Real Analysis - I
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Week 10 and 11
Lecture - 102
Introduction

Welcome to weeks 10 and 11 of this course on Real Analysis. In weeks 8 and 9, we had a whirlwind tour of both differential and integral calculus. Now, you might question what is the use of this theory of differentiation and integration of functions, when I do not have a good tool box of functions available for me? Well, weeks 10 and 11 are to remedy this situation.

We are going to look at the famous functions like exponential, logarithm, cosine, sine etcetera which are the tool box using which physics is conducted for instance in a deeper light. For that we need the machinery of power series. Power series are infinite expressions of the form $\sum a_n x^n$. This looks very similar to a polynomial except there are infinitely many terms, so loosely you can think of it as an infinite degree polynomial.

Because they look like polynomials, you might expect that these power series are also functions that behave like polynomials, but first of all is a power series even a function? If you look at the expression $\sum a_n x^n$other than x equal to 0 it is not clear that the series converges at all.

So, the first order of business is to understand at what points does the power series converge and is there some way by which we can describe the region in the real line on which this series converges. We will undertake this project.

The second question is now that we know that the power series does define a function, does that function defined by the power series behave nicely? Since polynomials are the nicest type of functions, you would expect that power series being nothing, but infinite degree polynomials continue to be nice.

This intuition is really good and it will continue to be true that this function defined by power series has a number of nice properties, but we need to study something called uniform convergence to understand this.

Essentially the convergence of the series $\sum a_n x^n$ is not just point wise. There is something uniform about it and that uniformity is what is going to make the limit function nice.

So, we study uniform convergence at some length especially the facts about the behavior of uniform convergence under both differentiation and integration. You will learn that uniform convergence behaves much better with respect to integration than with respect to differentiation.

So, this is a mistake that students often make they think that the uniform limit of functions, like summation I mean $\sum a_n x^n$ will continue to be as nice by interchanging differentiation and summation.

That is not quite true. We have a precise result that is weaker than what you would expect. Once we have this machinery of uniform convergence and power series, we can go on and define exponential sine, cosine, logarithm etcetera in the usual way.

You have already seen series expansions of exponential and the trigonometric functions in school and we will make that the definition of these functions at this higher perspective.

Once having defined these functions using power series we will prove the basic properties using the results developed in week 9. This will be the. Sorry, week 10 we will define I have used the various properties proved in week 10 to prove the various properties of exponential, logarithm, sine, cosine etcetera in week 11.

Now, one thing I will not do is prove the various trigonometric identities that you are tortured with in 11th and 12th standard. There are literally hundreds of them there is an entire book on trigonometric identities.

This is a fruitless exercise in my opinion, simply because when you eventually get the opportunity to learn complex analysis you will learn that using the complex exponential all of these identities can be proved more or less trivially.

So, you need to know only the very basic properties of the trigonometric functions for your scientific career and I am going to focus on those. What I will do is do a careful analysis of what is pi, how, why does the graph of sin look like what it looks like we can do all this using the tools of calculus that we have developed.

It will also be a very instructive exercise because we will be taking derivatives and second derivatives and seeing where the function is increasing where the function is decreasing. So, it is a fruitful exercise.

So, this sums up the content of weeks 10 and 11. It is not very difficult material. We will be revisiting high school level material from a higher perspective. The only new thing is uniform convergence and power series, which I urge you to pay careful attention to.

All the best for weeks 10 and 11.