Introduction to Probabilistic Methods in PDE
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Lecture 11
Further Properties of Stochastic Integration

In the last lecture, we have seen the definition of stochastic integration with respect to a square integrable continuous martingales where integrand is chosen from the class of progressively measurable processes L star we denote it in this way.

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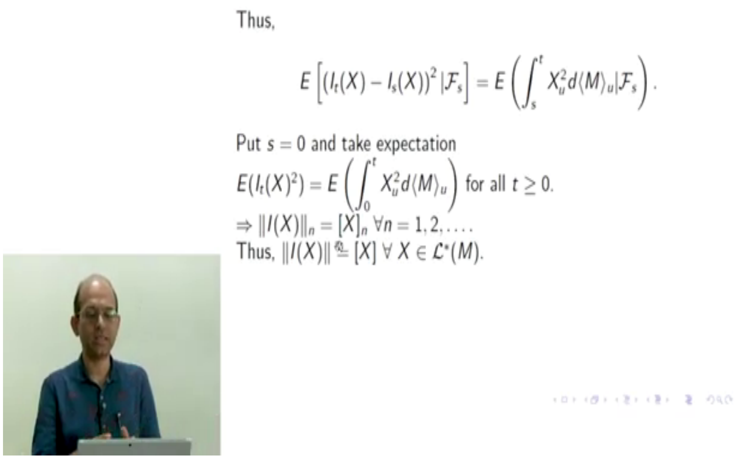

And then we have proved the stochastic integration is an isometry from the space of L star. Okay, from the space of progressively measurable processes and the space of square integral continuous martingale. So, this is that equality which says that okay there I mean this I is an isometry.

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- As X and M are progressively measurable, $\int_s^t X_u^2 d\langle M\rangle_u$ is $\mathcal{F}_t$ measurable for any $0 \le s < t < \infty$.

$$\begin{aligned} E(I_t(X)^2|\mathcal{F}_s) &= E\left[(I_t(X) - I_s(X) + I_s(X))^2|\mathcal{F}_s\right] \\ &= E\left[(I_t(X) - I_s(X))^2|\mathcal{F}_s\right] \\ &\quad + 2I_s(X)E\left[I_t(X) - I_s(X)|\mathcal{F}_s\right] + I_s(X)^2 \\ &= E\left[\int_s^t X_u^2 d\langle M\rangle_u|\mathcal{F}_s\right] + I_s(X)^2 \end{aligned}$$

as $I(X)$ is a martingale. Therefore, by subtracting from both sides

$$E\left[I_t(X)^2 - \int_0^t X_u^2 d\langle M\rangle_u|\mathcal{F}_s\right] = \left(I_s(X)^2 - \int_0^s X_u^2 d\langle M\rangle_u\right) \cdots (*)$$

Thus,

$$E\left[(I_t(X) - I_s(X))^2|\mathcal{F}_s\right] = E\left(\int_s^t X_u^2 d\langle M\rangle_u|\mathcal{F}_s\right).$$

Put $s = 0$ and take expectation

$E(I_t(X)^2) = E\left(\int_0^t X_u^2 d\langle M\rangle_u\right)$ for all $t \ge 0$.

$\Rightarrow \|I(X)\|_n = [X]_n \; \forall n = 1, 2, \ldots$.

Thus, $\|I(X)\| = [X] \; \forall X \in \mathcal{L}^*(M)$.

Next, we consider X and M okay X is from L star M and M is continuous square integrable martingale and then we know that if one has an adapted process which is right continuous or left continuous then that adapted process becomes also progressively measurable okay and X is anyway progressively measurable.

So, X and M both are progressively measurable. And therefore, if we consider the quadratic variation of M that would also be progressively measurable. And then if we consider this integration is not a stochastic integration. This is integration with respect to quadratic variation of M. Since M is progressively measurable the quadratic variation of M is also progressively measurable and therefore, this integration s to t X u square d<M>u is Ft measurable, okay.

So, we have that this integration is Ft measurable okay where Ft is the sigma algebra at time t ok. So, Ft is the sigma algebra at time t. So, now we consider these conditional expectation of square of integration square of the integration at time t. So, you can think that t is future, s is the present okay. So, conditional expectation or square of the future integration from 0 to the future t given in function till from 0 to s okay.

So, that what we do we just add and subtract I s first okay. So, for the integration s t less okay I s X subtract and add and then that is equal to so, we are just using the a plus b whole square formula, we have I t minus I s whole square plus 2 times I s into I t minus I s plus I s whole square okay.

Here this you know the condition is on sigma algebra Fs. So, the integration I s is Fs measurable as I have indicated above, due to this reason that we know that this I s is Fs measurable, so, conditional expectation of I s square X given Fs is I s square itself and here when you have the product two times this I s into I t minus I s, I t minus I s I can take I s outside of the integration because the condition is given with respect to sigma algebra Fs.

Now, here we realize we are going to apply the martingale property of the stochastic integration. Since I t is a martingale so conditional expectation I t given Fs is I s and I s, I s you know they subtract, then you get 0. So, 0 multiplied this so that it contributes nothing, okay. That will be 0 here.

Student: when we will prove I X martingale.

Professor: That we have actually defined stochastic integral as a map from L star to M 2 c. So, since stochastic integral is you know the image of the progressive measurable integrand under integration is square integral continuous martingale. So, integration is martingale here is coming from the definition.

Okay, so here we come back again. So, now this I t X minus I s whole square this given Fs this term appears and this term from the earlier result we know that conditional expectation is equal to this is equal to expectation of integration small s to t X u square the quadratic variation of Mu given Fs. So, let us go back to that result where we have proved already so, this is the result correct, this is the result we have already proved earlier so, we are using this result here okay.

So, now, now we subtract both side by this integration 0 to t Xu square d<M>u, given Fs so that thing we subtract from both sides, so right hand side I have integration for s to t okay but I am subtracting from 0 to t both sides. So, anyway so let us write down say s to t is there, but I am subtracting from 0 to t so I would get minus s 0 to s would remain correct because in 0 to t I can write down this integration 0 to s and s to t this s to t integration would cancel this okay but 0 to s would remain within second negative sign, so, I would get all on the right-hand side this quantity.

And I s X square was already present so this is here okay. So, conditional expectation, expectation of I t X whole square minus integration 0 to t Xu square integration with respect to quadratic variation M given Fs is equal to exactly what is there inside the integration instead of t you are writing s here and if I now consider this whole thing that this random variable, I mean this random process with respect to time t as a Y okay Yt. So, then let me read what is written here conditional expectation of Y t given Fs is equal to Y s ok. So, that is the criteria for martingale okay.

And anyway, so, this quantity has a finite expectation also. So, because I is of L 2 square integral martingale so square is finite expectation this is finite expectation. So, it satisfies the property of martingale therefore, this whole thing inside is a martingale as a function of t okay as a function of t this process, okay I have I t I have I s here is 0 to t is 0 to s. So we named this as star and we are going to use it later.

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On the other hand, $\langle I(X)\rangle$ is defined as A in the following $D-M$ decomposition.
If $I_t(X)^2 = \bar{M}_t + A_t$, then

$$E(I_t(X)^2 - A_t) = E(\bar{M}_t|\mathcal{F}_s) = \bar{M}_s = I_s(X)^2 - A_s$$

Such A is a unique adapted natural increasing process with $A_0 = 0$.
Thus by comparing this and 9(*)

$$A_t = \int_0^t X_u^2 d\langle M\rangle_u.$$

Hence

$$\langle I(X)\rangle = \int_0^{\cdot} X_u^2 d\langle M\rangle_u.$$

- As X and M are progressively measurable, $\int_s^t X_u^2 d\langle M\rangle_u$ is $\mathcal{F}_t$ measurable for any $0 \le s < t < \infty$.

$$\begin{aligned}E(I_t(X)^2|\mathcal{F}_s) &= E\left[(I_t(X) - I_s(X) + I_s(X))^2|\mathcal{F}_s\right]\\ &= E\left[(I_t(X) - I_s(X))^2|\mathcal{F}_s\right]\\ &\quad + 2I_s(X)E\left[I_t(X) - I_s(X)|\mathcal{F}_s\right] + I_s(X)^2\\ &= E\left[\int_s^t X_u^2 d\langle M\rangle_u|\mathcal{F}_s\right] + I_s(X)^2\end{aligned}$$

as $I(X)$ is a martingale. Therefore, by subtracting from both sides

$$E\left[I_t(X)^2 - \int_0^t X_u^2 d\langle M\rangle_u|\mathcal{F}_s\right] = \left(I_s(X)^2 - \int_0^s X_u^2 d\langle M\rangle_u\right)\cdots(*)$$

On the other hand, this quadratic variation of I(X) that I have not discussed earlier, I have discussed the definition of quadratic variation. Now, that we have defined for square integral martingales. And here I(X) is stochastic integration of X with respect to square integral martingale, however, I(X) itself is also square integral martingale continuous square martingale.

So, I can talk about it is a quadratic variation also. So, if I talk about quadratic variation of I(X) what is it? It is the it is defined to be the process the increasing natural process A which appears in the Doob–Meyer decomposition of I(X) square, of the I(X) square. So, if I(X) square is written as M bar t a new martingale okay plus A t.

Okay, that is the result of the Doob–Meyer decomposition where A is chosen to be a natural process, then it is a unique decomposition and then this A t is the quadratic variation of I(X). So, then A t is exactly this. So, let us see that. So, now if I subtract A t from both sides, I get I t square minus A t and then then you take expectation then right hand side okay given Fs.

So, this is missing so I should have written conditional expectation of this given Fs is equal to expectation of M bar t given Fs. But that is that is Ms M bar s and M bar s from this definition is I s square minus A s okay so I again have that conditional expectation this given Fs is I s square minus A s. Now, we can compare this thing with this star here also have I s I t squared minus something okay and this thing.

Okay so we have such A is unique adapted natural increasing process with A 0 equals to 0. So, now by comparing we obtain that this process A t is exactly equal to this process so this is A t okay so you have identified what is A t okay, so A t is this or A t is nothing but the quadratic variation.

So, we have obtained the expression of the quadratic variation of the stochastic integral. So, quadratic variation of the stochastic integral is obtained by just taking the square of the integrand and integrating that with respect to quadratic variation of the martingale. So, what you obtained is a quadratic variation of the stochastic integral.

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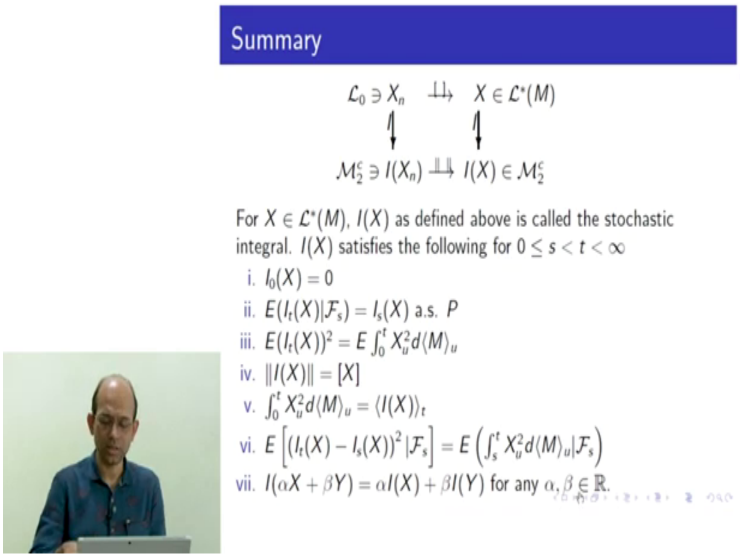

Now, we summarize whatever we have discussed till now. So, here given a stochastic process X which is progressively measurable this is the class of the progressively measurable processes, we have shown that there exists a sequence of simple processes we call X n which converges to the progressively measurable X in the box norm Okay.

And then for each and every X n, which is simple process we can define a to stochastic integral in a very intuitive manner, okay because simple process has a only you know on some time interval it has some bounded adapted value and then further the integration has a very natural meaning.

We define some integration of X n in that fashion and then this integration okay becomes again a square integral continuous martingale and then we have shown that okay this integrals okay as the X n converts to X this integral also convergence, converges somewhere okay we name this as I(X) okay to converges somewhere in Mc2. How do you do that? We actually show that Since Xn converges to X is Cauchy and since I is an isometry. So, I(Xn) is also Cauchy in this you know parallel norm in M c 2.

And then the limiting process what we obtain, we call that as the stochastic integral X. To call that as stochastic integration of X, we had to clarify that this limit does not depend on the choice of Xn okay and that we have justified and that is how we have obtained this integration okay integration of X.

So, that was the summary of definition of stochastic integration of X with respect to a square integral continuous martingale. Now, we list here some important properties, some of which we have already proved, some of which we are just mentioning here, so, for X in L start M that means progressive measurable process, the I(X) is defined as above stochastic integral and then this satisfies these, these properties that for, at time 0 I0(X) is equal to 0.

Conditional expectation of I t given Fs is I s almost surely with respect to probability P, third property is that the I t square the square of the integrals and expectation of that is expectation of, expectation of integration of the square of the integrand with respect to quadratic variation. Next is that norm the parallel norm of I(X) is same as box norm of X the isometry okay isometry property.

Next is so, here we talked about expectation after taking expectation you get this, but without expectation what is it? So, expected without expectation so integration 0 to t X u square d<M>u. So, this is just a random variable for a few fix time t and if you vary t is a stochastic process.

So, this stochastic process is nothing but the quadratic variation of I(X) Okay. And then also this property actually this property was used to prove fifth one that different, the conditional expectation of the square of the difference of the integration is equal to conditional expectation., integration of the Xu square d<M>u.

And then this is a property which you have used earlier also, but we have never mentioned explicitly that stochastic integration is a linear operator, I of alpha X plus beta Y is equal to alpha I(X) plus beta I(Y) for any alpha beta in real okay.

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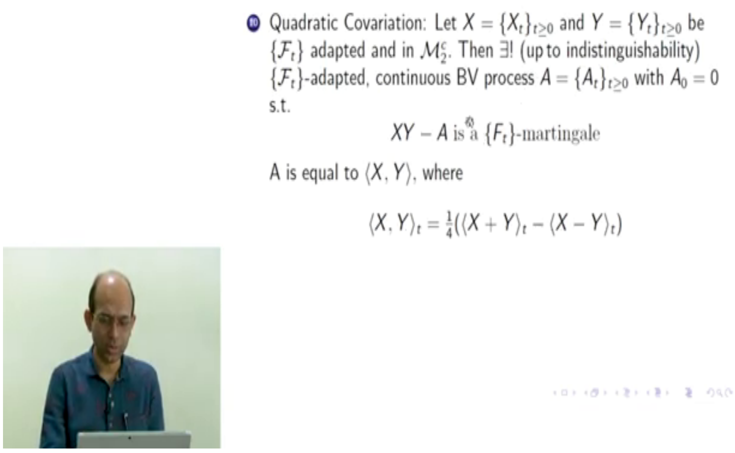

So, next we talk about what do you mean by quadratic covariation. So, let X is a stochastic process X t t is from 0 to infinity and Y is also another stochastic process both are adapted and both are square integrable continuous martingale. Then there exists after indistinguishability on Ft adapted continuous bounded variation process A with A 0 is equal to 0 such that X Y minus A is Ft martingale okay.

So, remember what we have done earlier we had X and Y both same there correct, X square minus A was Ft. So, that thing was the definition of quadratic variation, but here we have two different process we take a product of two processes so Xt into Yt for all t. So, that is X Y. So, this X Y minus A is Ft martingale and then this A is we are going to call this A as quadratic covariation of X and Y. What is this this you can actually also get in this for following formula. This is one fourth of X plus Yt quadratic covariation of X plus Y minus quadratic covariation of X minus Y.

If you subtract these two quadratic variation you also get this. So, the one can obtain this way or one can also obtain in this way both are same. So, that is a result we are not proving these results, we are just stating this result.

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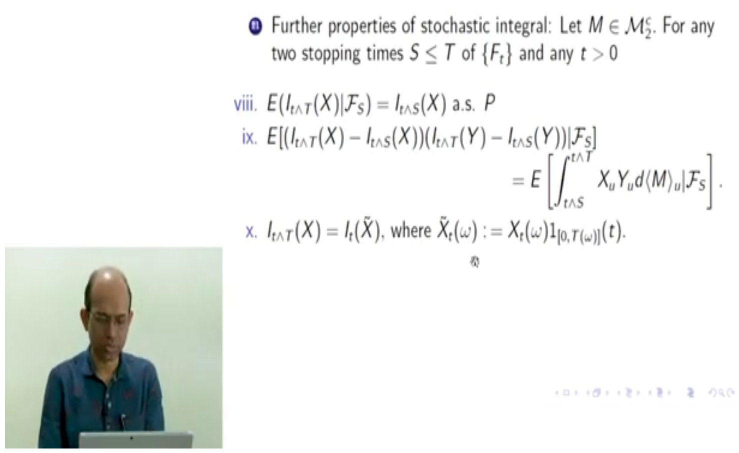

Next, we here quote some other properties of stochastic integral okay. So, these are these properties are the behavior of the integral with respect to the stopping times okay. So, optional sampling theorem these are called okay. So, let M is square integral continuous martingale and S and T are two Ft. So, descriptive okay this is the filtration S and T are two Ft stopping time stopping time with respect to Ft as filtration.

Such that S is less than or equal to T almost surely okay then for any T positive if I take I t means small t minimum capital T that means that the stochastic process I okay I(X) I would integrate from 0 to small t minimum capital T okay. So, what does it mean, it means that you for some certain omega if T omega is I mean if T omega is less than small t, then the process I(X) whatever it is I would just obtain its value at T omega, but if it’s value T omega values is more than t, then I would just observe the value of I t okay.

So, that is a meaning. So, here the meaning of this one should understand that first I have the stochastic process define it is not that okay for every omega I am obtaining t capital T omega and then I obtain that interval and that on that interval I am now integrating no one can do that why cannot one do that because, we have the way we have different stochastic integration we can we cannot define it well for each and every omega okay.

We have not defined that way, we have the defined stochastic integration as a process Okay, which is the limit of some other processes. So, one first obtain the stochastic process, the stochastic integral and after obtaining the stochastic integral, then we are putting this in you know small t minimum capital T omega. That is the way to understand this this symbol, then conditional expectation of stochastic I small t minimum capital T of X given FS is equal to capital I of t minimum capital S of X okay almost surely will be P.

So, this looks like you know martingale property correct. However, here you have capital T and capital S as a stopping time. Next is conditional expectation of this the product of these two differences. What is this that I small t medium capital T of X minus I t small minimum capital S of X.

So, this you can view as an increment correct? You have stopping time S and you have stopping time capital T, S is always less than or equal to capital T. And then during this time interval, what is the increment of a stochastic process, during this two time, what is the increment of this stochastic integral when the integral where the X is the integrand of the integral.

And you can also consider similar increment of stochastic integral where capital Y is the integrand. So, now the increment of the stochastic integral of X and increment of a stochastic integral of Y if you take product of that and you take conditional expectation of the product given FS then what you obtain is that is conditional expectation of X into Y d<M> and this quadratic variation of M given FS.

So, so this thing is actually more general than the one we are which we have obtain earlier, earlier we had say here here we had deterministic X n T and X and Y were all same we had X square here correct, where X square here. So, next we go to this another property condition. This is a stochastic integration of t minimum capital T of X.

So, that is I t of X tilde. So, what is X tilde here. So, the main purpose here is that in this stochastic integral that has a lower limit was 0 of course that upper limit was t minimum capital T so upper limit is a random variable and then and that you can write down as one integration where upper limit is deterministic small t okay but by changing the integrand how are you change the integrand your integrand is X tilde here new integrand, which is X times indicator function of small t and this indicator function of the set 0 to capital t omega so if you do consider this as integrand then you integrate it with respect to M and then you are going to get exactly the same value here.

This intuitive result, but this result also needs a proof we are not going to see the proof but we are quoting it because we might need these results in our following course okay following topics okay.

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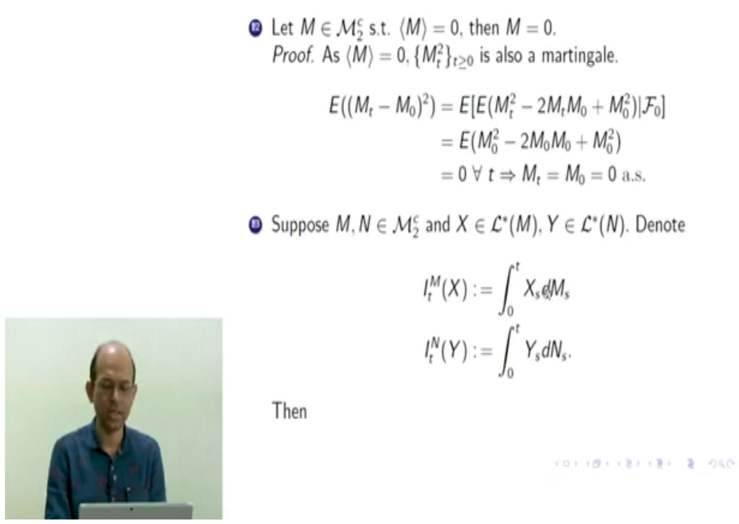

So, next we here quote another result this is a very important result. So, I prefer to give you proof of this result the proof is also not very long. So, the result says that if M is a square integral continuous martingale such that quadratic variation of M is 0, 0 means identically 0 okay so, it stays 0.

Because quadratic variation is a process at time t is equals to 0 of course 0, but then it moves correct and it never decreasing correct it moves it is non decreasing process. But if your M is such that the quadratic variation remains 0 all the time, then one can conclude that M is equal to 0 then it is this is a 0 martingale.

In general if one so here let me clarify one thing that our notation is such that when we say M is MC 2 square integral continuous martingale that is not the complete description this symbol has another more condition which I do not say explicitly that is starting from 0 okay that is there inside that.

Otherwise if you have M as a stochastic as a martingale you add one particular finite constant, then that also becomes another martingale. And then that martingale would have the same quadratic variation of M. So, if M has quadratic variation 0 that martingale would also have quadratic variation 0.

However, that M is just constant okay. So, you can see that that is just constant not 0, but here we are getting 0 because of the definition of MC 2. So, let us see the proof that quadratic variation of M is 0 okay and, and we know that Mt square is also a martingale why is it so because, quadratic variation M is the increasing process in the Doob-Meyer decomposition of Mt square, so Mt square Doob–Meyer decomposition you have this is equal to some martingale but plus increasing but increasing part is 0.

So, therefore, so Mt square itself is the martingale, now expectation of Mt square minus m 0 whole square you can write down the square as a square minus 2 a b plus b square form and then you would obtain that Mt square is a martingale so expectation of Mt square given F 0 is 0 M 0 itself. So, you get M 0 square minus 2 M 0 whole square plus M 0 square cancels everything cancels and you get 0 here.

So, for all t Mt is equal to M 0 and that is 0. Okay if and only if condition that under the condition that M is a continuous square integrable martingale, continuous square integrable martingale then the quadratic variation is 0 implies M is equal to 0 and other side is trivial if it is 0 then M quadratic variation is 0.

So, this side is trivial.

Student: actually it defines a norm

Professor: I mean the reverse side is trivial because M is 0 then quadratic variation is 0.

Student: it defines a norm

Professor: But which norm? Ha, ok. So, basically that is a good point. Actually, one can one can actually take this quadratic variation as norm like correct and then one can this is a L 2 space okay one can make L 2 with that norm actually and the quadratic covariation will be inner product kind of thing.

So, in that appropriate space okay and now. So, let me clarify another thing that if we have a function which is a continuous function then for that continuous function so, if you have a continuous function okay and if that for the continuous function the quadratic variation is nonzero that implies that its total variation is infinity correct that is a result okay.

So, is a quadratic variation is non 0 positive then total variation is infinity and that means that for square for square integral continuous martingale if you take any final intervals say 0 to 1 and then you look at the look at any particular path, whatever the path you choose okay, and then you find its total variation.

You would observe the total variation is infinity with probability 1 if I mean that martingale is not trivial. If you see that quadratic that if you see that okay you do not the total variation is not infinity that is the only case when the martingale is just a constant martingale okay it does not change at all. That means that when martingale changes is not a trivial one, then it fluctuates too much, fluctuates a lot okay. Total variation is infinity with probability one.

Now, we consider two different square integral continuous martingales M and N and we check, we consider X and Y, two different progressively measurable processes, and we denote mean this you know I t this is just a notation which I have also introduced earlier. So, it I am briefly emphasizing that the reason is that the earlier notations I have written I t of X I have not written in the super script anything, but I am writing it because sometimes when we are integrating with respect to two different martingales in the same context in the same you know, same discussion to distinguish between these two integrals.

So, so one has to specify the integrands. So, so we actually write this is very you know standard notation, this notation you know Newtonian notation actually integration 0 to t Xs dMs, integration 0 to t Ys dNs we also write down this way. Okay, that integration with respect to the N martingale and this integration with respect to M martingale.

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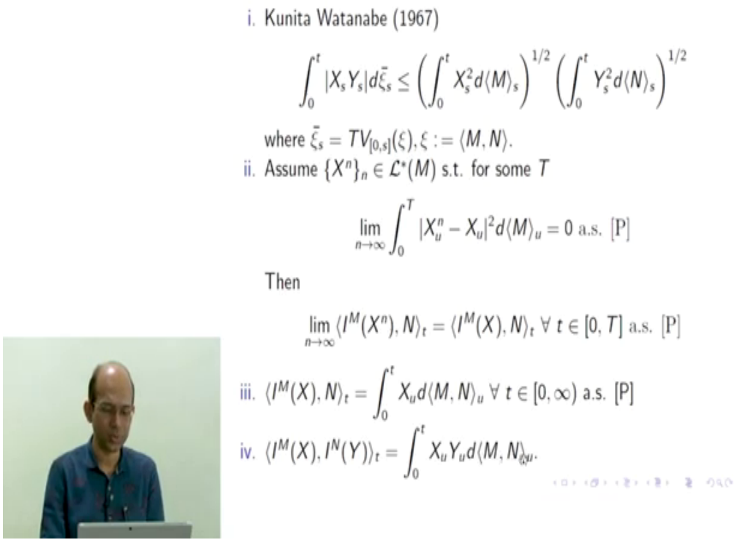

So, then we have this Kunita Watanabe inequality, this Kunita Watanabe inequality says that if I consider the Xi bar what is Xi bar? Xi bar is a total variation process of Xi okay, where where Xi is the quadratic covariation of M and N in quadratic covariation of M and N is actual a bounded variation process, so, total variation of Xi on interval 0 to s would be a finite number with probability 1 and that we call the Xi bar of s.

And if we integrate this mod of XsYs with respect to Xi bar of s. So, there it is interesting correct because here integrand is also product and integrator is also coming from both M and N correct , one can ask that, can he have any relation with respect to the integration of X with respect to M and Y with respect to N.

Yes, there is a relation, so one has this inequality that this you know, this sounds like Cauchy Schwarz inequality type of things correct. So, integration 0 to t X square dMs to the power of half into 0 to t Y square dNs to the power of half. Okay, next to assume that you have a sequence Xn which is progressively measurable L star okay such that for some capital T, we have limit n tends to infinity, 0 to capital T, X n minus X whole square.

So, d<M>u is equal to 0 this limit goes to 0 okay in this norm okay. So, with probability 1. So, if we have such sequences Xn and we have one X in also L star M such that there square of the difference and the square the integration of that with respect to quadratic variation of M that limit goes to 0, then we can actually pass the limit inside what is that the limit n tends to infinity of integration of Xn and M quadratic variation of integration of Xn with respect to M and the martingale N here.

Okay that quadratic variation if I pass to the limit n tends to infinity that converges to I M of X n where you know this X is the limit of Xn in this sense okay. This is actually the sense of the box correct? This is same as the box norm correct.

Student: So, here quadratic covariation is symmetric.

Professor: Yes, quadratic covariation is symmetry from the definition it is symmetry. From the polar identity correct. So, when I have obtained this polar identity.

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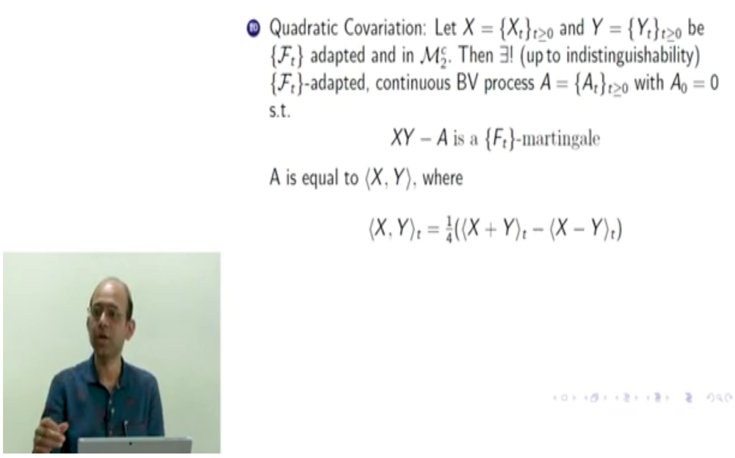

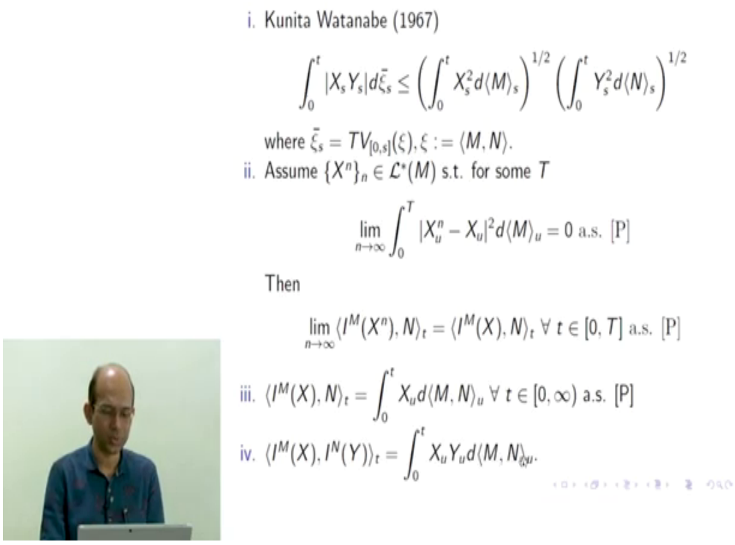

So, here X plus Y but X minus Y is there when I have when we write down Y minus X here. But then in the definition of quadratic covariation, it is a quadratic variation X minus Y and Y minus is the same, why because quadratic variation X minus Y is the Doob-Meyer decomposition X minus Y whole square, so X minus Y whole square is same as Y minus X the whole square so, that is the reason Okay.

Okay, so, we were discussing this then. So, we have already discussed this second point, the third point is that the quadratic variation of integration of X with respect to M and the martingale N is same as integration of X with respect to the quadratic variation of M and N.

So, this is also one just these are intuitive result it seems that okay you are X d, I mean dM was there and N was there and then you are getting it.

So, one can also write down these you know the intuitive manner like you know quadratic variation of M N and D basically d<M,N> one can think that d<M> into d<N> okay, actually the researchers who work in geometrical side of stochastic processes, so, this stochastic geometer’s they actually prefer to use that notation d<M> into d<N> okay.

So, here it is like you know X integration X d<M> and here you have N and the quadratic variation. So, this is integration of 0 to t of d of this thing. So, integration of d of I X is X d<M> and d<N>. So, that is X d<M> into d<N>. Okay so, that intuitive algebra holds here.

And now if you have the quadratic variation of I M X and I N Y and that is also integration 0 to t X Y X into Y into quadratic variation of M N okay. So, is that is the last slide. So, what I was trying to say let me write down here because I do not have board here. So, let us write down what I was trying to indicate here the left hand side.

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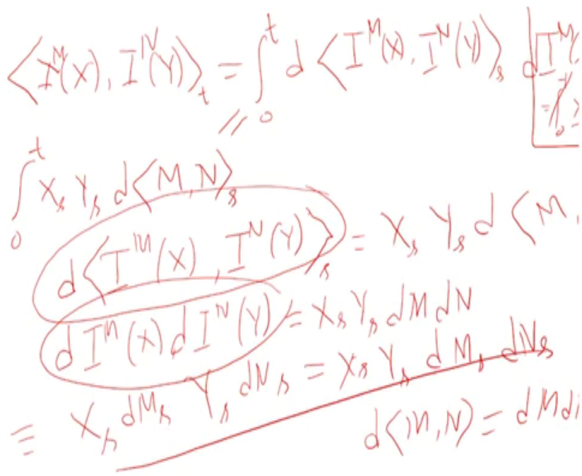

So, left hand side is I of X I of so here M, here N Y t. So, this thing is equal to integration 0 to t d <IMX INY> s and right hand side was 0 to t integration 0 to t, Xs Ys, d< MN >u correct. So, <MN> s so, in that last line of the slide we had this two integrators equal. So, in other words we have d <IMX INY >s is equal to Xs Ys d<MN>s for all s greater than 0.

Now, if you use the notation that quadratic variation d<MN> is d<M> into d<N> allow me to use that notation to say that what is there, so, d I M X into d I N Y is equal to Xs Ys dM dN. However, I M X is integration for all 0 to t process X with respect to dM. So, here we are recollecting IMX or at time t is integration 0 to t of X dM.

So, now d of this okay of t is here you do not have integration X dMt Xtdt. So, here we can from this consideration we can write down this as a Xt dMt and here it is Yt dNt and here what you have obtain is that sorry t is according to my notation here, my running time is not t but s okay.

So, s, s, s, s, so here we have obtained this dMs dNs, so, you understand that okay this is this is a trivial identity, okay, so this is the algebra one can obtain

Student: So how we quote dM quadratic dN?

Professor: Which one, this one this quadratic variation? So, this is I am saying that if you allow me to write down this quadratic variation of two processes, as dM into dN then we get this equal this you know, this identity which is you know, which is trivially true. Okay. So, that also gives the justification that one can write I mean one can think that quadratic variation of two processes is nothing but dM into dN.

Okay, so this is also another reason that why some authors use this notation to you to write down this okay, some authors write down d<MN> as dM into dN. This notation is more popular among the stochastic geometer's. Okay, thank you very much.