Approximate Reasoning using Fuzzy Set Theory

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Lecture – 16
Fuzzy Implications

Hello and welcome, to this first of the lectures, in the fourth week of the course titled, approximate reasoning using fuzzy set theory. A course offered over the NPTEL platform. In this week we will look at yet another important fuzzy logic connective, that of fuzzy implications, it is important by itself also in the context, that we are looking at in this course that of approximate reasoning using fuzzy set theory.

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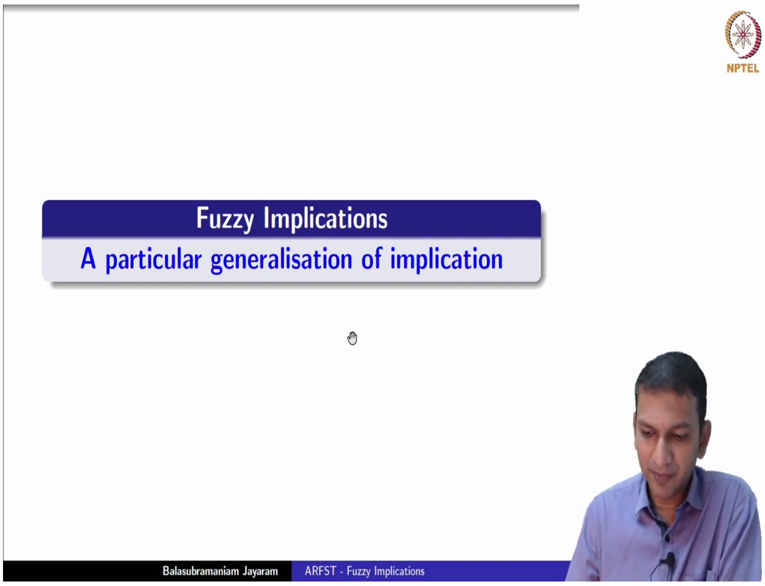

We will look at one particular generalisation of an implication that is from the classical implication to the fuzzy implication.

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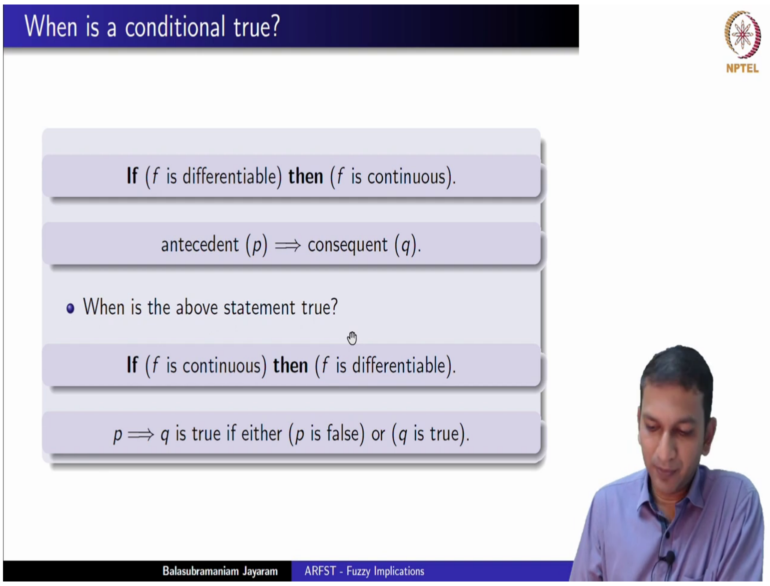

To understand this, let us look at, when is a given conditional true? Let us consider this statement, if f is differentiable, then f is continuous. Of course, we know that this statement is true, but now looking at it as an antecedent implying a consequent that is like a p implies q, the logical sense, we are wondering, we are questioning ourselves, when is p implies q true?

Let us once again, consider this statement, if f is differentiable, then f is continuous.

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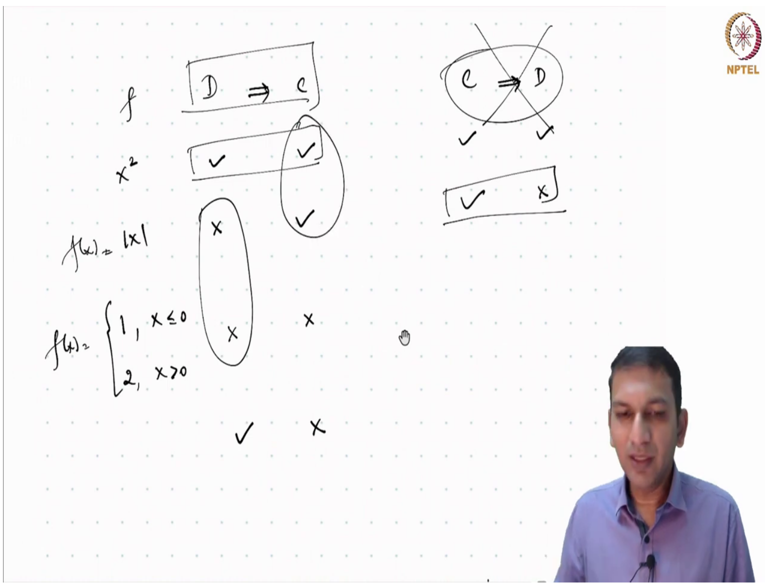

So, now, let us pick up a few examples of functions f. Consider for just the function x square. We know, this is differentiable and this is also continuous. Let us look at the function mod x. We know that, this is not differentiable at the point 0, the at the origin, but; however, it is continuous on its entire domain, which is r. So, now, given these two examples, the first example, it reinforces the conditional, that whenever it is differentiable, it is continuous.

The second example, does it contradict, the statement that f is differentiable implies f is continuous, no it does not contradict; because, the function itself is not differentiable. Let us consider another function f(x). So, it is 1, if x is less than or equal to 0 and 2, if x is greater than 0. Now this function, clearly is neither containing differentiable nor continuous. Now does this example contradict the statement differentiability implies continuity? Once again it does not.

So, only thing that could contradict, the statement is if you could have a function, which is differentiable, but not continuous; However, we know that such a function or r general does not exist. Now let us consider the in some sense, the conversion of this one statement. If f is continuous, then f is differentiable; that means, now we are looking at the statement, C implies D. Once again let us consider the function x square, we know it is continuous, we know it is differentiable.

So, it appears that the statement, continuity implies differentiability, looks true, given this example x^2; however, let us consider the next example, f(x) is equal to |x|. It is continuous, but as we have seen it is not differentiable at their origin. So, it is not differentiable, over its entire domain of r. This immediately tells you that this statement is no more true; that means, we have an example where the antecedent is true, but the consequent is not true.

Thus, this conditional statement is actually false. So, if you look at it, if you ask the question when is p implies q true, we see that whenever p is false, it is true because immediately the antecedent of the conditional statement does not come into picture or it is true, if the consequent is true. So, that is how we have captured it p in place q is true if either p is false or q is true.

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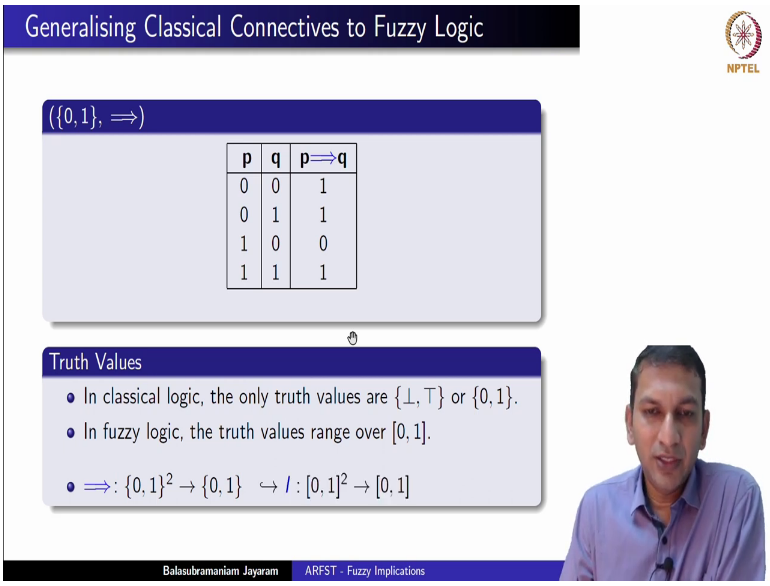

Now, if you are looking to generalize a classical connective to fuzzy logic connective, we have seen that we have to maintain the truth table of the corresponding classical logic connective. On the {0, 1} set, the truth values of the classical logic connectives, if you take the implication, then the truth table of the implication looks like this; sorry not once again that p and q we can take either 0 or 1 and p implies q again it can takes it can take either 0 or 1.

Now, this is the truth table of the fuzzy of the classical implication. We know that, in classical logic we have only these two truth values, indicated either with these two symbols or as 0, 1 in fuzzy logic, these truth values range over the entire interval [0,1]. So, now, we need to generalize this connective, the classical implication on the set $\{0, 1\}^2$ to {0, 1}, to the interval $[0, 1]^2$, to [0, 1]. Just like how we have done in the case of t norms negation and other such connectives ok.

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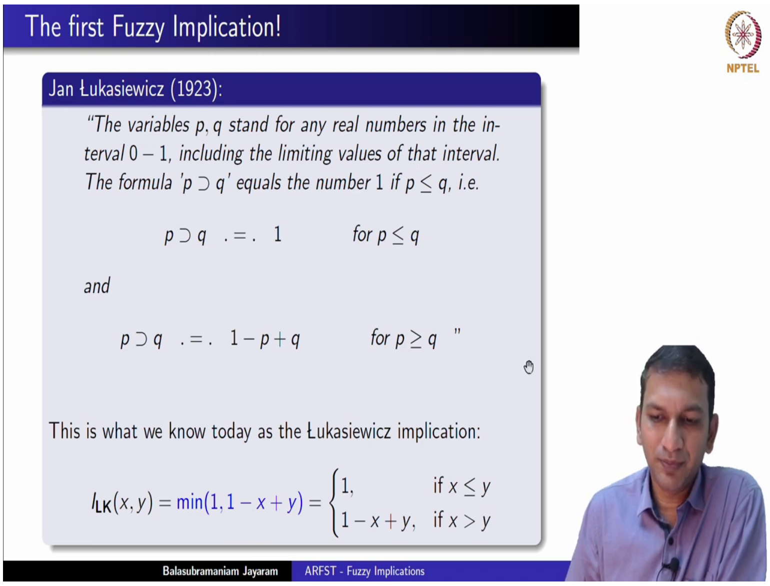

Let us look at what was the very first fuzzy implication that was proposed or what could be considered as the first fuzzy implication that was proposed. It was way back in 1923, almost 100 years from you know before you know today a polish logician Jan Lukasiewicz, proposed this in one of his papers and this is in some sense a good translation from what would have been written in a polish language, wherein he said, the variables p q let them stand for any real numbers in the interval [0, 1] including the limiting values of that interval.

The formula p using the symbol q equals the number 1 if p is less than or equal to q, p that is p what we would you called as implication, p implies q is 1 for p less than or equal to q and it is one minus p plus q, whenever p is greater than or equal to q and you can see that when p is equal to q, they satisfy ok both these conditions. But at that point, they actually coincide it is one.

So, this perhaps was the very first fuzzy implication multivalued implication that was proposed and was proposed almost a century ago together. This is what we know today as the Lukasiewicz implication, in the fuzzy set theory literature. And when you translate it in terms of x, y variables over that unit interval, this is how it looks like. This is one way of writing it this is how Lukasiewicz had written, and it could also be written equivalently in this form right.

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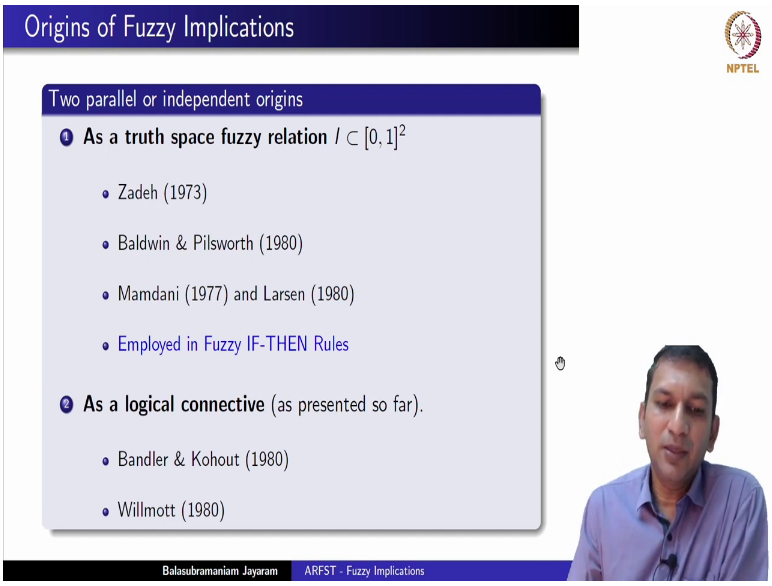

If you were to look at fuzzy implications themselves, say from early 70's or mid 60's when fuzzy set theory started catching up you could perhaps trace back its origins to parallel or independent origins, first as a truth base truths truth space fuzzy relation we have we are yet to see about fuzzy relations, which we will take up in the next week, but as a truth space fuzzy relation, this is how many of the early works on fuzzy implications, especially during the 70's explored this particular connector.

And remember, these were actually employed in fuzzy IF - THEN rules, which play an important role they are an important component, in approximate reason. We have seen that, seen this in some of the early lectures, how conditionals play a role and how we will interpret conditionals capturing knowledge in this course as fuzzy IF - THEN rules. So, they were actually used to capture fuzzy if-then rules.

Similarly, they were also explored, explored as a logical connective like how we have seen about the T-norm so far. This were this was also a way fuzzy implications were explored. So, they have had two independent and parallel origins and exploration, we will of course, take the second approach for the moment, but as you see, that it is it plays a very important role, when we are talking about approximate reasoning using fuzzy set theory.

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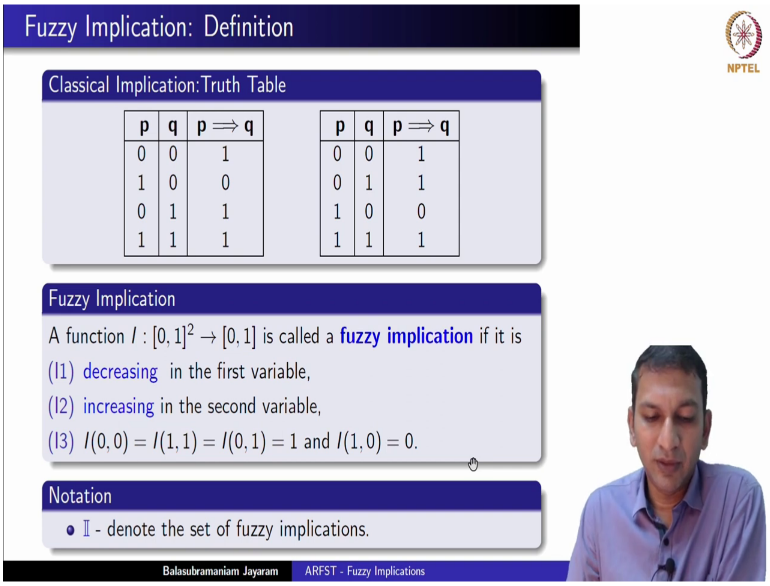

Well, let us try to come up with a generalization from the classical implication to the fuzzy implication. It's the same classical implication truth table, but only just slightly rearranged. So, that we would see how the definition has come into existence. So, we say a function I, the binary function binary operation on the unit interval 0, 1 to 0, 1 we call it a fuzzy implication, if it satisfies these three property. Firstly, it should be decreasing in the first variable.

Now, what has led to this observation. Look at this, if you fix the second variable, to be 0 and increase from 0 to 1, you see that the overall truth value moves from 1 to 0, it actually decreases. If the second variable is fixed at one, and if you increase it remains a constant, but in the case when the second variable is 0, as you move from 0 to 1, what we see is that actually the overall truth value decreases.

So, this is how what has been captured in this as saying that (Refer Time: 11:00) an implication operation is expected to decrease in the first variable. And once again, it is not in the strict sense, essentially it should be non increasing. What about in the second variable? Let us fix the first variable, for instance 0 here and as we increase the second variable from 0 to 1, it remains a constant; however, if we fix the first variable to be 1 and if we move from 0 to 1, we see that it is increasing.

So, this is what is captured as saying that it is increasing in the second variable. Note that, this is in stark contrast to what happens in t norms, where it is increasing in both the variables. Finally, we ask that all these boundary conditions are satisfied. Essentially we are rewriting the truth table in this form I(0, 0) is 1, I(1 ,1) is 1, I(0, 1) is 1, and I(1 0) is 0. Notation here we will use the script I to denote the set of all fuzzy implications henceforth ok.

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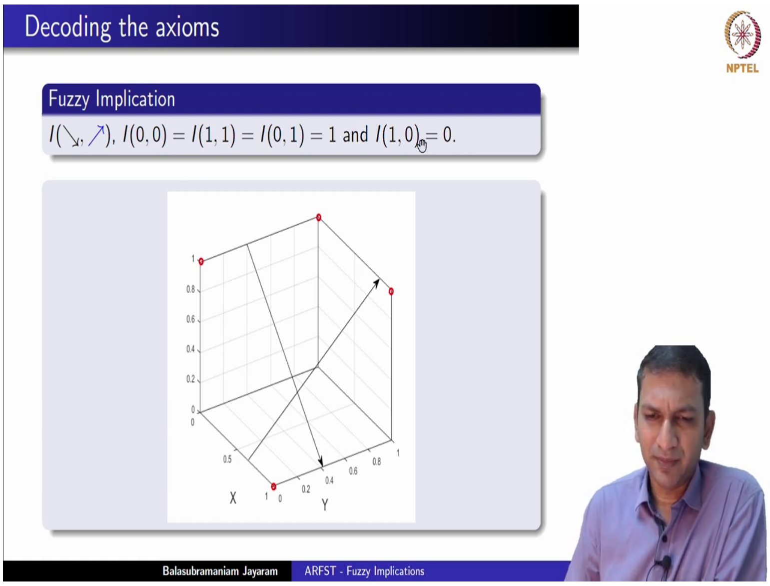

Next, let us try to decode in this definition. So, what we have here, in symbolism? I is decreasing in the first variable and increasing in the second variable and satisfies these 4 boundary conditions.

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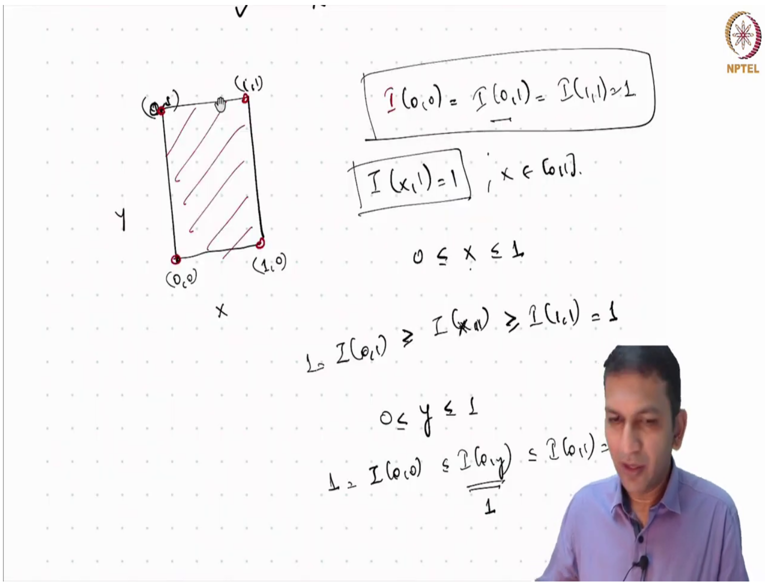

We have seen, that when you are looking at classical logic connectives, this is your x, this is your y. Then what we are looking at is, what happens at these points. So, this was what is defined in the case of classical logic connectives. So, on these vertices of the unit square the classical logic connectives are defined.

Now, what we want to do is, we want to extended this to this entire unit square. Now let us look at a 3D view of a fuzzy implication, now once again note, we have seen that in the definition of fuzzy implication, we have not asked for any algebraic property like symmetry, associativity or even neutrality. So, we have kept the definition extremely simple, we only abstracted the properties that we are able to see in the truth table.

Now, in the case of t norms, the perspective that we took for the plots, was from the origin to 0, 0; however, in the case of fuzzy implications, the perspective that we take is keeping the point 1, 0 as being the nearest point to us and we take this perspective because, this is the perspective that gives maximum information, about the implication from the graph of the implication.

Clearly it is a two variable function, which means the graph is in 3D. So, you see the origin 0, 0 is here. But the closest point to us in our in terms of our perspective what we see is the point 1, 0. So, now, let us look at how to understand the axiomatic definition of a fuzzy implication. First let us look at the boundary conditions. What does it say at 0, 1, it is 1, this is y is equal to 1 and x is also x is equal to x is 0 here, y is equal to 1 here.

So, it is 0 1 at I of 0, 1 is one and this is y is one here and x is also 1. So, I of 1, 1 is 1 here. So, on these three points it is 1, and this is the point x is equal to 1 and y is equal to 0. So, I of 1, 0 is 0. So, these are the four vertices that have been fixed, by the classical implication for us. Now what happens with the monotonic one.

So, it says it is decreasing in the first variable; that means, if we fix a y; that means, if you are on this axis we have fix a y and move along this direction from x is equal to 0 to 1, we expect that it should be decreasing. Similarly, if you look at the second variable; that means, you will fix the first variable fix an x and if you move along the y direction, from as y increases from 0 to 1, we expect that this should be increasing.

So, essentially this is all that we require of a fuzzy implication. At the corners of this unit square, we fix up the boundary condition and we say it is decreasing in the first variable and increasing in the second variable. Now what is interesting is, these four boundary conditions, can be actually reduced to just one of them.

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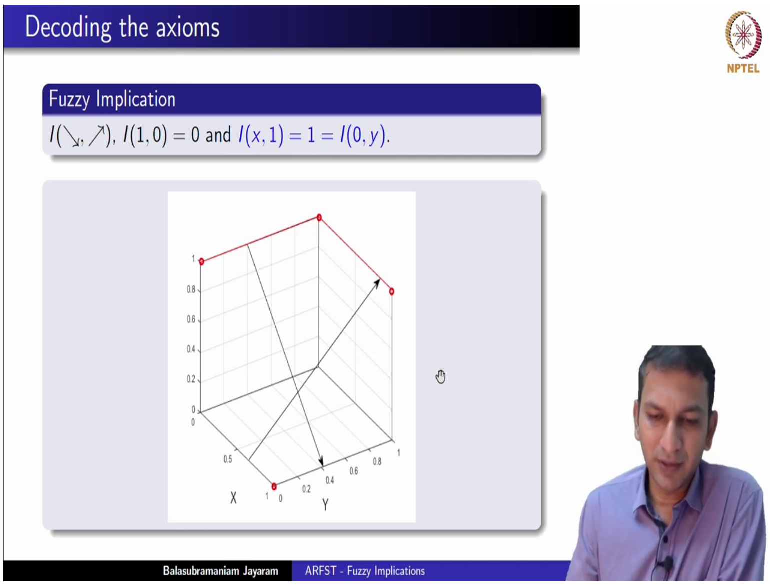

I of 1, 0, 0 and you could actually come up with an alternate way of or equivalent definition by adding these two properties. And note that, these properties can be obtained from the previous three boundary conditions. It is quite simple, let us look at what they mean to us.

So, what we have is the boundary conditions I of 0, 0 is equal to I of 0, 1 is equal to I of 1, 1 is equal to 1. Now this says I of x 1 is equal to 1 for any x element of 0, 1. Now let us look at I of x 1, now since x belongs to 0, 1 and then there is calculate them. So, we know that 0 less than or equal to x less than or equal to y.

Now, if you take I of x 1, we know that I is decreasing in the first variable which means I of x 1 is greater than or equal to I of 1, 1 right. Because as x 1 is bigger than x, I of x 1 is greater than or equal to I of 1, 1 that this is equal to 1. And we know that the function I itself is from 0s 1 square to 0, 1

So, nothing can be greater than 1, but if you would want, you could also bound this like this which is 1. So, from here it is clear, that I of x 1 is equal to 1 for every x amount of 0, 1. So, this is what we have seen here. Similarly, if we look at I of 0, 1, once again y is between 0 and 1, now we know that, in the second variable it is increasing. So, now, if you take I of 0, 0 less than or equal to I of 0 y which is less than or equal to I of 0, 1.

But note here, I of 0, 1 is 1, I of 0, 0 is 1 and essentially stating that this is also equal to 1. So, from these three conditions here, we actually get that I of x, 1 is 1 and I of 0, y is 1. Of course, I of 1, 0 cannot be obtained from anything else not from the monotonicity and the other boundary conditions. So, this has to be preserved.

Now, what does it mean in terms of geometry? It means, that we are talking about these boundaries. Look at this, this when x is equal to 0 and as we go along the y direction, it is 1 that is this boundary and this is y is equal to 1 boundary and as you walk along this boundary from x is equal to 0 to 1, we say that it is 1.

Because of monotonicity, these three points can only be linked at the height of 1. So, geometrically if you see, the left boundary is 1, the top boundary is 1 and the point closest to you or essentially the right bottom point is at 0.

This is all that we insist, on a fuzzy implication and this also comes above because of the boundary conditions and the monotonicity that we see here. Having this kind of a geometric perspective, helps us in graphing the given implication and also be able to visualize, any given fuzzy implication because some of these things are actually fixed for us and it also makes it interesting to construct implications with some specific properties.

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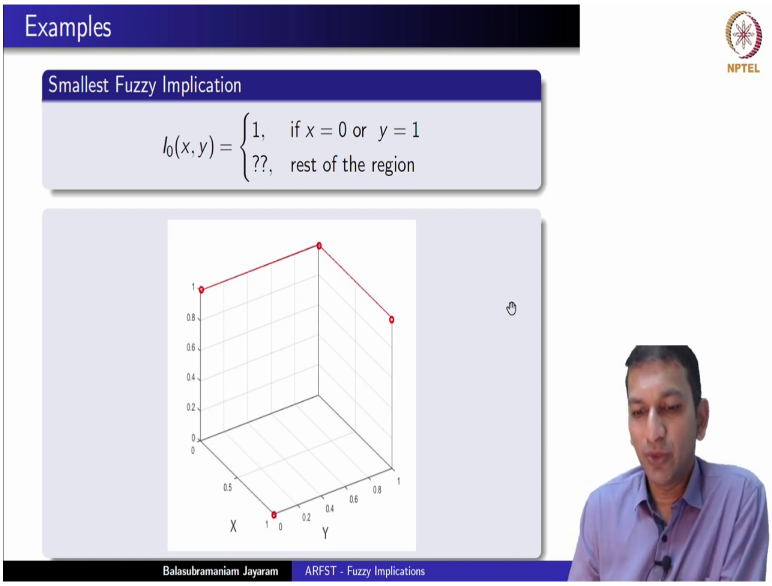

Let us look at a few of them. We can ask the question, is there a smallest fuzzy implication? Smallest in the sense of the point wise ordering of functions we know this that when x is equal to 0, this region that is 1 and y is equal to 1 it is 1 and at this point it is 0. Nothing else is specified, of course, overall we know that it should be increasing along y and decreasing along x.

But nothing else is specified. So, if you ask the question, what could be the smallest fuzzy implication? Then we keep these and then try to push the values in the rest of the region as small as possible. How low can we clearly we cannot go below 0.

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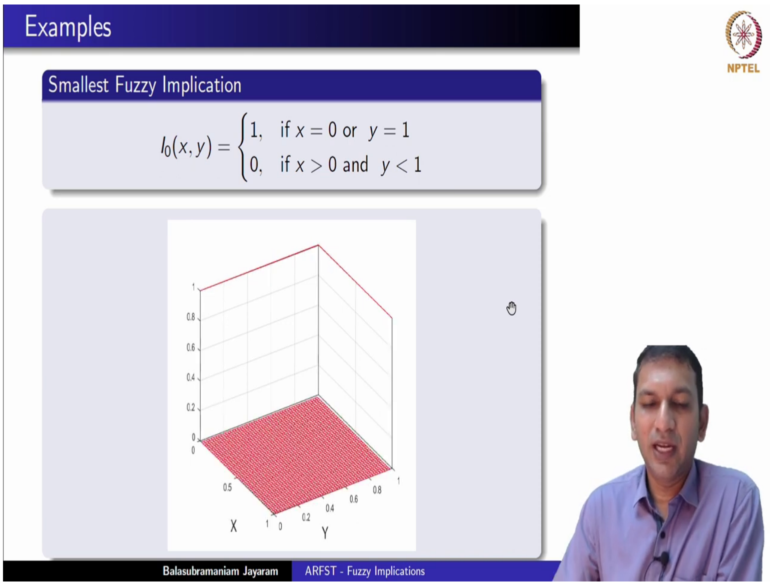

So, essentially, we just make it 0. How would the graph of this fuzzy implication look like? it would look like this. And in the literature, this is typically indicated by the implication symbol I 0. Essentially saying ok this is the least or the smallest fuzzy implication that you could have, where it is 0 almost everywhere except on the left and the top boundary, where it has to be 1.

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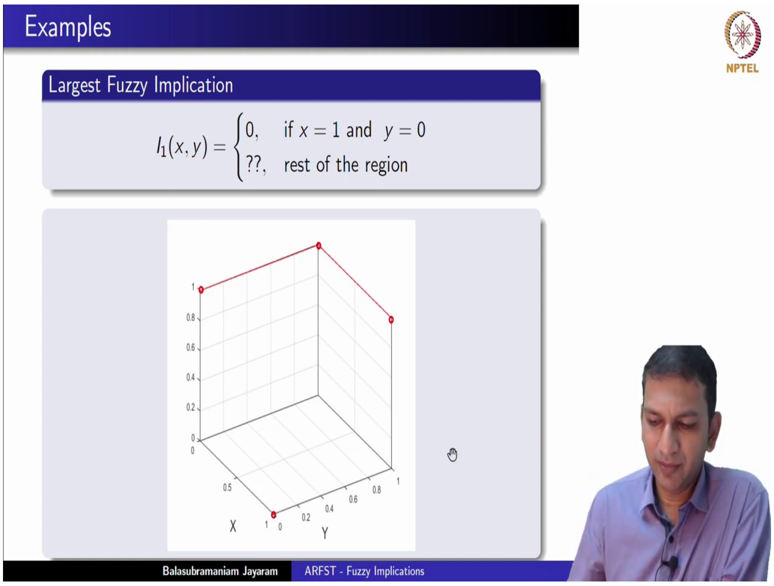

Similarly, we could ask the question, what would be the largest fuzzy implication. Once again the left and the top boundaries are fixed at 1 and the point closest to us, which is 1, 0 is 0.

Now, if you want it to be largest in terms of point wise ordering, then we want this function value to be as high as possible. Clearly it cannot go above 1. So, what we do is in rest of the places except at this point.

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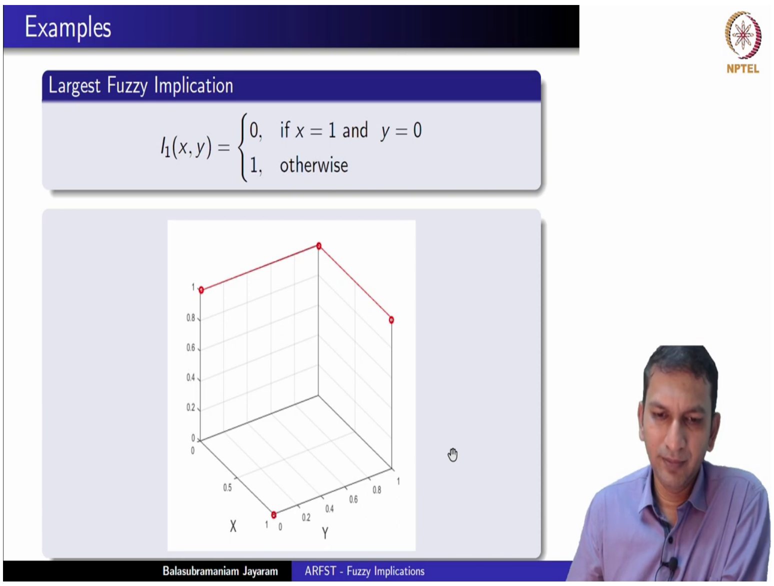

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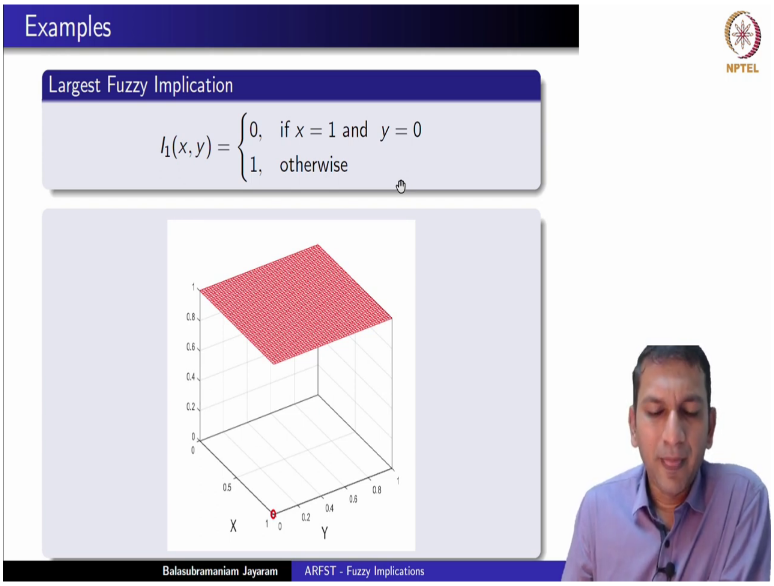

We make it 1. So, the graph of this largest fuzzy implication would look like this, it is almost everywhere 1 except at the point 1, 0.

At x is equal to 1 and y is equal to 0 which is this point which is 0 elsewhere it is 1 and typically it is denoted as I 1 these 2 notations actually come from considering set of all fuzzy implications as a not just as opposed with respect to point wise ordering, it can be shown it is also a lattice, but perhaps we may either see it a little later, when the situation warrants or maybe not also right.

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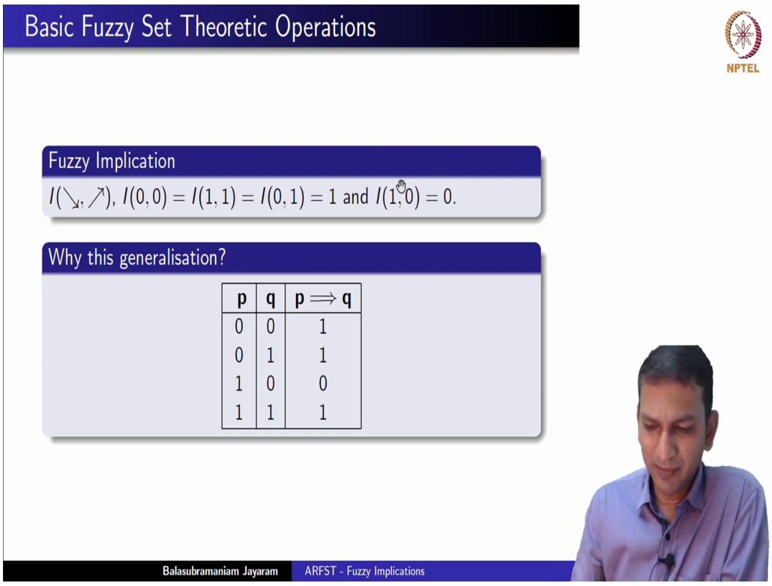

Now, let us look at why in the literature, they have gone for this particular generalization. Once again, I am going back the definition; that means, it is decreasing in the first variable and increasing in the second variable, and these four boundary conditions. Now why this generalization very clearly we have seen that we have got it from the truth table of the corresponding classical implication.

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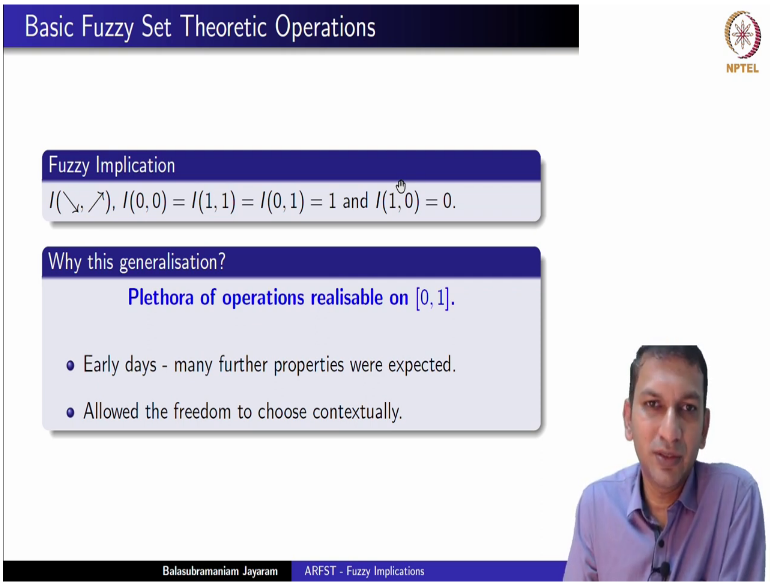

There is also another reason, if you maintain only this note that we have not asked for commutativity; obviously, we do not have commutativity because, the type of monotonicity that we have is mixed monotonicity its not the same, we have not looked at some form of associativity we have not asked for it to be having a neutral element, none of those algebraic properties we have asked. Because, this definition captures the essence of the classical implication truth table and allows us to have a plethora of operations. Which are useful in different contexts especially from the point of view of approximatism.

It should also be noted, that in early days, many other properties were expected of fuzzy implication, but slowly people realized that the freedom that having these basic axiom set alone as part of this definition, allowed them especially in different contexts. Hence, currently in the literature, this is accepted as the standard definition of fuzzy implication.

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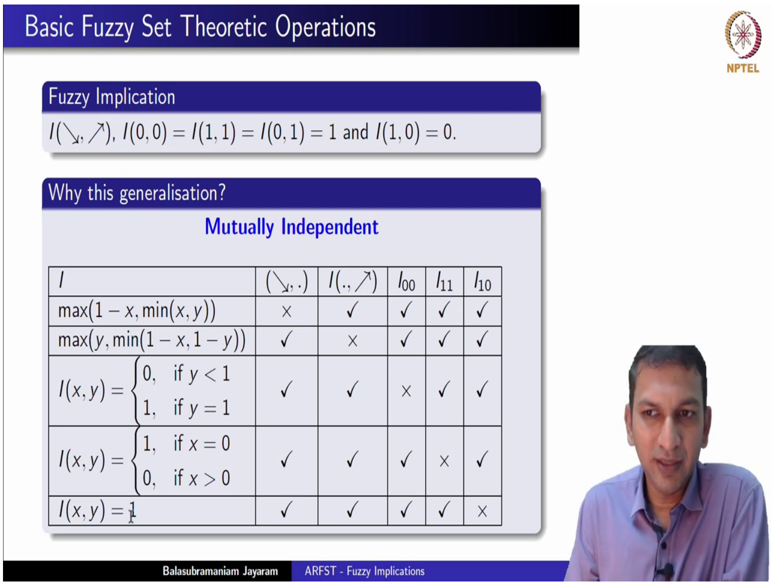

Finally, we can also show, that these properties are mutually independent. For instance, let us look at it, these are the five properties that we could consider, I 0, 0 means, whether at I of 0, 0 it is 1, at I 1, 1 means whether I of 1, 1 it is 1 and clearly I of 0, 1 will be 1 and we are looking at I of 1, 0 whether it is 0. Let us look at this function, max of 1 minus x, min of x, y. What we see is, this function has these four properties except for decreasing in the first variable. Now, this is not very difficult to see.

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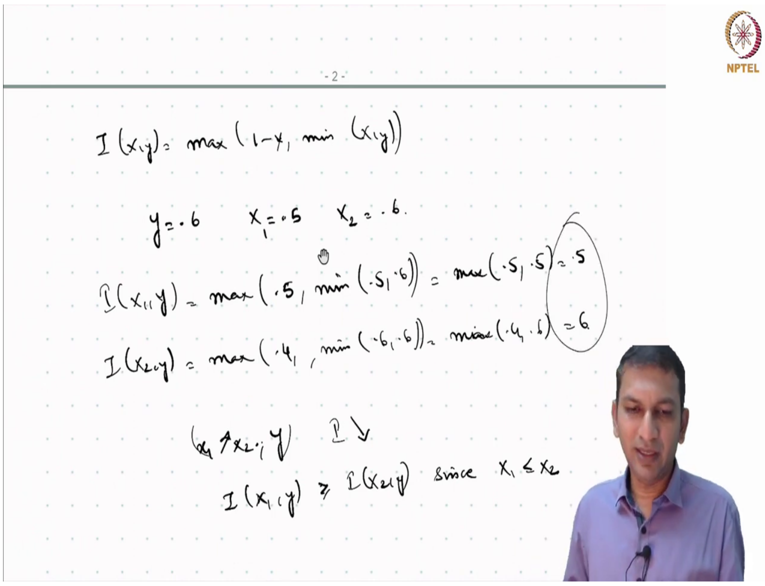

Let us look at this function I of x, y is max of 1 minus x, min of x, y. So, what we want to show is, that this function is actually not decreasing in the first variable. So, for this let us fix the second variable to be 0.6 and the first variable let us move from x 1 is equal to 0.5 to x 2 is equal to 0.6.

So, let us look at what is I of x 1, y. Now this is max of 1 minus x, x 1 is 0.5, 1 minus x 1 to that is 0.5, minimum of 0.5, 0.6. Now this will be max of 0.5, 0.5 which is 0.5, what about I of x 2, y? Is max of 0.4 this 1 minus x 2, min of 0.6comma 0.6, which is max of 0.4, 0.6 which is 0.6.

Now, remember what we want is as fixing y, we are seeing that, x 1 has gone up to x 2, fixing y and what we want is, the implication should actually be decreasing; that means, I of x 1, y, should be greater than or equal to I of x 2, y since x 1 is less than or equal to x 2 this is what we want for I to be decreasing in the first variable. But as you can see here it is actually increasing.

It can be easily verified, that if you fix x here as you increase y, it will increase I of 0, 0 is actually 1, put 0 here it is max of 1, something which will be 1, if you put 1 here, this will be 0, but min of 1, 1 is 1. So, max of 0, 1 is 1 and 1, 0 max of 1 minus 1 is 0, minimum of 1, 0 is 0, it is max of 0, 0.

So, I of 1, 0 is 0 you know also the single I of 0, 1 is 1, we put 0 here is essentially it just becomes 1 ok. Now, if we consider this function almost the dual of what we have seen earlier, it can be shown that this function is decreasing in the first variable, but it is increasing it is not increasing in the second variable.

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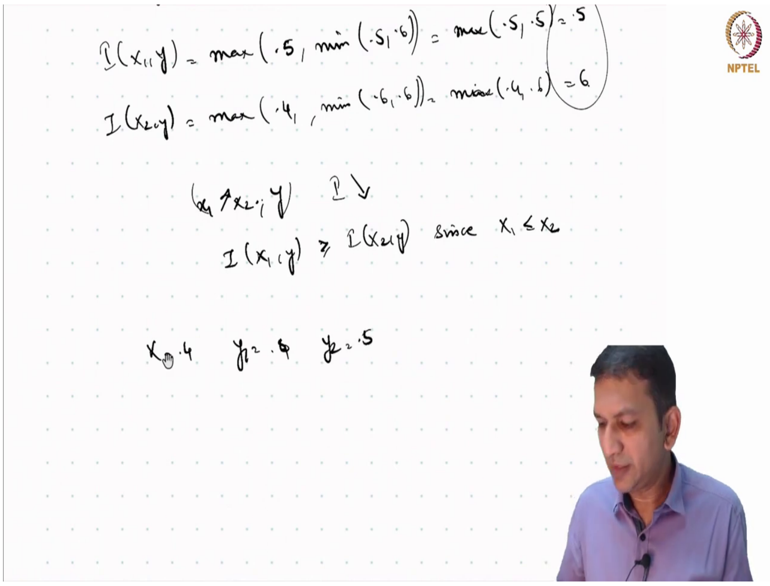

Once again we can take some selective values for instance and if you take the value x is equal to 0.4 and y 1 is equal to 0.4, y 2 is equal to 0.5, I think that easy to see for this values that it is not increasing in the second variable. Now if you consider this function, I of x y is given like this, once again you can see that, it is decreasing in the first variable, increasing in the second variable as the other properties except that and I of 0, 0 it is not 1.

When y is less than 1; that means, the moment y is 0, it is 0. If you consider this function we see that I of 1, 1 is not 1 because when x is 1, it is 0. Finally, this simple function shows that, you can have all the first four properties, but I of 1, 0 it is actually not equal to 0. So, you see here, the way we have obstructed these properties they are also mutually independent ok.

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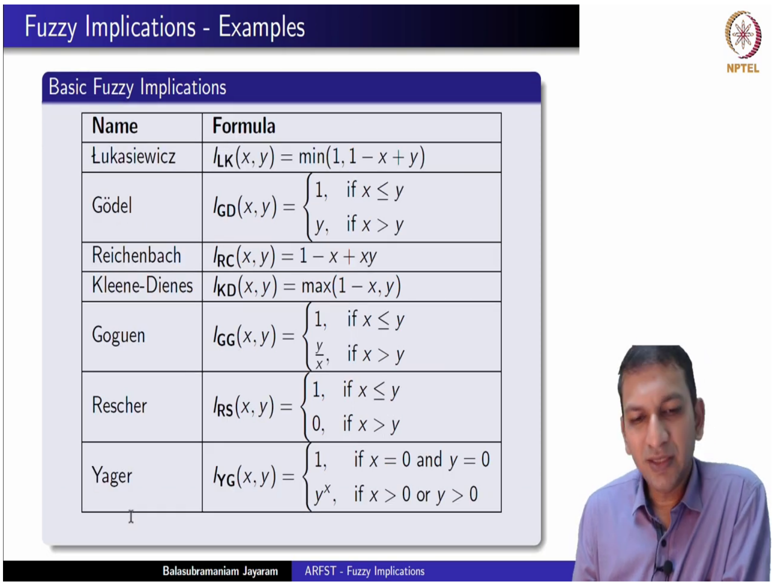

Just as in the case of t norms, where we had 5 basic t norms, the drastic Lukasiewicz product minimum and the Fodor t norm, the nilpotent minimum. Here also, we have what we would consider as basic fuzzy implications, but not just 5 of them, but 9 of them, and if you add also the smallest and largest fuzzy implications, it turns out to be 11.

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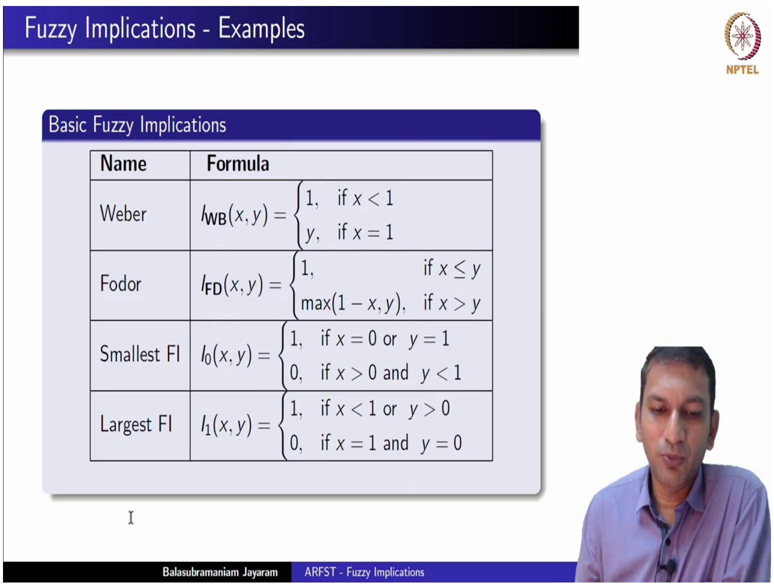

So, these are considered as basic, as we will see later on just like how these basic t norms played a role in characterization and representation, these fuzzy implications will also play a similar role, both in theory and application.

To better appreciate this formulae, let us look at how the graph of these fuzzy implications would look like ok.

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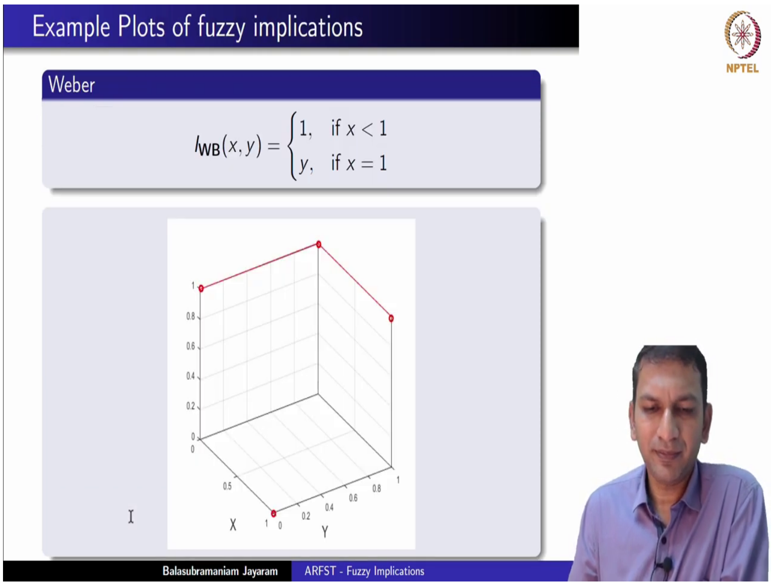

So, now, the Weber implication is given as this, the graph that you see on the screen is the skeletal graph of a fuzzy implication. Where we have only specified how it should be on the left boundary and the right boundary and also at the point 1, 0.

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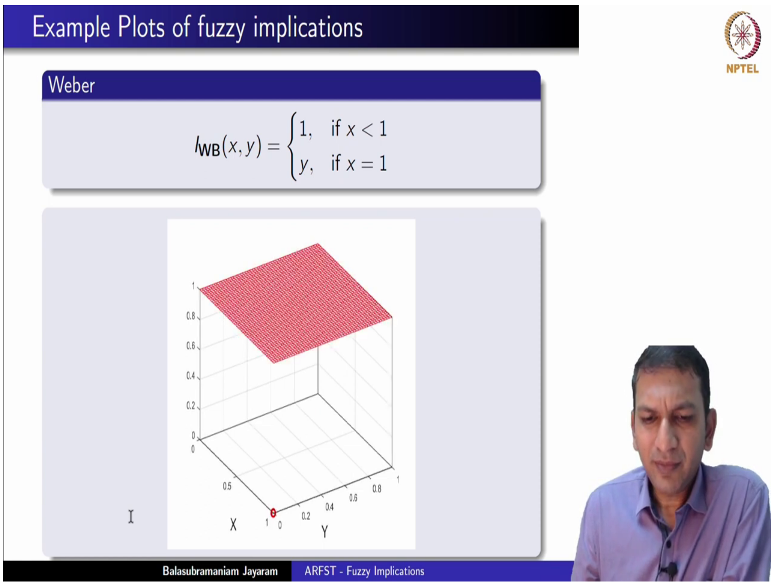

Now this was the largest fuzzy implication that we saw, Weber implication is slight modification of this. It is one everywhere, except at this point 1, 0. but when x is equal to 1, we make it the identity function.

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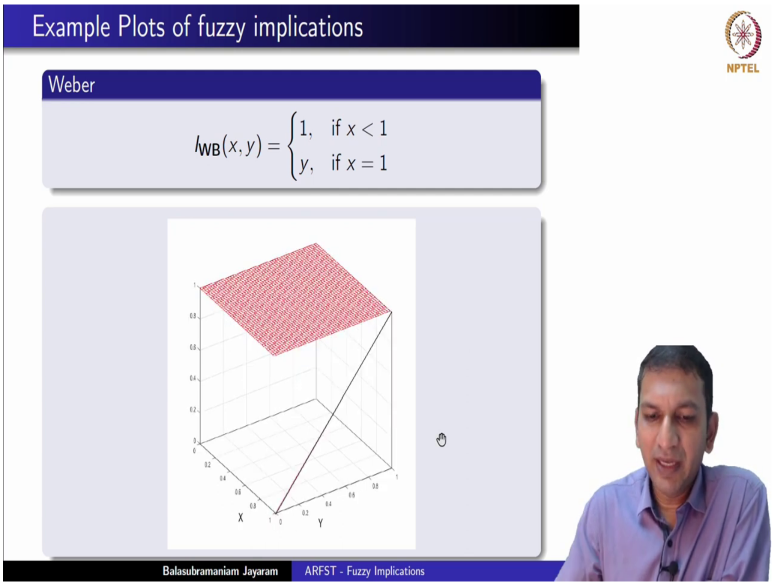

So, essentially the y of 1, 0 is 0, I of 1, 1 is still 1 and what we are doing is when x is equal to 1, we just making it y. So, this is the Weber implication.

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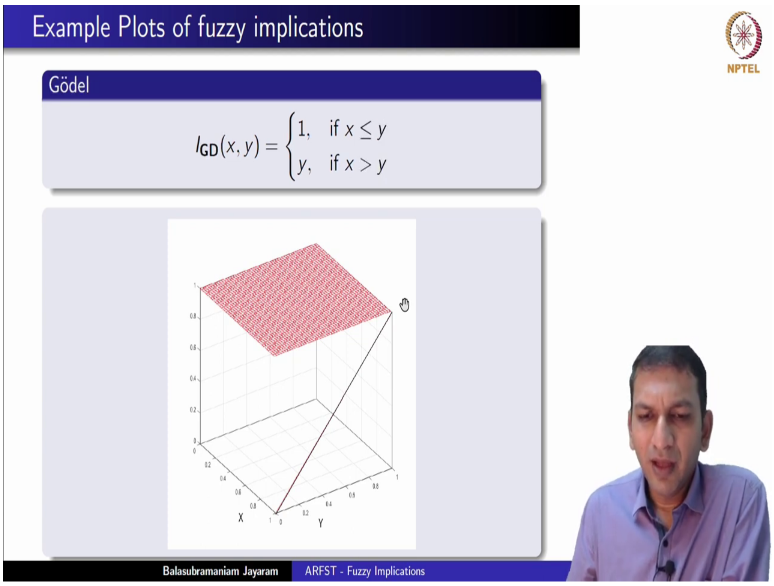

The Godel implication what it does is in some sense, it presses the part of the roof onto this line here.

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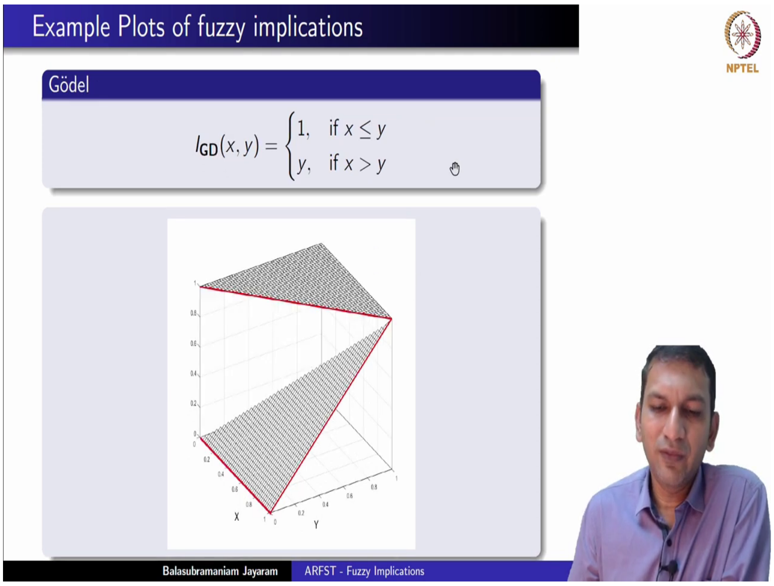

So, this is how the Godel implication will look like, when x is less than or equal to y it is 1 and anyway outside of this domain it is actually 1.

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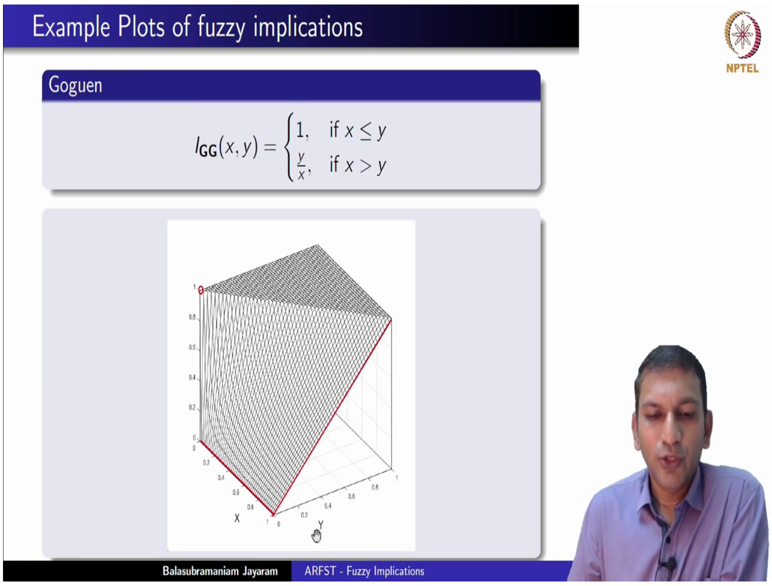

The Goguen implication can be looked at as somehow trying to make this graph look continuous of course, it cannot be continuous because at 0, 0 it is 1, but when x is not 0, but y is 0 we see that it is actually 0.

So, you have a point of discontinuity at 0 0, but otherwise in other places, it is 1 and the formula is given like this in the case of Godel, when x is greater than y it is y in the case of Goguen, it is y by x and this is how the plot of this fuzzy implication would look like.

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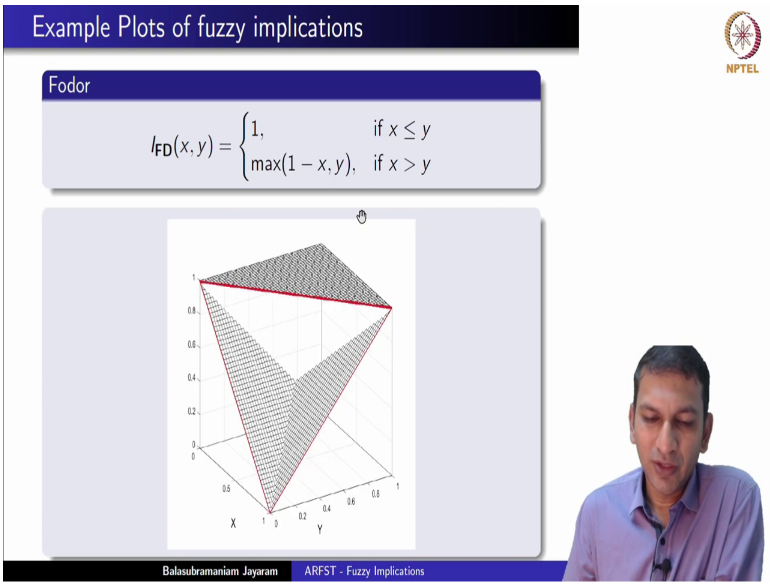

Now, this is what we call the Fodor implication, where you have one above the diagonal, essentially x is equal to y is the diagonal. So, above the diagonal you have 1. And below the diagonal to the right of the diagonal, this is how the graph looks like.

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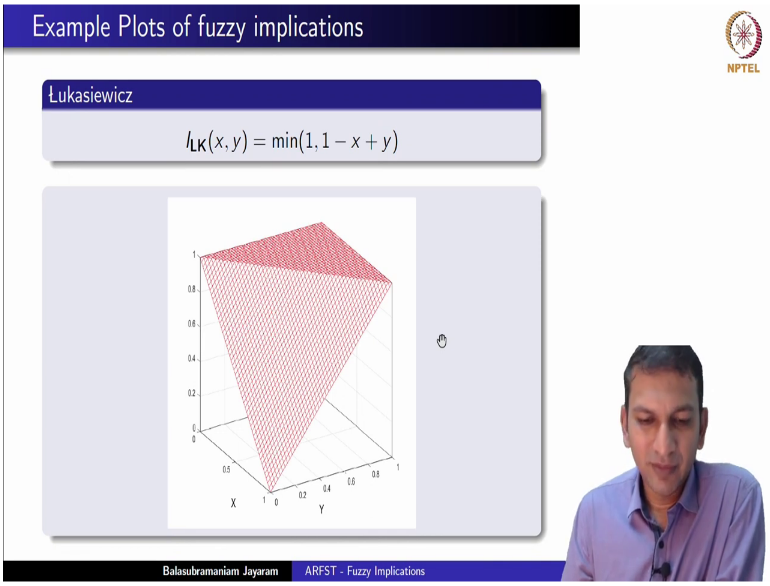

From here, in some sense the continuous complication I mean completion you could think of as would give you the Lukasiewicz implication, but not necessarily in the same way, but just only in terms of taking a visual view.

Finally, from this Fodor implication you could also get other kinds of applications.

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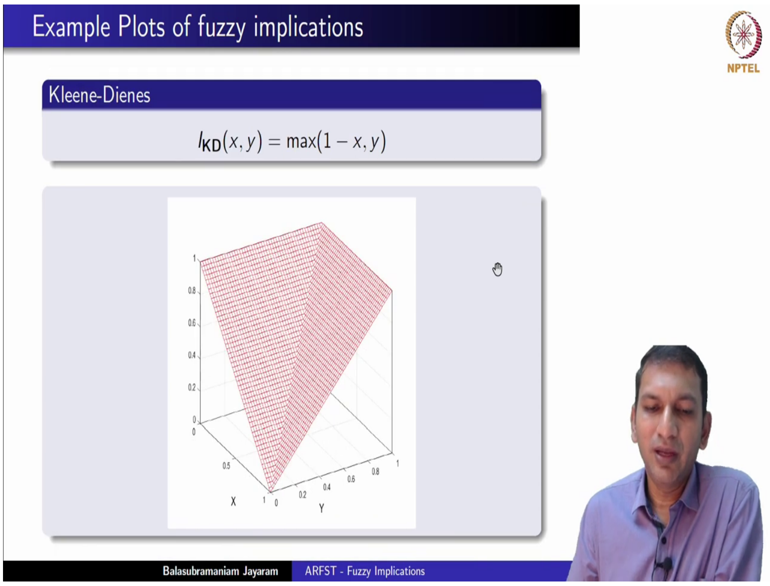

For instance, extended without having 1 in no other place other than on the left hand top boundaries, extending it you would get what we call the Kleene dienes implication.

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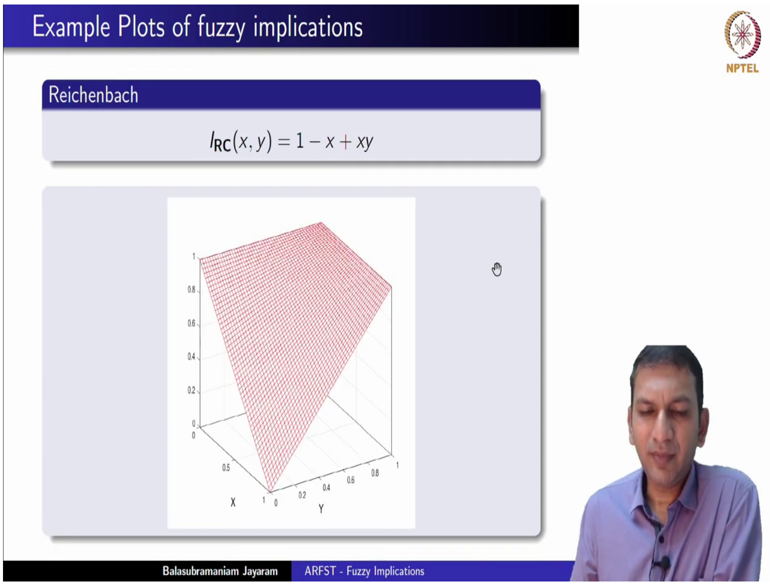

First, there is the Reichenbach implication.

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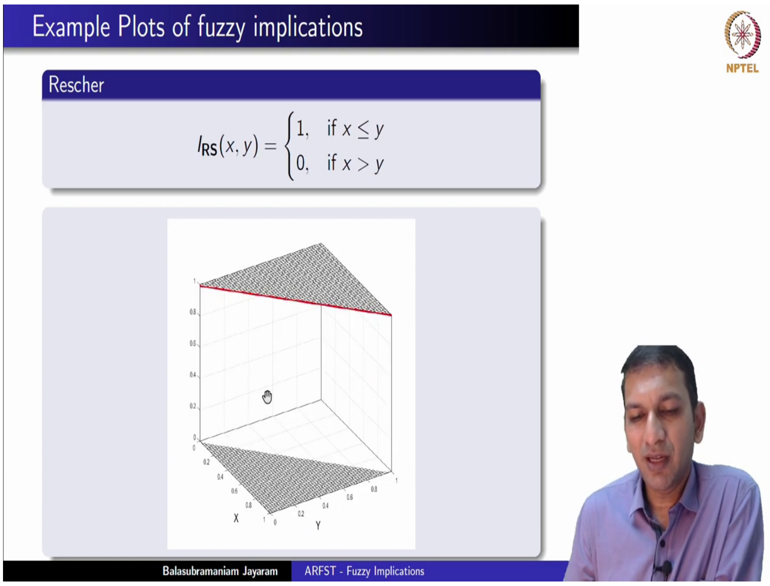

And if you push this part, down you would get the Rescher implication. So, earlier here outside of the diagonal we had some values. So, all it says is above the diagonal we have 1 and below the diagonal we have 0 ok.

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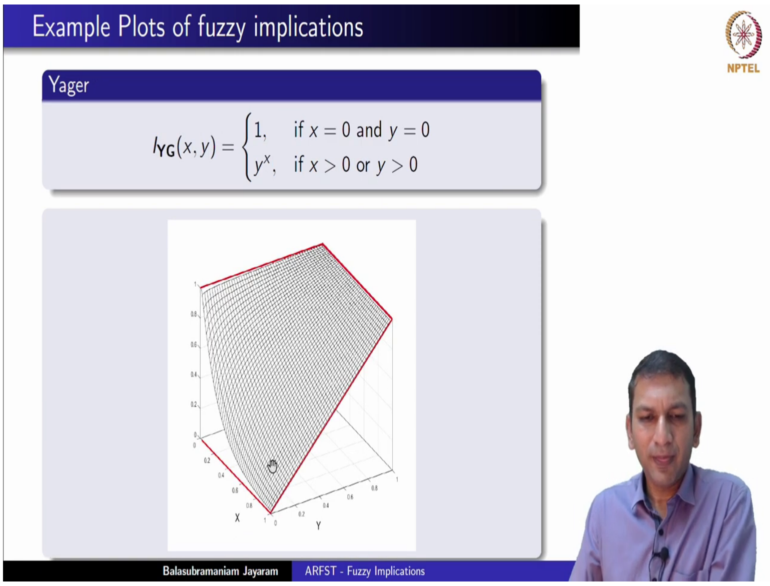

This is another important implication, proposed by Yager and this is how which looks like. Of course, these will keep appearing again and again and we would slowly get comfortable with this formulae.

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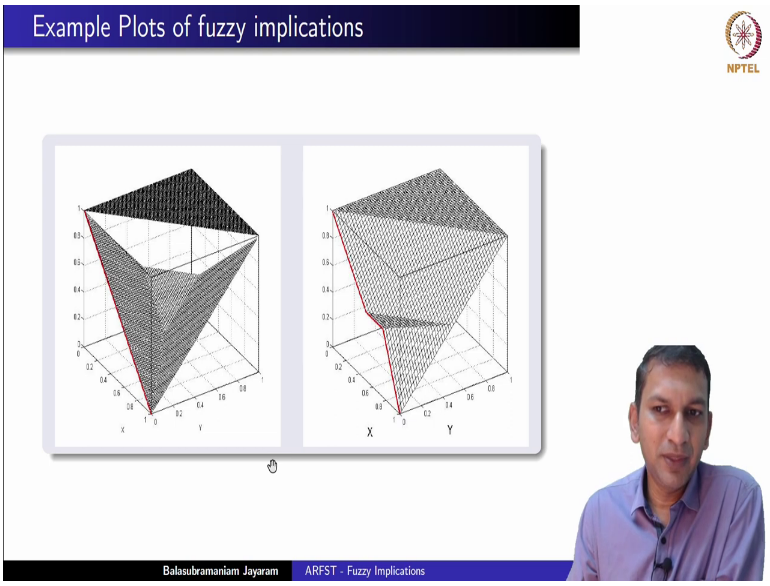

But these are not the only kind of implications I said these are the basic fuzzy implications which keep appearing again and again, but you could also have fuzzy implications of this type ok.

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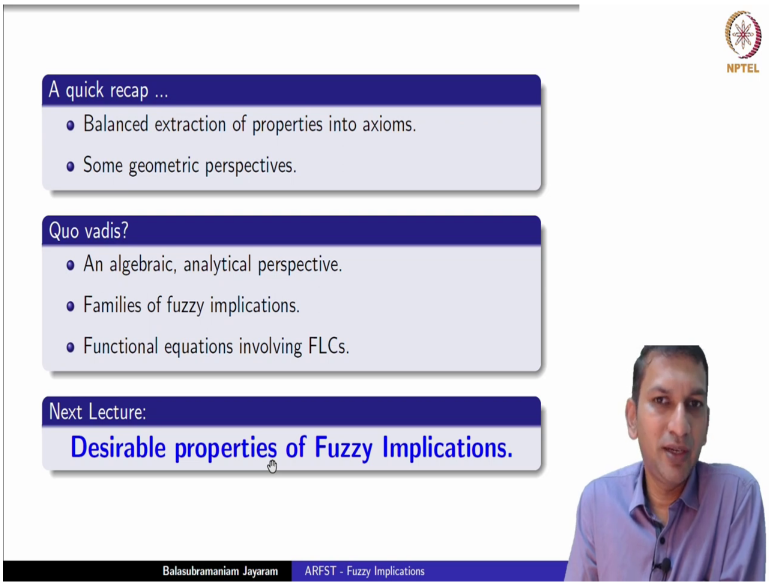

A quick recap of what we have seen in this lecture. Once again, we have made a balanced extraction of properties of classical implication into an axiomatic definition for fuzzy implication, we have also seen some geometric perspective. Especially of the basic fuzzy implications. Now what lies ahead in the next few lectures to come during this week, we will look at an algebraic and analytic perspective of fuzzy implications themselves, we will look at constructions of fuzzy implications which lead to the families of fuzzy implications.

And we will also look at some functional equations, involving fuzzy logic connectives; specifically implications. In the next lecture, we will look at some desirable properties of fuzzy implications. As was mentioned during this lecture, initial days the definition of fuzzy implication included many other property especially neutrality and exchange principle, which we will see as properties which are desirable of course,

But you may not find a place in an axiomatic definition of the fuzzy implication and there are no more such desirable properties, some of them which are useful considering the context of this course, the focus of this course, we will deal with those on those alone rest of them as and when required perhaps we will see.

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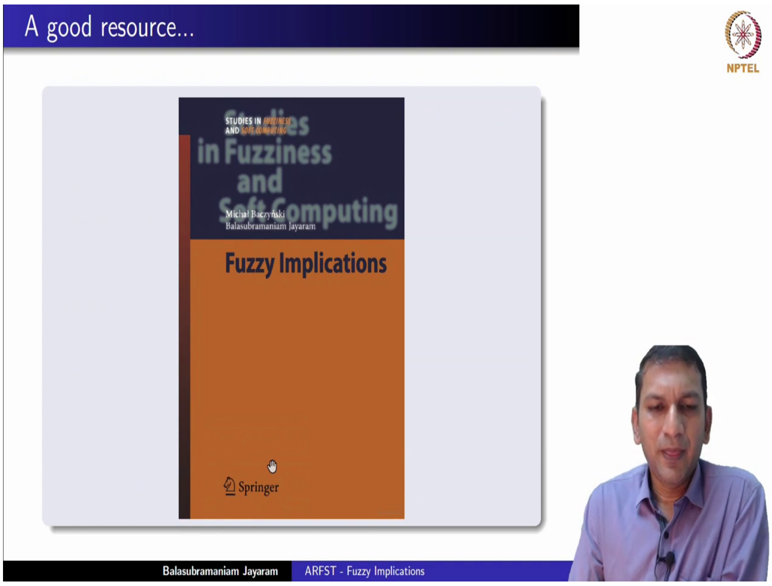

Finally a good resource for what has been covered in this lecture, is the book of fuzzy implications, you would find more examples more plots and also some interesting properties dealing with even the basic fuzzy implications. Which we may not be able to cover in this course, because we would want to stay true to the title of this course that of approximate reasoning using fuzzy set theory.

So, we pick the content keeping that in mind keeping that as our focus in mind, but there are also many other very interesting things that you can discuss about fuzzy implications and at least definitely what has been covered in this lecture, you will find that a little bit more in terms of explanation and some interesting examples in this book on fuzzy applications.

Thank you for joining me in this lecture and hope to see you in the next lecture.