Discrete Mathematics
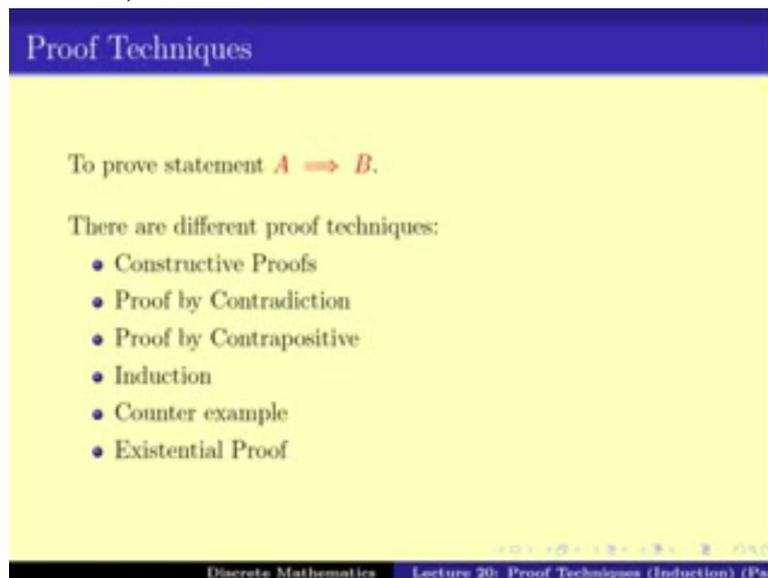**Prof. Sourav Chakraborty**
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Lecture - 20
Mathematical Induction (Part 7)

Welcome everybody. The third video lecture in week 5. Now we have been studying the induction technique.
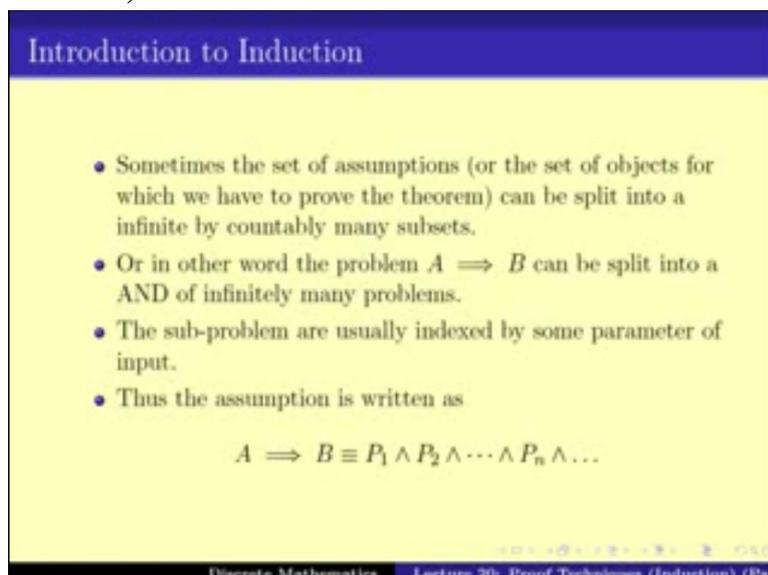
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So till now we have looked at various other two techniques, namely constructive proof, proof by contradiction, proof by contrapositive and so on and we were looking at the very power technique called induction.

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So the idea is of course that if you have a problem we can split up the assumptions into possibly infinite number of subsets and this will in turn split the problem into an AND of infinitely many problems. Now usually the sub problem gets indexed by some parameter of the input and thus the problem A implies B looks like P1 AND P2 and so on till infinity.

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Principle of Mathematical Induction

Problem
For all $k \geq 1$ prove that P_k is TRUE.

- Since there are infinitely many sub-problems one cannot expect to solve all the sub-problems.
- Idea is to solve the first one, namely

  Prove that P_1 is TRUE

- And prove that,

  if for any $k \geq 1$, P_k is TRUE then P_{k+1} is TRUE.

- Then for any $n \geq 1$ the problem P_n is true and hence proved.

Discrete Mathematics   Lecture 20: Proof Techniques (Induction) (Part

So the problem becomes something of this form that for all k, prove Pk is TRUE. Now there are many ways of proving this thing. One obviously is to prove that P1 is TRUE then P2 is TRUE and P3 is TRUE and so on. The only problem is that there are infinitely many sub problems and hence one cannot expect to solve all of them. So we have to come up with a nice technique of solving all the problem in one go.

So the idea is, first prove P1 is TRUE and then assuming the Pk is TRUE prove that Pk plus one is TRUE. Now this you prove this, we expect that we prove Pn for all n, right? So P1 is TRUE, so P1 is TRUE implies P2 is TRUE; P2 is TRUE implies P3 is TRUE, P3 is TRUE implies P4 is TRUE and so on and so forth.

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Principle of Mathematical Induction

$$\forall P, \ [P_1 \vee (\forall(k \geq 1) P_k \implies P_{k+1})] \implies [\forall(k \geq 1) P_k]$$

• There are different versions that one can use.

So it is that, that this does prove for all n is given by this principle of mathematical induction which basically states that proving for all k Pk is basically same as first proving P1 and for all k proving Pk implies Pk plus one. Now this particular mathematical induction, principle of mathematical induction has various defined versions and depending on the problem we might have to apply in usual defined versions.

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Mathematical Induction: Version 1

Problem
For all $k \geq 1$ prove that P_k is TRUE.

Proof using Mathematical Induction:

• **Base Case:** Prove that P_1 is TRUE

• **Induction Hypothesis:** Let P_k be true for some $k \geq 1$

• **Inductive Step:** Assuming Induction Hypothesis prove P_{k+1} is TRUE.

• Then for any $n \geq 1$ the problem P_n is true and hence proved.

Now the first version is of course the one that we just now told which is that if we have to prove that k is greater than or equal to one, prove that Pk is TRUE. Now here we have first proved the P1 is TRUE, we call it the Base Case. Then we have the Induction Hypothesis which states that let Pk be TRUE for some k and the Inductive Step states that okay assuming the Induction Hypothesis prove that Pk plus one is TRUE. And this will prove us the whole.

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Mathematical Induction: Version 2

Problem
For all $k \geq r$ prove that P_k is TRUE.

Proof using Mathematical Induction:

- **Base Case:** Prove that P_r is TRUE

- **Induction Hypothesis:** Let P_k be true for some $k \geq r$

- **Inductive Step:** Assuming Induction Hypothesis prove P_{k+1} is TRUE.

- Then for any $n \geq r$ the problem P_n is true and hence proved.

Now if the problem is not asking us to prove for all k greater than or equal to one but all k greater than or equal to some r, the Induction Hypothesis is basically saying except that we have the Base Case shifted. Namely the base case here becomes proof that Pr is TRUE and the induction hypothesis says the same thing that Pk be TRUE and then prove that Pk plus one is TRUE and that will be for all.

Now sometimes proving that Pk is TRUE in place Pk plus one is TRUE might not be that easy. We might be able to prove Pk is TRUE then Pk plus two is TRUE, that might be easier step to prove.

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Mathematical Induction: Version 3

Problem
For all $k \geq r$ prove that P_k is TRUE.

Proof using Mathematical Induction:

- **Base Case:** Prove that P_r and P_{r+1} is TRUE

- **Induction Hypothesis:** Let P_k be true for some $k \geq r$

- **Inductive Step:** Assuming Induction Hypothesis prove P_{k+2} is TRUE.

- Then for any $n \geq r$ the problem P_n is true and hence proved.

And when that happens then we have to have a different version. And this version says that first we have to prove Pr and Pr plus one is TRUE and for any Pk if we assume it is TRUE

then prove that Pk plus two is TRUE and this would take care of all the points. The main idea as I told in many of the course classes till now is that we have to ensure that all the cases are taken care of. In other words, if this is r and we have to prove for all the things greater than r.

So Pr is TRUE if something makes the Base Case, then in fact it states that Pr plus two is TRUE and Pr plus two is TRUE then Pr plus four is TRUE and so on. But that does not take care of the simple cases, this case further. But that the r plus one case, but that is taken care of by the Base Case here and it is not hard to see that if you prove that Pr and Pr plus one is TRUE then Pr plus two is TRUE.

Since Pr plus one is TRUE therefore Pr plus three is TRUE. Since Pr plus three is TRUE therefore Pr plus five is TRUE and so on and so forth. So in other words, we can pick any version that will cover all the cases and that is the most important thing to take care of in induction. Most of them are very (()) (06:17) in nature. So here we have seen that Pk is TRUE, that Pk plus two is TRUE and then we get the whole thing.

Sometimes unfortunately we might not be able to prove Pk is TRUE implies Pk plus two is TRUE but may be something more stronger is required.

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So here is another technique for that case that if you can assume that Pk and Pk plus one is TRUE, then we can assume the two in the Induction Hypothesis and Pk plus two is TRUE. Again by the same argument we should be able to cover all the cases. Note that, here the most

important thing is that which version to take depends up on what kind of Inductive Step I can call solve and depending on that, this Induction Hypothesis gets solved.

Depending on that the base case is solved just to ensure that all the cases are solved. So usually this is the way of choosing a particular recurrent version. First see what can you prove and then that would make sure various base cases and so on. Sometime looking at the base case is and the smaller cases may also lead you to choosing the right version. We would be seeing a reasonably complicated problem today in this video lecture and we will see how to choose the right version.

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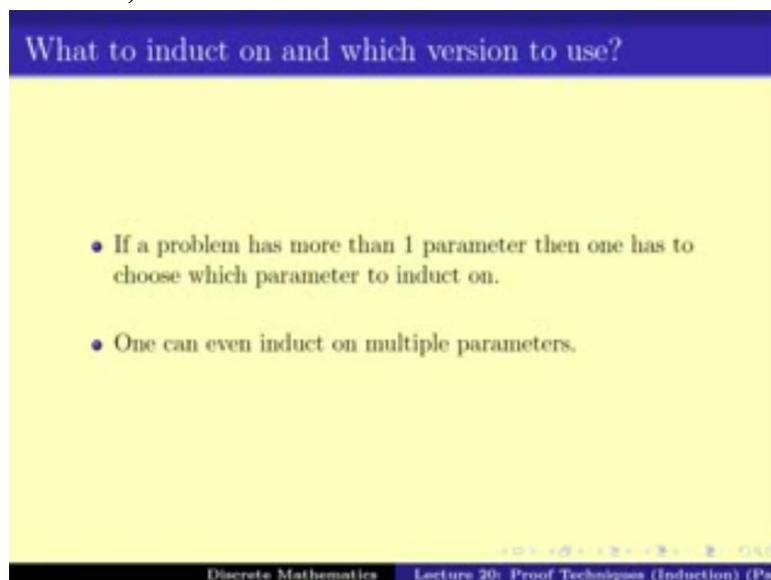
Now other than this sole version, one can come up with many other versions, but then there is also the question that if a problem has more than one parameter, then which parameter to induct on. And we see in the last video that one can also induct on the multiple parameters.

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So in fact this is the one on the third step is that if we have to prove that for all p, q, some problem that is parameterized by p, q is TRUE then one way of going about it is may be first prove that 1,q is TRUE and assuming that p, q is TRUE prove that p plus one comma q is TRUE and this would take care for all p, q. Now again here just like the earlier cases the idea is to ensure that all the possible cases are covered.

Here all the possible cases, are two dimensional integer grid and not just the integers. And so that has to be taken care of. And there can be various ways of covering all the cases.

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So here is one more thing that we saw here, first assume that P one comma q is TRUE and p of p comma one is TRUE and then if we have p comma q is TRUE then p plus one or q plus

one is TRUE. Convince yourself that again this technique covers all the case. By doing so for all the p, q greater than or equal to one we have P of p, q is TRUE.

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Different Versions for two dimensional Induction: Possibility 3

Problem
For all $p, q \geq 1$ prove that $P_{(p,q)}$ is TRUE.

Proof using Mathematical Induction:
- **Base Case:** Prove that $P_{(1,q)}$ and $P_{(p,1)}$ is TRUE for all $p, q \geq 1$
- **Induction Hypothesis:** Let $P_{(p,q)}$ for all p, q such that $p + q \leq k$ be true.
- **Inductive Step:** Assuming Induction Hypothesis prove $P_{(p',q')}$ is TRUE when $p' + q' = k + 1$.
- Then for any $p, q \geq 1$ the problem $P_{(p,q)}$ is true and hence proved.

Discrete Mathematics   Lecture 20: Proof Techniques (Induction) (Part

A third version that we saw an application of last class was again the same base case but where they induct not on p or q but p plus q. In other words, if p plus q is less than k then it is TRUE, then can we prove that if p prime or q prime is k plus one then it is TRUE. So here we induct on p plus q and again it shows that it can be a valid Induction Hypothesis or version of induction.

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Different Versions for two dimensional Induction: Possibility 4

Problem
For all $p, q \geq 1$ prove that $P_{(p,q)}$ is TRUE.

Proof using Mathematical Induction:
- **Base Case:** Prove that $P_{(1,q)}$ and $P_{(p,1)}$ is TRUE for all $p, q \geq 1$
- **Induction Hypothesis:** Let $P_{(p,q)}$ for all p, q such that $\min p, q \leq k$ be true.
- **Inductive Step:** Assuming Induction Hypothesis prove $P_{(p',q')}$ is TRUE when $\min p', q' = k + 1$.
- Then for any $p, q \geq 1$ the problem $P_{(p,q)}$ is true and hence proved.

Discrete Mathematics   Lecture 20: Proof Techniques (Induction) (Par

Page one more that we saw where we look at the minimum of p, q and induct on minimum of p, q.

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- If a problem has more than 1 parameter then one has to choose which parameter to induct on.

- One can even induct on multiple parameters.

- Choose a version of the Induction Hypothesis which will help us solve the inductive step.

- And, ensure that all the cases are covered.

Now given that we have so many versions that are available to us and possibly one can come up with many other versions, let us go back on the same question that we asked. What to induct on, which version to use? A problem again one multiple parameters, which parameters to induct from, one can of course induct on multiple parameters. The main idea is you should always choose a version of inductive hypothesis which help us solve the inductive step and ensure that all the cases are covered.

So these are the two most important things. Choose the Induction Hypothesis that help us solve the Inductive case easily and to choose the Base Case and so on to ensure that all the cases are covered. So why it is not necessarily a very easy thing to do, what is the right person to choose? But you can have some better feel of it by doing a lot more problems.

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Example: AM-GM Inequality

Problem

For all $n \in \mathbb{N}$ and all positive real numbers $a_1, \ldots, a_n$ prove that

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \ldots a_n}$$

So in today's video let us look at this very interesting problem. It is called the AM-GM Inequality. You might have seen this particular inequality stated in your class 12 or class 11 books. What it states is that for all n and if I give you any n positive real numbers then the average, the summation of ai by n is greater than or equal to the n th root of product of a1 to an.

This one is known as the arithmetic mean which is AM and this one is known as the geometric mean which is GM and the problem says that arithmetic mean is greater than or equal to geometric mean. I am sure this was stated in some books or some course in your class 11 or class 12 but it is unlikely that you have seen a proof of this one. So in this video we will see how to solve or approve this statement.

Now, one way of trying to solve any of this problem is by induction, but again what version, which version of induction to choose, what parameter to choose from. As you can see here, there are many parameters although there is n, there is this ai and so on, which parameter to use from.

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So in choosing the correct version, the first thing to note is that identify parameters on which one can induct on. And the most crucial thing is that you can only induct on a parameter that is an integer. You cannot induct on a mere number. So the parameters on which to induct must be an integer, right? So in the AM-GM Inequality there is only one integer which is the n, the number of ai.

So it is kind of obvious there that we have to induct on that particular n. Now a good way of starting it is always try the initial cases. Meaning case when n equals to one, two, three, four, five and so on. And this would possibly help to identify the correct version to use. And then let us try to prove the kth case by using whatever other cases we need to assume. It can be proved when n equals to eleven by choosing whatever is.

And that will also help us to solve the problem. At the end of the day we have to ensure that all the cases are answered or covered.

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Example: AM-GM Inequality

Problem

For all $n \in \mathbb{N}$ and all positive real numbers $a_1, \ldots, a_n$ prove that

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq \sqrt[n]{a_1 a_2 \ldots a_n}$$

We should induct on n.

So the P_k would be for all positive real numbers $a_1, \ldots, a_k$ prove that

$$\frac{a_1 + a_2 + \cdots + a_k}{k} \geq \sqrt[k]{a_1 a_2 \ldots a_k}$$

Discrete Mathematics   Lecture 20: Proof Techniques (Induction) (Part

So for in this problem, it is kind of obvious that we have to induct on n, good? So if we induct on n it is not hard to say that the Pk or the kth sub problem is for all positive real numbers, a1 to ak proves that a1 plus a2 till ak by k is bigger than or equal to kth root of a1 times a2 times dot dot till ak, right?

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Example: AM-GM Inequality

If P_k is for all positive real numbers $a_1, \ldots, a_k$ prove that

$$\frac{a_1 + a_2 + \cdots + a_k}{k} \geq \sqrt[k]{a_1 a_2 \ldots a_k}$$

Problem

Prove P_k is true for all k.

Can we prove the initial cases: $k = 1, 2, 3, \ldots$

And thus we have to prove that for all k, Pk is TRUE. So this is the problem that clearly speaks up into these cases or these various sub problems. Now we should try to see whether we can solve this one for smaller values of k? Can we prove these initial cases, like k equals to one, k equals to two, k equals three and so on.

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Example: AM-GM Inequality

$k = 1$ To prove: $(a_1/1) \geq (a_1)^{1/1}$. Obvious

$k = 2$ To prove: $(a_1 + a_2/2) \geq \sqrt{a_1 a_2}$.

$(\Rightarrow) \quad \dfrac{(a_1 + a_2)^2}{4} \geqslant a_1 a_2$

$(\Leftrightarrow) \quad a_1^2 + a_2^2 + 2a_1 a_2 \geqslant 4 a_1 a_2$

$(\Leftrightarrow) \quad a_1^2 + a_2^2 - 2a_1 a_2 \geqslant 0$

$(\Leftrightarrow) \quad (a_1 - a_2)^2 \geqslant 0 \qquad \text{This is known } a_1, a_2 \in \mathbb{R}$

So let us start with k equals to one. And k equals one we have to prove that a1 by one is greater than a1 power one by one and this is completely trivial. It is fairly, if you see that it is obvious. Now let us go the second one, k equals to two. Already things are pretty interesting. How we will prove that a1 plus a2 by two is bigger than or equal to square root of a1 times a2? So this was something that was even in your assignments and quiz.

So like to prove it here again. So the idea is that to prove this statement it is same as saying I require on both sides, I guess a1 plus a2 whole square by four is bigger than a1 a2. And now if I open up the a1 plus a2 whole square I get a1 square plus a2 square plus twice a1 a2 is bigger than or equal to let me take four on this side, so I get four a1 a2. So this is same as a1 square plus a2 square.

I am going to take this four to the left, I get minus of a1 a2 greater than zero and this is of course fine because this is a1 minus a2 whole square greater than zero and since a1 and a2 are real numbers, so a1 minus a2 is a real number. So a1 minus a2 whole square must be greater than zero and this is TRUE as a1 and a2 are real numbers. I hope you have proved this one yourself earlier.

So this also was the application of the backward proof that we have talked about when we talked about the direct proof techniques. So in other words k equals two follows not to hard way. Okay, that is one good thing. So we know how to prove the k plus.

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Example: AM-GM Inequality

$k = 1$ To prove: $(a_1/1) \geq (a_1)^{1/1}$. Obvious

$k = 2$ To prove: $(a_1 + a_2/2) \geq \sqrt{a_1 a_2}$.

$k = 3$ To prove: $(a_1 + a_2 + a_3/3) \geq \sqrt[3]{a_1 a_2 a_3}$.
Check that this is indeed tough.

Discrete Mathematics   Lecture 20: Proof Techniques (Induction) (Par

Now can you prove the k equals to three case. So in the k equals three case we have to prove that a1 plus a2 by three is greater than cube root of a1, a2, a3. Of course what we can do is that now queue both sides and then try to see. Let us leave it as an exercise to check that this particular case is not at all easy. This is indeed pretty tough. It is not at all if you see that one can prove this case.

So okay, k equals to 3, we do not know how to prove at this point and we still have of course no idea which version to choose.

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Okay, next moving on, what about k equals to four. So in k equals to four we have to prove that a1 plus a2 plus a3 plus a4 by four is greater than fourth root of this thing. So note that a1 plus a2 plus a3 by four can be written as a1 plus a2 by two plus a3 plus a4 by two by two. And this is useful because both these two terms individually looks like the k equals to two.

So we know that a1 plus a2 by two is greater than square root of a1 a2 and a3 plus a4 by two is greater than a3 a4. Thus this quantity meaning a1 plus a2 plus a3 plus a4 by four is greater than square root of a1 a2 plus a3 a4 by two and this is again, this is a real number, square root of a1 a2. This is another real number a3 a4. So I can apply the case k equals to two once again and I get this equal to square root of this two term product with square root of a1 a2.

And a3 a4 and note that square root of a1 a2 times a3 a4 is actually the fourth root of a1 a2, a3 a4 and hence k equals to four is proved. Now this is not the most easiest way of getting the proof. But these are nice problem, what is it? I have used k equals to two in this for proving k equals to four. So in some sense what I proved is that P2 implies P4. Using two k lying through P4 case and I know that P2 is correct.

Now convince yourselves that what you can solve it actually much stronger you can solve that Pk implies P2k. This same proof technique will go through. So thus we have got some progress. Let us see what we have.
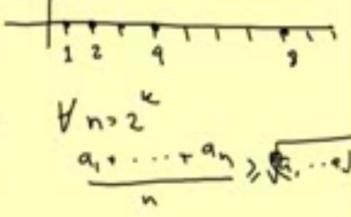
Thus for proving this AM-GM Inequality, we know that k one and two is TRUE, that we know. We have also got that Pk is TRUE then Pk plus two is TRUE, that is P2k is TRUE. But does it cover all the cases? So let us go back to our real line. K equals to one, I know how to prove, k equals to two I know how to prove, now let us see which all cases I know how to prove. Since I know two I know four because of k and 2k.

Since I know four I know eight. Since I know eight I will know sixteen and so on. But still I don't know how to prove k equals to 3. So this is some progress. So what have we proved till now? We have technically proved that for all n which are of the form two power k we can prove that a1 plus till an by n is greater than the product of them power 1 by n, the n th power root of this.

So we know that we have k we have not solved for all n but we are solved for all the powers of two. We know how to call Pk for all the powers of two. But we have now, we had talked about how to improve week 6.

Example: AM-GM Inequality $P_k \Rightarrow P_{k-1}$

$[k = 3]$ To prove $(a_1 + a_2 + a_3/3) \geq \sqrt[3]{a_1 a_2 a_3}$.

We know that for all a_1, a_2, a_3, a_4

$$\frac{a_1 + a_2 + a_3 + a_4}{4} \geq \sqrt[4]{a_1 a_2 a_3 a_4}$$

Let $a_4 = \frac{a_1 + a_2 + a_3}{3}$. Then $\frac{a_1 + a_2 + a_3 + a_4}{4} = \frac{a_1 + a_2 + a_3}{3}$ *check*

Then we have

$$\frac{a_1 + a_2 + a_3}{3} \geq \sqrt[4]{a_1 a_2 a_3 \frac{a_1 + a_2 + a_3}{3}}$$

$$\Rightarrow \left(\frac{a_1 + a_2 + a_3}{3}\right) \geq \sqrt[3]{a_1 a_2 a_3}.$$

So to understand let us go back to the k equals to three case and let us see, if I can prove P3 using any of the other Pk. So I told you earlier that I do not think that this very interesting to prove directly but can we use something else. So at this point what we know is that, we know that for k equals to four, we have the solution. It means we have a1 plus a2 plus a3 plus a4 times four is greater than fourth root of a1 multiplied by a3, a4.

Can I use a equals to 4k to prove the k equals to 3 case? The idea here is that let us apply, let us assume ar equals to a1 plus a2 plus a3 by 3. Now a1, a2, a3, a4 are any real numbers, so I can put anything there. So I put a4 equals to a1 plus a2 plus a3 by three. Note that if I do so then a1 plus a2 plus a3 by a4 by four does become a1 plus a2 plus a3 by three. This is something that you need to check.

And once I have that, that means I have to prove, sorry, not to prove, we already have we probably know in the case of, because we know this thing. So we have a1 plus a2 plus a3 by three is greater than fourth root of a1, a2, a3 time Pk4. This is already good because this left hand side looks like very correct, similar to actually exactly same as the left hand side we have to get, but the right hand side is still bit different.

Here we have a third root and we have a fourth root. But in any case let us now, so this is the case, which means that I can take fourth power in both side and I get a1 plus a2 plus a3 by four, whole power four is bigger than the product of a1, a2, a3 times a4. I am just taking the fourth root power from both sides, we used to power both the sides here. Now once we have that as you can see this is a1 plus a2 plus a3 by four.

Sorry this is actually not four, this should be three, I made a mistake here. This is three, correct? Well it has been the fourth power, right? And this is also a one a two plus a three by three, so this has a fourth power, this has no power. So I can of course therefore write it as in this should be the third power, a1 plus a2 plus a3 by three, the whole cube is bigger than a1 a2 a3.

And once we have that, now we can take the cube root of both sides and we get, this is a third power, a1 plus a2 plus a3 is greater than cube root of a1 a2 a3. And this is exactly what we wanted to solve. Thus we have been able to prove k equals to 3k using the k equals to 4k. Now I leave you with an exercise here that proves that using Pk you can prove Pk minus one is just like where p4 to prove P3 you can use P8 to P7, P7 to prove P6 and so on.

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Okay, so now back to where do we stand. So till now we have the case that for k equals to one and TRUE I know how to solve it. Pk is TRUE then 2k is TRUE, we saw that one and now we have that the Pk is TRUE then Pk minus one is TRUE. Now does it till cover the whole case, all the cases, let us go back to all lines, one, two, three, four, five, six, seven, eight, nine, ten, eleven, twelve, thirteen, fourteen, fifteen, sixteen.

Now let me use red dot to identify which of these things that I have proved. So this again tells that k equals one is done, k equals two is done. Now second one says that okay because two is done therefore four is done. Third one says that if four is done therefore three is done. Since four is done therefore eight is done. Since eight is done by the third rule, seven is done.

Again since eight is done by third rule six is done and six is done again by third rule five is done.

And the second rule eight is done that means sixteen is done, and since sixteen is done therefore fifteen is done, therefore fourteen is done, therefore thirteen is done, twelfth is done, therefore eleven is done, ten, nine and all so on. So in other words this will help us to cover all big bases. It is not the most obviously all covering all the cases, you first cover all the powers of two, then you go backwards.

This is sometimes known as a backward induction because we are not going from k to k plus one necessarily but we are going from k to 2k plus one and then going from k to k minus one. The idea is that this version works perfectly fine because we are able to cover all the cases. So using this pretty interesting inductive technique we can now prove that for all n, AM is greater than or equal to GM.

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So we have seen quite a number of induction version for solving various problems mostly in equations and numbers and so on. But induction can also help and use to solve various interesting combinatorial problems. So we will be seeing one such one example in the next video. Thank you.