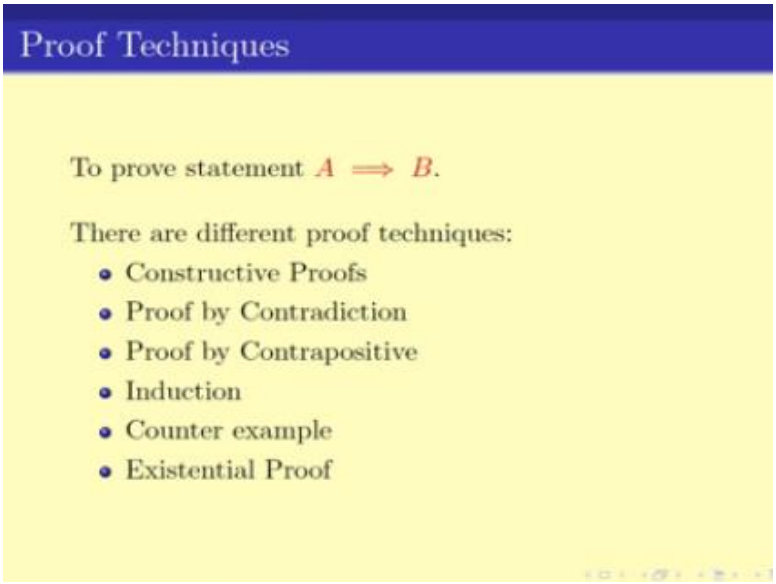
Discrete Mathematics
Prof. Sourav Chakraborty
Department of Mathematics
Indian Institute of Technology - Madras

Lecture - 13
Proof by Counter Example

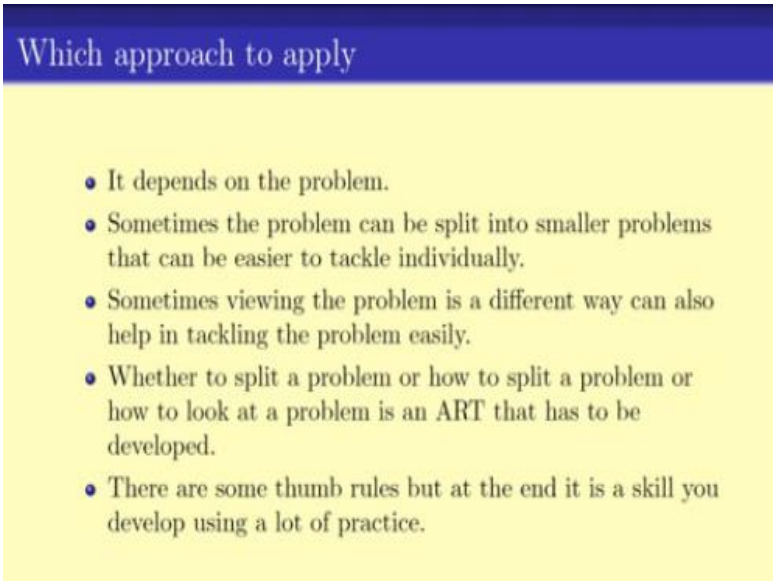
Welcome everyone to this last video lecture in week three. This week, we have looked at various proof techniques and we will continue to look at more proof technique in this particular video lecture.

(Refer Slide Time: 00:17)



So till now we have seen that we have rather talked about to prove A implies B. There can be many proof techniques, namely Constructive Proof, proof by Contradiction, proof by Contrapositive, induction, counter example and Existential Proof. Till now we have seen constructive proofs, proof by contradiction and proof by Contrapositive.

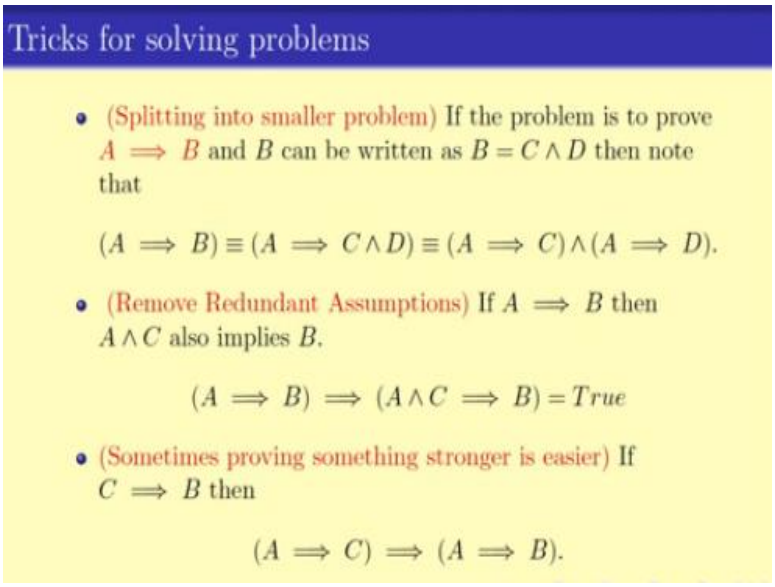
(Refer Slide Time: 00:57)



Now here I again repeat this, I repeat this in every of this video lectures, namely which proof to apply which problems is something that you have to decide for yourself. Some of the problems can be split into smaller problems that can be easier to tackle while some of them can be viewed in a different way and by viewing so one can make the problem easier. But which problem to split and how to split it or how to look at it is an art in itself, that we have to develop.

In this particular set of video lectures, we will be giving you thumb rules on which one to use proof technique use for which problem, but at the end of the day you will have to make a choice for yourself. Many of the problems can have multiple different proof techniques that is good enough for it. We also looked at the some of the simple techniques of how to split the problems in to smaller cases.

(Refer Slide Time: 02:00)

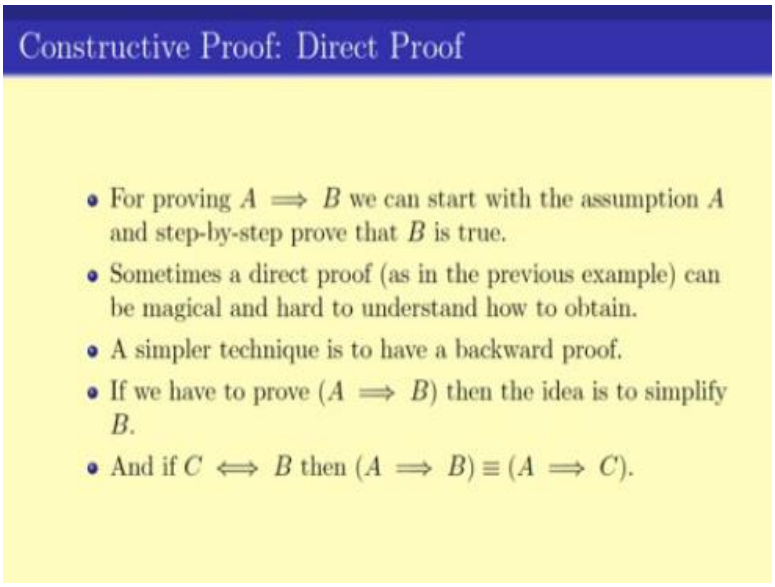


Particularly, we looked at the following two techniques, first of all if B can be written as C and D then proving A implies D is same as proving A implies C and A implies D. And in this case one can split the problem of proving A implies B into this two parts A implies C and A implies D. There is also the other thing of removing the redundant assumptions, in other words sometimes some essentials are there which are not necessary.

Say in other words says if I have been asked to prove A and C implies B, one might throw away the C and end up proving A implies B, which is good enough for us. So in other words what I am saying is that, one should extract the most relevant set of assumptions that are necessary to prove B. That would simplify the proof or rather they will simplify the problems and help in getting the proof easier.

The third part was that sometimes proving something stronger is easier. So we might have like C implies B and we have to proof A implies B, it might be easier to prove A implies C instead, which would be a harder problem in itself, but might be easier to prove. So in that case, making the problem harder can make our life of getting a proof is easier. Other than these three tricks, we also solve some proof technique.

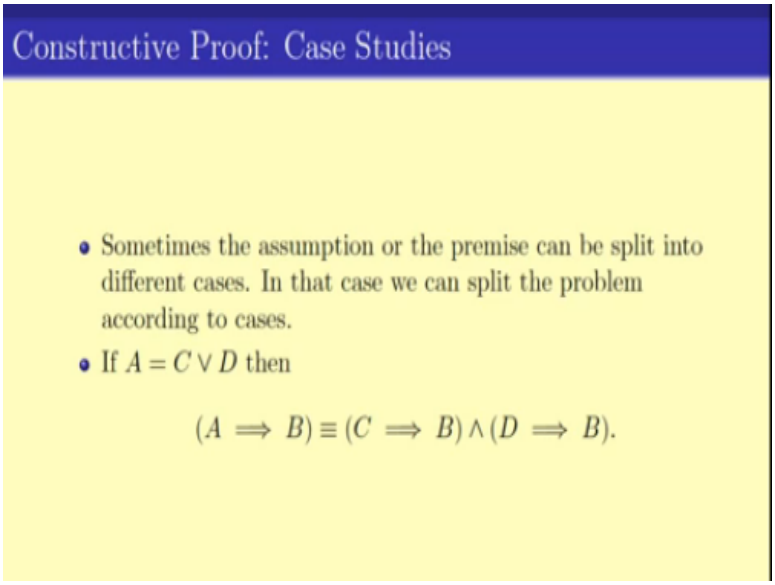
(Refer Slide Time: 04:05)



In particularly, we looked at constructive proof and in that we saw direct proof, in direct proof the idea is that proof A implies B. We start with the assumption A and step by step go on to prove B. But sometimes by doing so or doing such a thing the proof can became very magical and hence one can imagine that it is harder to get such a proof. Another technique that can be there is what is known as backward proof.

So in this technique, to prove A implies B, the idea is to simplify B slowly till the time it is simplified to a different form, so namely if I simplify B to C then A implies B would basically mean A implies C. And since A implies C is a simplest statement, it will be easier to get that proof. So this was the Constructive proof and in which case we saw the direct proof technique, there was the other proof technique in the Constructive proof, which is called the case studies.

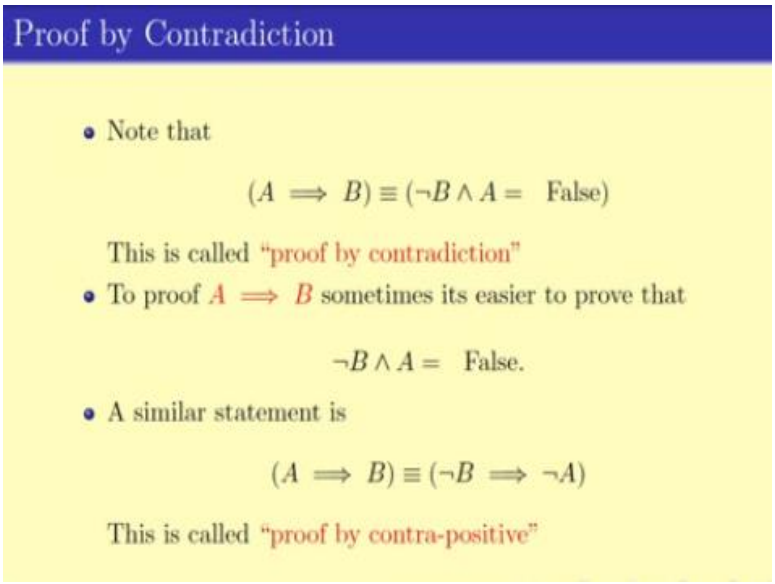
(Refer Slide Time: 05:34)



The idea is that sometimes the assumptions or the premise can be split into different cases. And in that case we can split the problem according to cases namely if A can be written as C or D then A implies B is same as proving C implies B and D implies B. Here it is important to note that how to split A into this two things C or D, is something that require some of amount of art, an understanding of the problem.

One would like to split up into C or D in such a way that proving C implies B and D implies B becomes easier.

(Refer Slide Time: 06:40)

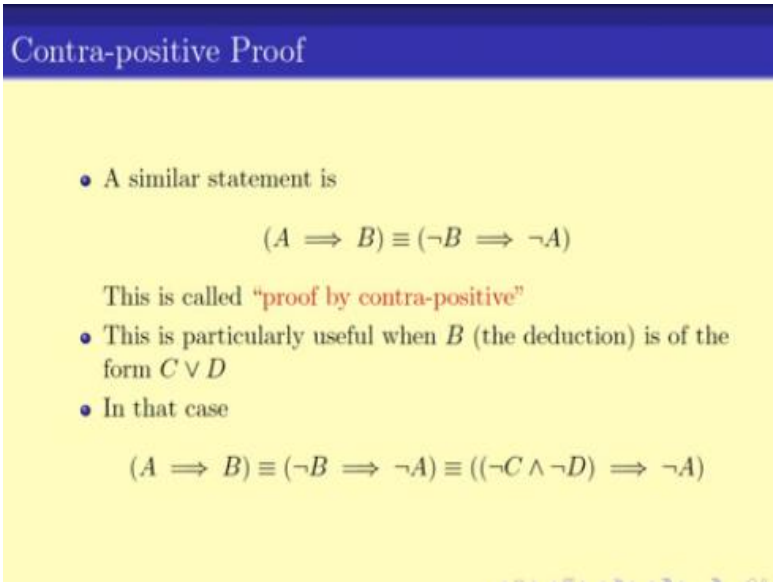


We have of course seen example to all the various problem case studies that we have seen or various proof technique that we have seen till now. Now the third one that we saw the proof technique or proof by contradiction, the idea was that to prove A implies B one can also end up proving not B and A is false, so this is called the proof by contradiction. So there we

assume that B is not true and you work our ways till you get some real statement which is always false.

And very similar statement to this same technique is what to known as proof by contrapositive, then the idea is that proving A implies B is same as proving not B implies not A and sometimes proving not B implies not A can be easier to prove.

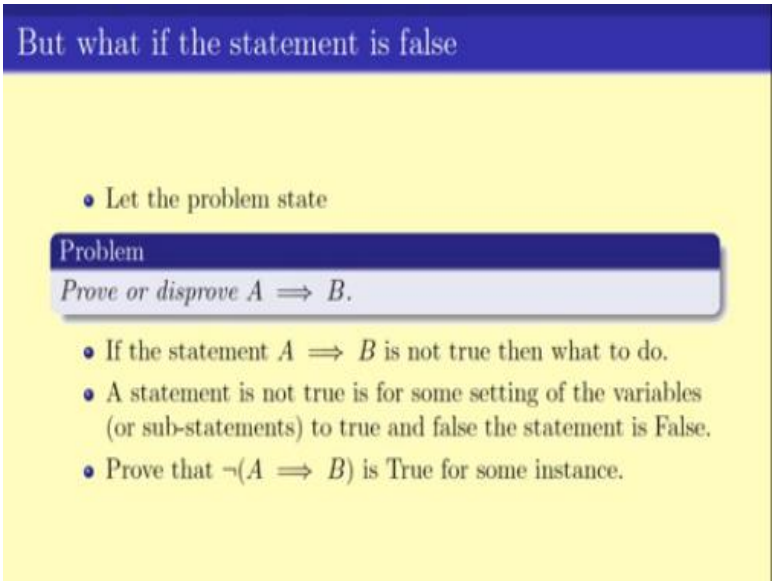
(Refer Slide Time: 07:37)



So in particular, if B can be written in the form of C or D, in that case A implies B, which is same as not B implies not A is same as not B, not C and not D implies not A. So this form of not C and not D implies not A is sometimes easier to prove. One thing notice that, all of these things are similar, all the proof technique are similar in the sense that they all are different ways of writing A implies B.

How all problems can be solved with any of this proof techniques? But it so happens that some of this proof technique is easier to get or work with for certain problems and that is what we are trying to till you. Now this was what we have done till the last week.

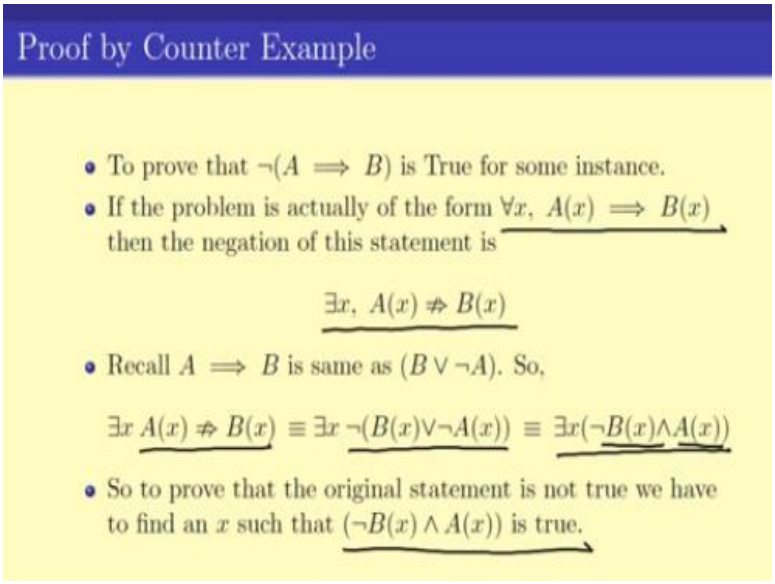
(Refer Slide Time: 08:51)



In this particular video, we will look at case when the statement that we are asked to prove is actually false, for example we can get some problem like prove or disprove A implies B. And let us assume that this technique is actually false, that means A does not implies B. So if A does not implies B or if A implies B is not true then what you will do? A thing to note here that a statement is not true if for some setting of variables to true or false.

The statement is false. Here can I somehow put the variables true and false and get some problem false statement. So in something like this true implies false of this statement. So in other words, we have to prove that if not of A implies B is true for some instance, where in this instance for which we can see that A implies B is not true. So A implies B is not true or otherwise not of A implies B is not true.

(Refer Slide Time: 10:25)

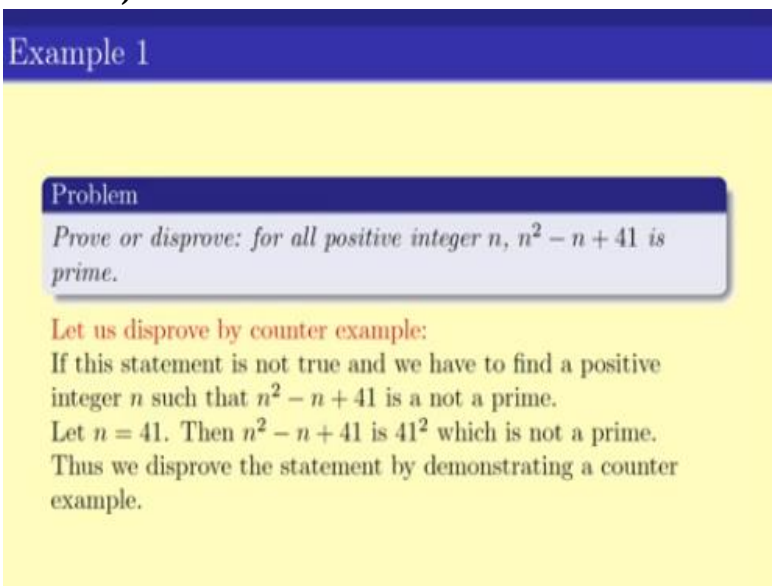


Now also to prove that not of A implies B is true, for some instance how we will go about it. So usually the problems are in the form for all x of their existence something happens. So say the problem is of the form for all x, prove that Ax implies Bx. The negation of this one as we have seen is that, it is their existence and Ax implies Bx opposing, or in other words Ax does not implies Bx. Now what we mean by Ax does not implies Bx? Recall call that A implies B as same as B or not A.

So that means the negation of the Ax implies Bx, or that means Ax does not implies Bx is same as not of Bx or not of Ax. So not of the Bx or not of Ax, which is like (()) (11:43) there exist x and not of Bx and Ax. So in other words we have to produce an x, such that Bx does not hold but Ax holds, right. So this is what we have to prove. So to disprove in other words disprove a statement of the form A implies B, which is the form there exist Ax implies Bx one has to produce at x such that Bx does not hold and Ax holds.

So to prove the original statement is not true, we have to find an x such that this statement is true. And this is what we call proof by counter example, this is one of the few cases where an example can give you a proof. So we will see a couple of examples, problems in this case.

(Refer Slide Time: 13:12)

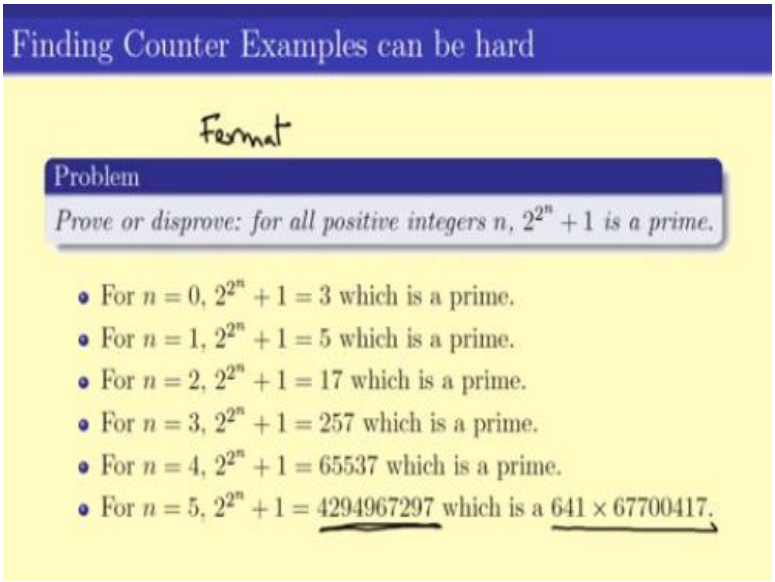


So look at the first problem, this is a problem that we started in our first video only. So further statement is prove or disprove for all positive integer n, n square minus n plus 41 is prime. Now how to disprove it, the way to disprove it is to produce an n, such that n square minus n plus 41 is not a prime. If this statement is not true and we have to find an n such that this number is not a prime.

Now to producing such an n is not necessarily easier object easiest of here. For example, one can try to prove that or one can check that, if I put n equals to one or two or three or four or five or so on, it will always turn out to be a prime. The lowest n for which this number is not a prime is unfortunately as high as 41. So only when we put 41 n equals to 41, we realize that this number is not prime.

Thus we disprove the statement by demonstrating a counter example here which is n plus 2 41. Finally, this counter example and not at all equal and sometime depending on the problem can take a lot of hard work. In fact, there are problems for which the counter example has been found after centuries of hard work.

(Refer Slide Time: 15:04)

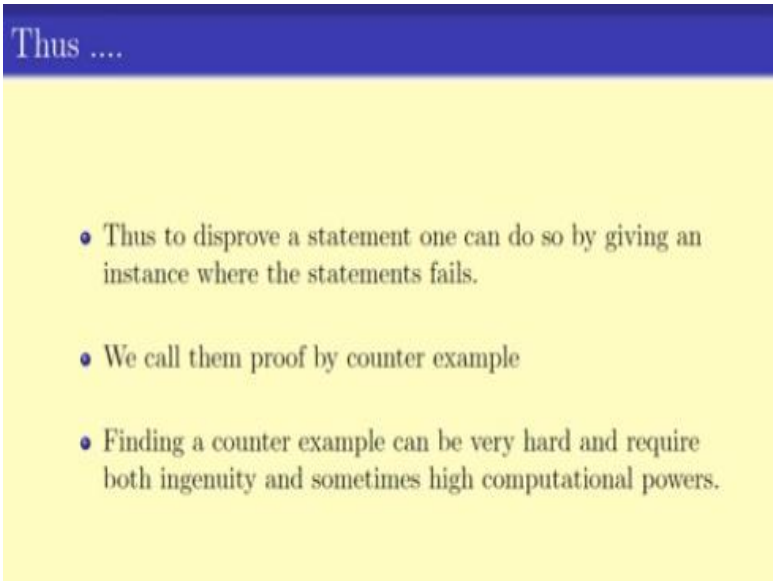


So one such examples is, say, prove or disprove for all positive integers n, 2 power 2 power n plus 1 is a prime. Now these are what are known as the Fermat practice, Fermat has been very famous French mathematician Fermat. He was there over a century ago and we have forced this problem. Unfortunately for n equals to 1 it is not that hard to calculate in over comes to 3 the prime.

For n equals to 1 it is 5 which is a prime, n equals to 2, is the number n, 2 power 2 power n plus 1 becomes 17, which is also a prime. n equals to 3 it is again a prime, n equals to 4 now we already can see the things are becoming very complicated, here it is 2 power 2 power n in other words 2 power 2 power 4 plus 1, which stands out to be 65537 is a prime. Now, what about n equals to 5.

And what it turns out is that n equals to 5 is tremendously big number 4294967297, which is actually not a prime, because it is product of this numbers. As you can see or imagine first of all proving that this number is not a prime, is not at all integer because it's prime factorization is quite complicated. So, to prove or disprove this statement whether 2 power 2 power n plus 1 is a prime is not necessarily what is in, but sometimes this is the only way of proving or disproving the statement.

(Refer Slide Time: 17:28)



Thus, what we can say is that, to disprove a statement one can do so by giving an instance where the statement fails. We can prove, we call this thing proof by counter example and finding a counter example can be very hard and require both ingenuity and sometimes very high computational problem. So this demonstrates one more example of proof technique, namely counter example, a very useful proof technique for proving or disproving a statement.

Note that proving a statement does not, is not necessarily the easiest job and neither is disproving statement. In both of the cases, we need proper proof techniques. Till now we have looked at, the proof by Construction or constructive proof and proof by Contradiction, Contrapositive and proof by Counter example. In the next week, we will continue our study of proof techniques by looking at proof by induction, an extremely powerful proof technique that we will be looking at. Thank you.