Our Mathematical Senses

The Geometry Vision

Prof. Vijay Ravikumar

Department of Mathematics

Indian Institute of Technology- Madras

Lecture-49

Video 10A: introducing the cross ratio

All right, let's introduce the long-awaited cross ratio. So the cross ratio is a numerical quantity that stays the same under perspective shifts. And I want to introduce it through an application. So I'm stealing this application from a Numberphile video, and it's due to Federico Ardila. And it concerns a very historic game of football in which Roberto Carlos scored a winning goal for Brazil, this is back in 1997, with a single kick. So this is a free kick.

And the question is, how far is the ball from the goal in this photograph? This photograph was taken just before the kick happens. And suppose we want to know how far the ball is from the goal, can we figure that out from this photograph? So this is the application of the cross ratio that I want to examine first. And in case you've seen the video, you'll notice that I'm using a slightly different definition of the cross ratio. And also, my use of it is going to be slightly different from what's done in that video.

So this is very much inspired by that, but I want to do it in a slightly different way. So first of all, we need more information in order to solve this problem. And luckily, we have more information. In other words, we can't do it strictly from the photograph. We need to know something more because from this photograph alone, we don't have any sense of what one foot is, or how big these people are, or anything.

But luckily, we do know the dimensions of a football field from a bird's eye view. We know that it's 5.5 meters from this first line to the next line. And this is 11 meters. And finally, we can just call the distance from the ball to this line x meters.

So drawing those lines on the photograph, we can take some measurements and get even more information. We can look at these corresponding lines in the photograph, and we can measure the pixel distance between them. So it's 50 pixels between these lines, 120 pixels between these lines, and 290 pixels between these lines. So in other words, we have meters on the one hand from this image, and we have pixels on the other hand from this image. And we also know that these represent the same things.

This is a photograph of this. But unfortunately, the ratios are clearly not preserved. So 11 is double of 5.5. But clearly, 120 pixels is not double 50 pixels.

So we can't solve for x by assuming that the ratio of x to 11 is 290 over 120. That's not going to work. We do know that this is a photograph of this. So these four points on the right hand side are related to the four points on the left hand side via a perspective. So if we call this A, B, C, and D, we know that although these ratios are not preserved, there is a certain very special ratio of ratios that is preserved.

Now just to be clear, not any ratio of ratios will be preserved. This is a highly specialized one that we're going to introduce now. And that's known as the cross ratio. So given four distinct collinear points, A, B, C, and D, we can define their cross ratio, ACBD, to be AB over BC divided by AD over DC, where all of these segments are signed lengths. So what do I mean here? AB is the distance from A to B.

BC is the distance from B to C. AD is the distance from A to D. But DC is the signed distance. These are all signed distances. But DC is going to be negative because we're going in this direction.

So you have to make a choice of preferred direction here. So let me just make the conventional choice that going to the right is our preferred direction. So that's why AB is positive, BC is positive, AD is positive, but DC is negative because it's going against the preferred direction. So that's what I mean by a signed distance. And what do we get as our.

.. So our cross ratio is literally this ratio divided by this ratio. But it's a bit strange, AB over BC. We have AB over BC all divided by AD over DC. So how do we remember this? It's a bit confusing. So here's one mnemonic device for remembering it.

I just mean a method of remembering it. Don't read too much into it. It's just something that I sometimes use to remember it myself. So again, we're looking at AB over BC. So what I'm doing here, my points are A, B, C, and D on the line.

But I'm going to draw them on a circle instead. I'm going to draw them around a circle.

And then to remember this formula, I just say, OK, it's AB over BC. So that's AB over BC divided by AD over DC, which is AD over DC. So I'm just going around the circle in this direction and then in this other direction.

So AB over BC all over AD over DC. And I'll write it over here. AB over BC all divided by AD over DC. So I'll go around in this direction, go around in this direction. This is, again, just a mnemonic device.

It's just a method of remembering it. So don't read too much into the circle. And then just compute this, keeping in mind that it's a signed distance. So DC, in particular, is a negative number. And I want to mention that these points, A, B, C, and D, can lie in any order on this line.

And the same definition will hold. So let's just use it. It's a bit confusing in the abstract, but when we actually start applying it, it'll make more sense. So what's the cross ratio of these four points? I'm just taking them out from the photograph, laying them out here. 50 pixels, AB is 50 pixels, BC is 120 pixels, and CD is 290 pixels.

So let's use the cross ratio to find where the ball is. So the cross ratio, I've written it out here. AB over BC divided by AD over DC. So now I can flip this second fraction and write this as a multiplication. So this is AB over BC times DC over AD now.

And now let's just substitute stuff in. AB is 50 pixels. BC is 120 pixels. DC, you have to add these up, 50 plus 120 plus 290.

Oh, no, no, I'm sorry. DC is negative 290. And finally, AD, adding all these up, is 50 plus 120 plus 290. That's 460. So we get this quantity here. Multiplying it together, we get this quantity, negative 14,500 over 55,200.

Put that in your calculator, and you get negative 0.26. Fine, it's a number, nothing special. What's the big deal? How is this going to help us? Well, the way it's going to help us is that, OK, we get this number based on the ratios of these lengths.

And this is just from the pixels. But we're going to prove soon that the cross ratio is preserved under projectivities. So the cross ratio of the corresponding points, A prime, B prime, C prime, and D prime from that aerial bird's eye view of the football ground, that's also got to be negative 0.26. It's preserved. It's invariant under changes in perspective.

So now let's look at these same, let's look at the points A prime, B prime, C prime, and

D prime. We know that this distance is 5.5 meters. This is 11 meters. And this is the distance from the ball to that first line.

That's going to be x meters. So the cross ratio, A B over B C divided by A D over D C, in this case we'll apply it to A prime, B prime, C prime, and D prime, and do the same calculation. We first flip this second fraction to get D prime C prime over A prime D prime. And then substitute in A prime B prime is 5.

5 meters. B prime C prime is 11 meters. D prime C prime, that's going to be a negative because it's a signed distance and it's going in the wrong direction. So that's negative x meters. Finally, the sum A prime up to D prime is 5.5 meters plus 11 meters, 16.

5 meters, plus x meters. So we get that. So multiplying that together, we get negative 5.5x over 181.5 plus 11x. And that has to be equal to negative 0.

26 by the invariance of the cross ratio. Since A prime B prime C prime and D prime are related to A B C and D via a perspective, via a photograph, we better get the same cross ratio from these points. So now let's solve this equation for x. Multiplying everything out, we get that 5.

5x is equal to 47.19 plus 2.86x. I'm literally just multiplying this out and grouping the terms. Rather, once I group the terms and solve for x, I get that x is equal to 47.19 divided by 2.

64. Putting that in the calculator, we get 17.9 meters. So we get an actual meter value for x, x being, again, this distance from C prime to D prime. It's the distance from the ball to that first line here.

So that's 17.9 meters. But we know that the rest of the stuff from this line up to the goal, the rest of that distance, is 16.5 meters, 11 plus 5.5.

So the total distance from the goal is 17.9 plus 16.5, which is 34.4 meters. So we get an actual distance in meters.