0$, $G = 1 - \alpha h$

$| G | \leq 1 \Rightarrow \ -1 \leq G \leq 1$

$-1 \leq 1 - \alpha h \leq 1 \Rightarrow \alpha h \geq 0$

$h \leq \frac{2}{\alpha}$

So, this amplification factor has to be less than equal to 1 other one so that means this put a restriction for the Euler method because we can see that the stability analysis in general situation is not a very easy task only for the linear.

We can say that some analysis we can make and what we will show that the implicit Euler implicit method on the other hand is unconditionally stable that is no condition need to be stable. So, that means no restriction for that is stable for any choice of h.

So, this is all about the initial value problem we wanted to discuss here. So, the other kind of methods like higher order predictor corrector method there is of course a one important method that we need to discuss in the next class that is the Runge–Kutta method that has not been talked about. We will just give a procedure. So, one of the explicit single step method. Now, obviously as we could see that explicit and single step method is very important, very easy to handle.

But need to be higher order explicit single step method that we will discuss and that will be the all whatever we wanted to talk about the IVP. So, that I will do in the next class. Thank you.